Quasiperiodic Solutions for Reversible Oscillators at Resonance *

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Abstract. We consider the existence of quasi-periodic solutions for the following nonlinear differential equation

$$x'' + f(x)x' + n^2x + \phi(x) = p(t)$$

where p is a 2π-periodic function and n is a positive integer. Under some assumptions on the parities of f, ϕ and p, we prove that there are infinitely generalized quasi-periodic solutions based on a result of S.N. Chow and M.L. Pei for Aubry-Mather theory for reversible mappings.

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1 Introduction and main results

In this paper, we shall be concerned with the existence of quasi-periodic solutions in generalized sense of the nonlinear Liénard equations

$$x'' + f(x)x' + n^2x + \phi(x) = p(t), \tag{1.1}$$

where f and g are smooth in x, p is a 2π-periodic function and n is a positive integer.

When $f \equiv 0$, the equation (1.1) is reduced to

$$x'' + n^2x + \phi(x) = p(t),$$

which is a special form of Duffing equations

$$x'' + g(x) = p(t). \tag{1.2}$$

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As the simplest but nontrivial Hamiltonian systems, (1.2) has been extensively investigated by many authors. Using Topological degree theory, variational methods, fixed point theorem with the analysis of phase plane and so on, the existence and multiplicity of periodic solutions have been obtained. Since 1970's, there are some results on the existence of quasi-periodic solutions as well as boundedness of solutions based on the celebrated Moser's twist theorem. The first result was due to Morris [10], in which he proved that there are infintely many quasi-periodic solutions for the equation $x'' + 2x^3 = p(t)$. We refer to [3], [5], [14], [6], [13], [12] and references therein for recent developments.

When $f(x) \not\equiv 0$, the system (1.1) is not Hamiltonian in general. In this case, under some symmetric assumptions on ϕ, f and p, this system becomes reversible. So, one may use KAM theorem for reversible systems to obtain the existence of quasi-periodic solutions. In [4], the authors proved that if

$$4\,|\phi(+\infty)| > \left|\int_0^{2\pi} p(t)e^{int}dt\right|,$$

then there are infinitely many quasi-periodic solutions and every solution is bounded. Moreover, if the above inequality is reversed, then there are unbounded solutions. A similar result for Duffing equations can be found in [6] and [13].

A natural question is what will happen if

$$4\,|\phi(+\infty)| = \left|\int_0^{2\pi} p(t)e^{int}dt\right|.$$

It is the task of the present paper. We will show that under some reasonable assumptions, Eq. (1.1) has also quasi-periodic solutions in generalized sense, that is, the Poincaré map of (1.1) has Mather sets.

Let F be the integral of f, that is, $F(x) = \int_0^x f(t)dt$, and suppose that

(a1) $f, \phi, p \in \mathcal{C}^2(\mathbb{R})$.

(a2) f, ϕ and p are odd, p is 2π-periodic in t.

(a3) $\lim_{x\to+\infty} \phi(x) =: \phi(+\infty) \in \mathbb{R}$, and $\lim_{|x|\to+\infty} x^2\phi''(x) = 0$.

(a4) $|x^k F^{(k)}(x)| \le M$, $x \in \mathbb{R}$, $0 \le k \le 2$, for some constant $M > 0$.

(a5) The limit

$$\lim_{x\to+\infty} x^\sigma(\phi(x) - \phi(+\infty)) = C_+,$$

exists, where $\sigma \in (0, 1)$ is a constant.

Remark 1. (1) The function F is even because f is odd.

(2) Since ϕ is odd, it follows that $\lim_{x\to-\infty} \phi(x) =: \phi(-\infty) = -\phi(+\infty)$ does exist.

(3) The condition (a3) implies that

$$\lim_{|x|\to+\infty} x\phi'(x) = 0.$$

(4) Since ϕ is odd in x, we have

$$\lim_{x\to-\infty} |x|^{\sigma-1} x(\phi(x) - \phi(-\infty)) = C_+.$$

Now we are ready to state the main results in this paper.

