Stable characters from permutation patterns (jt. w/ Christian Gaetz)

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Permutation Patterns

If σ is an element of the symmetric group S_k, we write σ in one-line notation:

$$\sigma(1), \sigma(2), \ldots, \sigma(k).$$

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Given $\tau \in S_n$, we say $(i_1, i_2, \ldots, i_k)$ is an *occurrence* of σ in τ if

- $1 \leq i_1 < i_2 < \cdots < i_k \leq n$
- $\tau(i_1), \tau(i_2), \ldots, \tau(i_k)$ is σ-*sorted*.

We let $N_\sigma(\tau)$ be the number of occurrences of σ in τ, and if $N_\sigma(\tau) = 0$, we say τ is σ-*avoiding*.

Pattern Avoidance

Example:
The only 21-avoiding element of S_n is $12\cdots n$ (the identity). In fact $N_{21}(\tau)$ is equal to the (Coxeter-theoretic) length of τ.

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Many properties of permutation are controlled by avoidance of permutation patterns.

Examples:

- Knuth's stack-sortable permutations are those that avoid the pattern 231.
- The Schubert variety associated to σ is smooth if and only if σ avoids both 3412 and 4231 (Lakshmibai–Sandhya theorem).

Pattern Avoidance in Conjugacy Classes

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> *"That really is hell."*
>
> *- Anonymous Combinatorialist on counting pattern-avoiding permutations in conjugacy classes.*

We show: the **average number** of occurrences of σ in a conjugacy class $\mathcal{C}$ of S_n is well behaved.

$$\frac{1}{|\mathcal{C}|}\sum_{\tau\in\mathcal{C}} N_\sigma(\tau)$$

Schur-Weyl Duality

Suppose V is a vector space. Then $GL(V)$ acts on $V^{\otimes d}$, which has a commuting action of S_d by permuting tensor factors. So $V^{\otimes d}$ has an action of $GL(V) \times S_d$. In fact:

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If M is a matrix with eigenvalues x_i, and $g \in S_d$ has cycle type $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$, then

$$tr_{V^{\otimes d}}(M \otimes g) = p_{\mu_1}(x)p_{\mu_2}(x)\cdots p_{\mu_l}(x) = p_\mu(x),$$

where $p_r(x_i) = \sum_i x_i^r$ is the r-th power-sum symmetric function.

Example

Suppose $M = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ with respect to a basis $\{v_1, v_2, \ldots, v_n\}$ of V. Also let $g = 213 = (12)(3)$, which has cycle type (2,1). Then, acting on $V^{\otimes 3}$:

$$(M \otimes g) v_i \otimes v_j \otimes v_k = (x_i x_j x_k) v_j \otimes v_i \otimes v_k.$$

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The only terms that contribute to the trace are those with $i = j$, which contribute $x_i^2 x_k$. Summing over i and k gives $p_2(x) p_1(x)$.

Partition Diagrams

An (r, s)-*partition diagram* consists of:

- a top row with vertices $1, 2, \ldots, r$
- a bottom row with vertices $1', 2', \ldots, s'$
- a partition of the set $\{1, 2, \ldots, r, 1', 2', \ldots, s'\}$ into subsets

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Usually we draw subsets as connected components of a graph (in an arbitrary way).

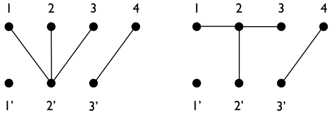

Figure: Two depictions of $\{\{1'\}, \{1, 2, 3, 2'\}, \{3', 4\}\} \in \mathrm{Par}_{4,3}$.

Multiplication of Diagrams

Fix a choice of $t \in \mathbb{C}$.

Given $\mathbf{x} \in \mathrm{Par}_{q,r}$ and $\mathbf{y} \in \mathrm{Par}_{r,s}$, define $\mathbf{xy} \in \mathbb{C}\mathrm{Par}_{q,s}$ as follows.

- Concatenate the diagrams of $\mathbf{x}$ and $\mathbf{y}$ by merging the rows with r vertices.
- This gives a (q, s)-partition diagram $\mathbf{z}$, possibly together with p components of the r middle vertices that are not connected to any vertex in $\mathbf{z}$.
- We take $\mathbf{xy} = t^p\mathbf{z}$

Example

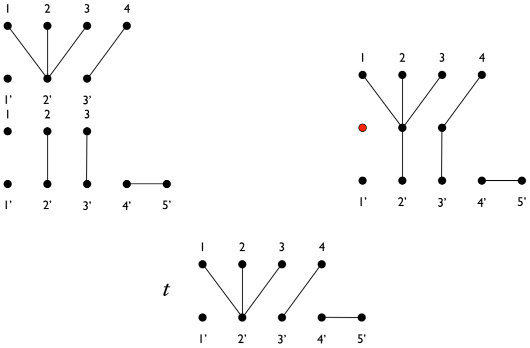

Figure: An example $\mathrm{Par}_{4,3} \times \mathrm{Par}_{3,5} \to \mathbb{C}\mathrm{Par}_{4,5}$:

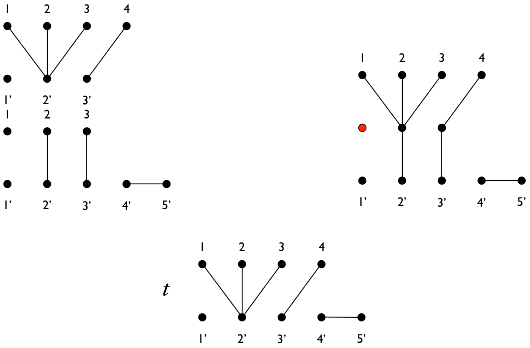

Figure: An example $\mathrm{Par}_{4,3} \times \mathrm{Par}_{3,5} \to \mathbb{C}\mathrm{Par}_{4,5}$:

The *Partition algebra* $P_k(t)$ is $\mathbb{C}\mathrm{Par}_{k,k}$ with the above multiplication.

Duality with Symmetric Groups

Let $V = \mathbb{C}\{e_1, e_2, \ldots, e_n\}$ with the symmetric group S_n acting by permutation. There is a map $\mathbb{C}\mathrm{Par}_{r,s} \twoheadrightarrow \hom(V^{\otimes s}, V^{\otimes r})$ defined by the following.

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Let $\mathbf{x} \in \mathrm{Par}_{r,s}$, and $v = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_s} \in V^{\otimes s}$. Label the component containing q' in $\mathbf{x}$ with i_q. If this labelling is inconsistent, $\mathbf{x}v = 0$.

Otherwise we obtain $\sum e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}$, where j_q is the label of the component containing q if it is labelled, and we sum over all possible values of j_q if it is not.

Example 1

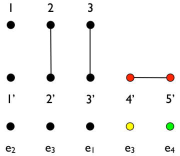

Figure: Inconsistent labellings: obtain zero.

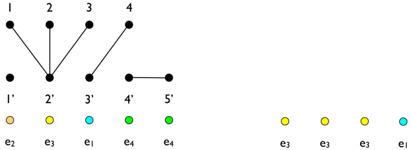

Figure: Consistent labellings: result nonzero.

Example 2

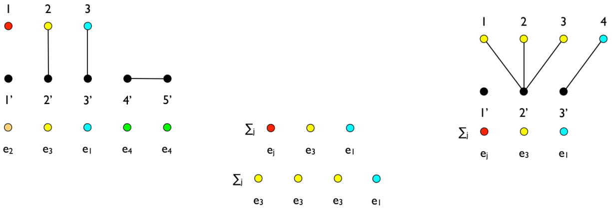

Figure: Two examples

Note that in the above, the final Σ_j just becomes n.

Theorem

Composition of maps of S_n representations coming from partition diagrams is satisfies the relations of composition of partition diagrams with parameter $t = n$.

Duality Theorem

If $V = \mathbb{C}^n$ with the permutation action of S_n, then we have an action of S_n on $V^{\otimes k}$. This has a commuting action of $P_k(n)$, leading to an action of $\mathbb{C}S_n \otimes P_k(n)$ on $V^{\otimes k}$. In fact,

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If $g \in S_n$ and $\mathbf{x} \in \mathrm{Par}_{k,k}$, we may as for the trace

$$tr_{V^{\otimes k}}(g \otimes \mathbf{x}).$$

There is a combinatorial formula that is a polynomial in $n, m_1, m_2, \ldots, m_k$ (where m_i is the number of i-cycles in the cycle type of g).