Theorem 1 *Suppose (a1)-(a5) hold, and*

$$4\,|\phi(+\infty)| = \left|\int_0^{2\pi} p(t)e^{int}dt\right|,$$

then there exists $\epsilon_0 > 0$*, such that for any* $\omega \in (n, n+\epsilon_0)$*, (1.1) possesses a solution* $(x_\omega(t), {x_\omega}'(t))$ *of Mather-Type with rotation number* ω*.*

(i) If $\omega = p/q$ *is a rational number, the solutions* $(x_\omega(t+2i\pi), {x_\omega}'(t+2i\pi))$*,* $0 \le i \le q-1$ *are periodic solutions with* $2q\pi$ *as period and*

$$\lim_{\omega\to n} \min_{t\in\mathbb{R}}(|x_\omega(t)| + |{x_\omega}'(t)|) = +\infty.$$

(ii) If ω *is an irrational number, solutions* $(x_\omega(t), {x_\omega}'(t))$ *are either usual quasi-periodic solutions or generalized quasi-periodic solutions.*

Remark 2. A solution is called generalized quasi-periodic if the closed set

$$\overline{\{(x_\omega(2i\pi), {x_\omega}'(2i\pi)), i \in \mathbb{Z}\}}$$

is a Denjoy's minimal set.

If $\phi(+\infty) = \int_0^{2\pi} p(t)e^{int}dt = 0$, we may get the classical quasi-periodic solutions provided that the functions f, ϕ and p are smooth. More precisely, we have

Theorem 2 *Suppose the following assumptions hold:*

(A1) $f, \phi, p \in \mathcal{C}^6(\mathbb{R})$.

(A2) f, ϕ, p *are odd and* $p(t + 2\pi) = p(t)$.

(A3) $\lim_{|x|\to+\infty} x^6\phi^{(6)}(x) = 0$ *for* $0 \le k \le 6$.

(A4) $|x^k F^{(k)}(x)| \le M$*,* $x \in \mathbb{R}$*,* $0 \le k \le 6$*, for some constant* $M > 0$.

(A5) The limit

$$\lim_{x\to+\infty} |x|^{\sigma-1} x\phi(x) = C_+,$$

exists, where $\sigma \in (0, 1)$ *is a constant. Then (1.1) has infinitely many quasi-periodic solutions, and every solution* $(x_\omega(t), {x_\omega}'(t))$ *is bounded, that is,*

$$\sup_{t\in\mathbb{R}} (|x_\omega(t)| + |{x_\omega}'(t)|) < +\infty.$$

2 Important transformations

Because the assumptions of Theorem 1 and 2 are similar, for simplicity, throughout this section, we assume that the conditions (A1)-(A4) in Theorem 2 and (a5) in Theorem 1 are satisfied. Without loss of generality, we also assume $n = 1$ in (1.1).

Rewrite (1.1) into an equivalent system

$$x' = y - F(x), \quad y' = -x - \phi(x) + p(t). \tag{2.1}$$

Passing to polar coordinates, $x = r\sin\theta$, $y = r\cos\theta$, we get

$$\begin{aligned} r' &= [p(t) - \phi(r\sin\theta)]\cos\theta - F(r\sin\theta)\sin\theta \\ \theta' &= 1 - [p(t) - \phi(r\sin\theta)]r^{-1}\sin\theta - F(r\sin\theta)r^{-1}\cos\theta \end{aligned} \tag{2.2}$$

Since p, ϕ, F are bounded, we have, for r sufficiently large,

$$\left|[p(t) - \phi(r\sin\theta)]r^{-1}\sin\theta + F(r\sin\theta)r^{-1}\cos\theta\right| < 1,$$

then $\theta' > 0$. That is, there exists $r_1 > 0$, for any $r > r_1$, $t \mapsto \theta(t)$ is a diffeomorphism, and then the equation is changed into

$$\frac{dr}{d\theta} = g_1(r, t, \theta), \quad \frac{dt}{d\theta} = g_2(r, t, \theta), \tag{2.3}$$

where g_1, g_2 are defined as

$$g_1 = \frac{[p(t) - \phi(r\sin\theta)]\cos\theta - F(r\sin\theta)\sin\theta}{1 - [p(t) - \phi(r\sin\theta)]r^{-1}\sin\theta - F(r\sin\theta)r^{-1}\cos\theta}$$

$$g_2 = \frac{1}{1 - [p(t) - \phi(r\sin\theta)]r^{-1}\sin\theta - F(r\sin\theta)r^{-1}\cos\theta}$$