Unification (1/4)

For $\rho \in S_k$, let $E_\rho : \mathbb{C} \to V^{\otimes k}$ be the linear map such that

$$E_\rho(1) = \sum_{(i_1, i_2, \ldots, i_k)\ \rho-\text{sorted}} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}.$$

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Using properties of trace,

$$tr_{\mathbb{C}}(E_\sigma^T \tau E_{\mathrm{Id}}) = tr_{V^{\otimes k}}(\tau E_{\mathrm{Id}} E_\sigma^T) = \frac{1}{n!} \sum_{g \in S_n} tr_{V^{\otimes k}}(g \tau E_{\mathrm{Id}} E_\sigma^T g^{-1})$$

Unification (2/4)

$$N_\sigma(\tau) = \frac{1}{n!}\sum_{g\in S_n} tr_{V^{\otimes k}}(g\tau E_{\mathrm{Id}} E_\sigma^T g^{-1})$$

If we average τ over a conjugacy class $\mathcal{C}$ of S_n, we get

$$\frac{1}{|\mathcal{C}|}\sum_{\tau\in\mathcal{C}} N_\sigma(\tau) \quad = \quad \frac{1}{n!}\sum_{g\in S_n} tr_{V^{\otimes k}}\left(g\left(\frac{1}{|\mathcal{C}|}\sum_{\tau\in\mathcal{C}}\tau\right)E_{\mathrm{Id}}E_\sigma^T g^{-1}\right)$$

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using the fact that conjugacy class sums are central.

Unification (3/4)

$$\frac{1}{n!}\sum_{g\in S_n} tr_{V^{\otimes k}}\left(\left(\frac{1}{|\mathcal{C}|}\sum_{\tau\in\mathcal{C}}\tau\right) gE_{\mathrm{Id}}E_{\sigma}^{T}g^{-1}\right)$$

$$= \quad tr_{V^{\otimes k}}\left(\left(\frac{1}{|\mathcal{C}|}\sum_{\tau\in\mathcal{C}}\tau\right)\left(\frac{1}{n!}\sum_{g\in S_n} gE_{\mathrm{Id}}E_{\sigma}^{T}g^{-1}\right)\right)$$

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Magical lemma:

$$\left(\frac{1}{n!}\sum_{g\in S_n} gE_{\mathrm{Id}}E_\sigma^T g^{-1}\right) = \sum_{\mathbf{x}\in\mathrm{Par}_{k,k}} a_{\sigma,\mathbf{x}}\mathbf{x}$$

for some constants $a_{\sigma,\mathbf{x}} \in \mathbb{Q}$ independent of n.

Unification (4/4)

So we have

$$\frac{1}{|\mathcal{C}|}\sum_{\tau\in\mathcal{C}} N_\sigma(\tau) = \frac{1}{|\mathcal{C}|}\sum_{\tau\in\mathcal{C}}\sum_{\mathbf{x}\in\mathrm{Par}_{k,k}} a_{\sigma,\mathbf{x}} tr_{V^{\otimes k}}(\tau\otimes\mathbf{x})$$

$$= \sum_{\mathbf{x}\in\mathrm{Par}_{k,k}} a_{\sigma,\mathbf{x}} tr_{V^{\otimes k}}(\tau\otimes\mathbf{x})$$

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So the original expression becomes a sum of polynomials in n and the cycle counts of τ (i.e. $n, m_1, m_2, \ldots, m_k$). So it is itself such a polynomial.

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We can modify the argument for d-th moments of $N_\sigma(\tau)$:

$$\frac{1}{|\mathcal{C}|}\sum_{\tau\in\mathcal{C}} N_\sigma(\tau)^d$$

which turns out to be a polynomial in $n, m_1, m_2, \ldots, m_{dk}$ (and we can bound the degree).

Conclusion

If we change "counting pattern avoidance" to "counting pattern occurrence", interaction with the group structure becomes tractable.

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If we define a class function f on S_n via

$$f(\mathcal{C}) = \frac{1}{|\mathcal{C}|} \sum_{\tau \in \mathcal{C}} N_\sigma(\tau),$$

then we have

$$f = \sum_{|\lambda| \leq k} a_{\sigma,\lambda}(n) \chi^{(n-|\lambda|,\lambda)},$$

where χ^μ is the character of S_n indexed by μ, and $a_{\sigma,\lambda}(n)$ is a polynomial of degree at most $k - |\lambda|$.