It is easy to verify that, by our assumptions

$$g_1(r, -t, -\theta) = -g_1(r, t, \theta), \quad g_2(r, -t, -\theta) = g_2(r, t, \theta).$$

So the system (2.2) is reversible under the involution $(r, t) \mapsto (r, -t)$. To estimate subsequent error terms, we need to introduce some convenient notations.

Definition. *Function $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times (0, \epsilon_0) \to \mathbb{R}$ given by $(r, t, \theta; \epsilon) \mapsto f(r, t, \theta, \epsilon)$ is called in $O_n(r^{-j})$, if it is smooth in (r, t), continuous in θ, periodic of 2π in both t and θ, and*

$$\left| r^{k+j} \frac{\partial^{k+l}}{\partial r^k \partial t^l} f \right| \le M, \quad 0 \le k + l \le n,$$

for some constant $M > 0$, and all (r, t, θ), uniformly in ϵ.

A function f is called in $o_n(r^{-j})$, if

$$\lim_{r \to +\infty} \left| r^{k+j} \frac{\partial^{k+l}}{\partial r^k \partial t^l} f \right| = 0, \quad 0 \le k + l \le n,$$

uniformly in (t, θ, ϵ).

Write the functions g_1 and g_2 in the form

$$g_1(r,t,\theta) = [p(t) - \phi(r\sin\theta)]\cos\theta - F(r\sin\theta)\sin\theta + h_1(r,t,\theta),$$

$$g_2(r,t,\theta) = 1 + [p(t) - \phi(r\sin\theta)]r^{-1}\sin\theta + F(r\sin\theta)r^{-1}\cos\theta + h_2(r,t,\theta).$$

By the assumptions (A3) and (A4), it follows that

$$\left| r^l \frac{\partial^l \phi(r\sin\theta)}{r^l} \right| + \left| r^l \frac{\partial^l F(r\sin\theta)}{r^l} \right| \le M, \quad 0 \le l \le 6,$$

so it is easy to verify that $h_1 = O_6(r^{-1})$.

Since $p(t)$ is periodic and C^6, we have $\left|\frac{d^l p(t)}{dt^l}\right| \le M, \quad 0 \le l \le 6$. So, $p(t) = O_6(1)$.

Similarly, we have $h_2 = O_6(r^{-2})$, and finally we get

$$g_1(r,t,\theta) = [p(t) - \phi(r\sin\theta)]\cos\theta - F(r\sin\theta)\sin\theta + O_6(r^{-1}),$$

$$g_2(r,t,\theta) = 1 + [p(t) - \phi(r\sin\theta)]r^{-1}\sin\theta + F(r\sin\theta)r^{-1}\cos\theta + O_6(r^{-2}).$$

Lemma 2.1 *Under the transformation:*

$$\begin{aligned} \lambda &= r + S(r,\theta), \\ S(r,\theta) &= \int_0^\theta [\phi(r\sin\psi)\cos\psi + F(r\sin\psi)\sin\psi] d\psi, \end{aligned}$$

The system (2.2) is transformed into

$$\begin{cases} \frac{d\lambda}{d\theta} = p(t)\cos\theta + O_6(\lambda^{-1}) \\ \frac{dt}{d\theta} = 1 + [p(t) - \phi(\lambda\sin\theta)]\lambda^{-1}\sin\theta + F(\lambda\sin\theta)\lambda^{-1}\cos\theta + O_6(\lambda^{-2}) \end{cases} \tag{2.4}$$

The new system (2.4) is reversible with respect to $(\lambda, t) \mapsto (\lambda, -t)$.

Proof. From the assumptions (A3) and (A4), we have

$$\begin{aligned} \left|\frac{\partial S(r,\theta)}{\partial r}\right| &\le \int_0^{2\pi} (|\phi'(r\sin\psi)\sin\psi\cos\psi| + \left|F'(r\sin\psi)\sin^2\psi\right|) d\psi \\ &\le \int_0^{2\pi} (\left|\frac{M}{r}\right| |\cos\psi| + \left|\frac{M}{r}\right| |\sin\psi|) d\psi \\ &= \frac{M}{r} \int_0^{2\pi} (|\cos\psi)| + |\sin\psi|) d\psi \\ &\le Cr^{-1}. \end{aligned}$$

Furthermore, one can verify that $S = O_6(1)$. Hence, there existing $r_2 > 0$, such that, for $r > r_2$, the transformation $(r, t) \mapsto (\lambda, t)$ with $\lambda = r + S(r, \theta)$ is a diffeomorphism. By a direct computation, under this transformation, the system (2.2) is changed into the system (2.4). On the other hand, by the assumption (A2), it follows that

$$S(r, \theta + 2\pi) = S(r, \theta), \quad S(r, -\theta) = S(r, \theta),$$

so the system (2.4) is reversible with respect to the involution $(\lambda, t) \mapsto (\lambda, -t)$. □

Lemma 2.2 *Under the transformation* $(\lambda, t) \mapsto (\lambda, \tau)$*, where*

$$\tau = t - L(\lambda, \theta), \quad L(\lambda, \theta) = \lambda^{-1} \int_0^\theta F(\lambda \sin \psi) \cos \psi d\psi,$$

(2.4) is changed into

$$\begin{cases} \frac{d\lambda}{d\theta} = p(\tau) \cos \theta + O_6(\lambda^{-1}) \\ \frac{d\tau}{d\theta} = 1 + [p(\tau) - \phi(\lambda \sin \theta)]\lambda^{-1} \sin \theta + O_5(\lambda^{-2}) \end{cases} \tag{2.5}$$

The new system (2.5) is reversible with respect to $(\lambda, \tau) \mapsto (\lambda, -\tau)$*.*

Proof. Note that $\int_0^{2\pi} F(\lambda \sin \psi) \cos \psi d\psi = 0$, we have $L(\lambda, \theta + 2\pi) = L(\lambda, \theta)$. Moreover, from the evenness of F, it follows that $L(\lambda, -\theta) = -L(\lambda, \theta)$.

Similar to the discussions in Lemma 2.1, we have

$$|\lambda L| \leq \int_0^{2\pi} |F(\lambda \sin \psi) \cos \psi| d\psi \leq C,$$

and

$$\left| \lambda^2 \frac{\partial L}{\partial \lambda} \right| \leq |\lambda L| + \int_0^{2\pi} |\lambda F'(\lambda \sin \psi) \sin \psi \cos \psi| d\psi \leq C.$$

Inductively, one can prove that

$$\left| \lambda^k \frac{\partial^k L}{\partial \lambda^k} \right| \leq C,$$

for $0 \leq k \leq 6$. Hence, $L(\lambda, \theta) = O_6(\lambda^{-1})$. Since

$$\frac{d\tau}{d\theta} = \frac{dt}{d\theta} - (\frac{\partial L}{\partial \lambda} \frac{\partial \lambda}{\partial \theta} + \frac{\partial L}{\partial \theta}),$$

we have

$$\frac{d\tau}{d\theta} = 1 + [p(\tau) - \phi(\lambda \sin \theta)]\lambda^{-1} \sin \theta + O_5(\lambda^{-2}).$$

From $L(\lambda, -\theta) = -L(\lambda, \theta)$, it follows that (2.5) is reversible with respect to the involution $(\lambda, \tau) \mapsto (\lambda, -\tau)$. □

Remark 3. If the functions f, ϕ and p satisfy the assumptions (a1)-(a5) in Theorem 1, it is not difficult to obtain similar estimates as in Lemmas 2.1 and 2.2.

3 The Proofs of the main results

In this section, we first give an expression of Poincaré map of (2.5) and then apply the result by Chow and Pei [2] to obtain the existence of Mather sets.

Introduce a new variable ρ and a small parameter ϵ by $\lambda^{-1} = \epsilon\rho$, $\rho \in [1/2, 2]$. Under this change of variables, the system (2.5) is transformed into

$$\begin{cases} \frac{d\rho}{d\theta} = -\epsilon\rho^2 p(\tau)\cos\theta + \epsilon^2 O_6(1) \\ \frac{d\tau}{d\theta} = 1 + \epsilon[p(\tau) - \phi(\epsilon^{-1}\rho^{-1}\sin\theta)]\rho\sin\theta + \epsilon^2 O_5(1). \end{cases} \tag{3.1}$$

We make the ansatz

$$\rho(\theta;\rho_0,\tau_0) = \rho_0 + \epsilon F_1(\theta;\rho_0,\tau_0,\epsilon), \quad \tau(\theta;\rho_0,\tau_0) = \tau_0 + \theta + \epsilon F_2(\theta;\rho_0,\tau_0,\epsilon)$$

for the solution of (3.1), then we get

$$\begin{cases} \frac{\partial F_1}{\partial\theta} = -(\rho_0 + \epsilon F_1)^2 p(\tau_0 + \theta + \epsilon F_2)\cos\theta + \epsilon O_6(1) \\ \frac{\partial F_2}{\partial\theta} = [p(\tau_0 + \theta + \epsilon F_2) - \phi(\epsilon^{-1}(\rho_0 + \epsilon F_1)^{-1}\sin\theta)](\rho_0 + \epsilon F_1)\sin\theta + \epsilon O_5(1) \end{cases} \tag{3.2}$$

$$F_1(0;\rho_0,\tau_0,\epsilon) = F_2(0;\rho_0,\tau_0,\epsilon) = 0.$$

Noticing that

$$\begin{aligned} F_1 &= \epsilon^{-1}(\rho - \rho_0) = \epsilon^{-1}\theta\int_0^1 \rho'(\xi\theta;\rho_0,\tau_0,\epsilon)d\xi \\ &= \epsilon^{-1}\theta\int_0^1 [-\epsilon\rho^2(\xi\theta;\rho_0,\tau_0,\epsilon)p(\tau) + \epsilon^2 O_6(1)]d\xi \\ &= O_6(1) + \epsilon O_6(1) = O_6(1), \end{aligned}$$

and similar steps on F_2, we get

$$F_1 = O_6(1), \quad F_2 = O_5(1).$$

Back to (3.2), we have

$$F_1(2\pi;\rho_0,\tau_0,\epsilon) = -{\rho_0}^2\int_0^{2\pi} p(\tau_0 + \psi)\cos\psi d\psi + \epsilon O_6(1),$$

$$F_2(2\pi;\rho_0,\tau_0,\epsilon) = \rho_0\int_0^{2\pi} p(\tau_0 + \psi)\sin\psi d\psi - \int_0^{2\pi}\phi(\epsilon^{-1}{\rho_0}^{-1}\sin\psi)\rho_0\sin\psi d\psi + \epsilon O_5(1).$$

Lemma 3.1 *If ϕ satisfies the condition (a5), then*

$$\lim_{\lambda\to+\infty}\lambda^\sigma[\int_0^{2\pi}\phi(\lambda\sin\theta)\sin\theta d\theta - 4\phi(+\infty)] = 2C_+\int_0^\pi \sin^{1-\sigma}\theta d\theta := C_0.$$

Proof. Because ϕ is an odd function, it follows from Lebesgue convergent theorem and (a5) that

$$\begin{aligned} \lambda^\sigma \left[\int_0^{2\pi} \phi(\lambda \sin\theta)\sin\theta d\theta - 4\phi(+\infty)\right] &= 2\int_0^{\pi} (\lambda^\sigma \sin^\sigma\theta)\left[\phi(\lambda\sin\theta) - \phi(+\infty)\right]\sin^{1-\sigma}\theta d\theta \\ &\to 2C_+ \int_0^{\pi} \sin^{1-\sigma}\theta d\theta, \end{aligned}$$

as $\lambda \to +\infty$. □

In the following, we assume C_0 is positive.

From this lemma, we have

$$\int_0^{2\pi} \phi(\epsilon^{-1}{\rho_0}^{-1}\sin\psi)\sin\psi d\psi = 4\phi(+\infty) + \epsilon^\sigma(C_0 + o(1)).$$

Since p is odd and 2π-periodic, we have

$$\int_0^{2\pi} p(\tau_0+\psi)\sin\psi d\psi = \cos\tau_0 \int_0^{2\pi} p(\psi)\sin\psi d\psi,$$

$$\int_0^{2\pi} p(\tau_0+\psi)\cos\psi d\psi = \sin\tau_0 \int_0^{2\pi} p(\psi)\sin\psi d\psi.$$

Then,

$$F_1(2\pi;\rho_0,\tau_0,\epsilon) = -{\rho_0}^2 \sin\tau_0 \int_0^{2\pi} p(\psi)\cos\psi d\psi + \epsilon O_6(1),$$

$$F_2(2\pi;\rho_0,\tau_0,\epsilon) = \rho_0\cos\tau_0 \int_0^{2\pi} p(\psi)\sin\psi d\psi - 4\phi(+\infty)\rho_0 - \epsilon^\sigma\rho_0 C_0 + \epsilon^{1+\sigma}O_5(1).$$

Finally,

$$\begin{aligned} \rho_1 &= \rho_0 - \epsilon\rho_0^2 \sin\tau_0 \int_0^{2\pi} p(\psi)\cos\psi d\psi + o_6(\epsilon), \\ \tau_1 &= \tau_0 + 2\pi + \epsilon\rho_0\left(\cos\tau_0 \int_0^{2\pi} p(\psi)\sin\psi d\psi - 4\phi(+\infty)\right) - \epsilon^{1+\sigma}C_0\rho_0 + o_5(\epsilon^{1+\sigma}), \\ \frac{\partial\tau_1}{\partial\rho_0} &= \epsilon\left(\cos\tau_0 \int_0^{2\pi} p(\psi)\sin\psi d\psi - 4\phi(+\infty)\right) - \epsilon^{1+\sigma}C_0 + o_4(\epsilon^{1+\sigma}). \end{aligned}$$

Since

$$4\left|\phi(+\infty)\right| = \left|\int_0^{2\pi} p(t)e^{int}dt\right|$$

and $\phi(+\infty) \geq 0$, we can choose ϵ sufficiently small, such that

$$\frac{\partial\tau_1}{\partial\rho_0} < 0.$$

This means that the Poincaré map has monotone twist property.

Using similar arguments in [11], one may extend the map to a new one which is globally monotone twist map and guaranteed by Aubry-Mather theory for reversible systems (see [2]). From the above discussion, we make to the conclusion that there exists $\epsilon_0 > 0$, such that for any $\omega \in (n, n + \epsilon_0)$, Eq. (1.1) possesses a solution $(x_\omega(t), x_\omega{}'(t))$ of Mather-Type with rotation number ω. The proof of Theorem 1 is completed.

Furthermore, when

$$4\,|\phi(+\infty)| = \left| \int_0^{2\pi} p(t)e^{int}dt \right| = 0,$$

we have

$$\rho(2\pi; \rho_0, \tau_0, \delta) = \rho_0 + \delta o_4(1), \quad \tau(2\pi; \rho_0, \tau_0, \delta) = \tau_0 + 2\pi - \delta C_0 \rho_0 + \delta o_4(1),$$

with $\delta = \epsilon^{1+\sigma}$.

Hence the existence of invariant curves is guaranteed by Moser's twist theorem with small twist. The solutions starting from these curves are quasi-periodic. Moreover, because these curves tend to infinity, all solutions of (1.1) are bounded.

Final Remark. The main results in this paper are true for the following general equation

$$x'' + F_x(x,t)x' + n^2x + \phi(x,t) = 0,$$

where the functions F and ϕ are bounded and periodic in t. However, in this case, some cumbersome estimates are needed. For the existence of invariant tori and classical quasi-periodic solutions, we refer to [8].

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