LECTURE NOTES OF MATH 2D

RUI WANG

CONTENTS

Date: March 23, 2016.

1. Plane curves I – Parametric curves

1.1. **Euclidean spaces.** Roughly speaking,

- a line, which you can imagine as a straight, infinitely long string, is called a 1-dimensional Euclidean space and denoted by $\mathbb{R}$ or $\mathbb{R}^1$;
- a plane, which you can imagine as a flat, infinitely large board, is called a 2-dimensional Euclidean space and denoted by $\mathbb{R}^2$;
- a space, which you can imagine as infinitely large surrounding, is called a 3-dimensional Euclidean space and denoted by $\mathbb{R}^3$.

In this chapter, we focus on $\mathbb{R}^2$.

1.1.1. *How to represent* $\mathbb{R}^2$*?* Draw the graph of $\mathbb{R}^2$ with x-axis, y-axis, origin.

1.1.2. *How to represent points in* $\mathbb{R}^2$*?* The Cartesian coordinate of the point in $\mathbb{R}^2$ is a pair of numbers (x, y), where x comes from the x-coordinate of the point, and y comes from the y-coordinate of the point. Every point can be represented by its Cartesian coordinate.

1.1.3. *What is the meaning of a curve in* $\mathbb{R}^2$*?* Intuitionally, this is a subset of $\mathbb{R}^2$ with points in it can move in 1 free degree. Examples from pictures:

- smooth curves vs curves with sharp points;
- simple curves vs multiple curves.

We only consider smooth simple curves in our course or the curves which can be decomposed into pieces of smooth simple curves. See more from Examples of parametric curves below.

Example 1.1. A graph of a function, say $y = f(x)$, is a curve in $\mathbb{R}^2$. It has the property that for every x, there is a unique corresponding y (However, there are many curves which do not satisfy this property, and the parametric curves is a generalization which removes this restriction.).

1.2. **Parametric curves.**

Definition 1.2. A parametric curve is a map from some interval $I \subset \mathbb{R}$ to $\mathbb{R}^2$:

$$r : I \to \mathbb{R}^2, \quad r(t) = (x(t), y(t)).$$

In another word, a parametric curve is two maps from I to $\mathbb{R}$, which are $x(t)$ and $y(t)$. (Here the interval I can be $(0,1)$, $[0,2]$, $[0,2\pi)$, $\mathbb{R}$ and so on.)

Example 1.3. (1) Graph the parametric curve

$$r(t) = (x(t), y(t)) = (\cos t, \sin t), \quad t \in [0, 2\pi].$$

(2) Graph the parametric curve

$$r(t) = (x(t), y(t)) = (\cos 2t, \sin 2t), \quad t \in [0, 2\pi].$$

In this example, (1) is simple, (2) is multiple.

Example 1.4. The cycloid

$$r(t) = (x(t), y(t)) = (t - \sin t, 1 - \cos t), \quad t \in \mathbb{R}.$$

This is not a smooth curve. However we can decompose it into pieces:

$$r(t) = (x(t), y(t)) = (t - \sin t, 1 - \cos t), \quad t \in [2k\pi, 2(k+1)\pi],$$

for every piece $k = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$.

Lecture 1 stopped here.

Example 1.5. Eliminate the parameter to find a Cartesian equation of the curve for

$$r(t) = (x(t), y(t)) = (t^2 - 3, t + 2), \quad t \in [-3, 3].$$

Solution: From

$$y = t + 2,$$

we can solve $t = y - 2$. Plug it into $x = t^2 - 3$, we get the Cartesian equation

$$x = (y - 2)^2 - 3.$$

From this equation, we know it is a parabola with vertex $(-3, 2)$ and open to right.

Then you need to figure out which part on this parabola corresponds to the parametric curve for $t \in [-3, 3]$. As we did in the class, this is $y \in [-1, 5]$ (or you can also write $x \leq 6$).

□

2. Plane curves II – Calculus for parametric curves

In this section, we see how we can use calculus to investigate more information of parametric curves. Most of time, we focus on the example of a piece of cycloid

$$r(t) = (x(t), y(t)) = (t - \sin t, 1 - \cos t), \quad t \in [0, 2\pi].$$

2.1. **Tangent.**

Definition 2.1. The tangent line (or simply tangent) to a curve at a given point is the straight line that touches the curve only at that point locally, and if you perturb it a little but fix this point, it will then touch the curve at more point(s) than this one.

(This word *locally* here means that you can find a small enough neighborhood of that point to make this work. It is OK for the line to touch the curve far away from this point.)

Recall, given a graph from the function

$$y = f(x),$$

the tangent line through the point $(x_0, y_0 = f(x_0))$ can be obtained through the following steps:

(1) Slope: $k(x_0, y_0) = \frac{df}{dx}|_{x_0}$;
(2) Equation for the tangent line:

$$y - y_0 = k(x_0, y_0)(x - x_0).$$

Our question now is, if a curve is given by parametric curve

$$r(t) = (x(t), y(t)),$$

how can we get $k(x_0, y_0)$, where $(x_0, y_0) = r(t_0)$, from $r(t)$ directly, even when we don't know how to eliminate t to get a Cartesian equation of x, y? With the help of the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt},$$

we can do it by expressing the slope as

$$k(t_0) = \frac{dy}{dx}|_{x(t_0)} = \frac{\frac{dy}{dt}|_{t_0}}{\frac{dx}{dt}|_{t_0}}.$$

Example 2.2. The tangent line of the cycloid at $t_0 = \frac{\pi}{2}$.

Solution: (1) Calculate the point at $t_0 = \frac{\pi}{2}$ which is

$$(x_0, y_0) = (x(\frac{\pi}{2}), y(\frac{\pi}{2})) = (\frac{\pi}{2} - 1, 1).$$

(2) Calculate the slope using

$$\frac{dx}{dt} = 1 - \cos t, \quad \frac{dy}{dt} = \sin t.$$

We get

$$k(\frac{\pi}{2}) = \frac{\frac{dy}{dt}|_{\frac{\pi}{2}}}{\frac{dx}{dt}|_{\frac{\pi}{2}}} = \frac{1}{1} = 1.$$

(3) Express the equation

$$y - 1 = 1 \cdot (x - (\frac{\pi}{2} - 1)).$$

□

What happens if the denominator $\frac{dx}{dt}|_{t_0} = 0$? Then instead of using $\frac{dx}{dt}|_{t_0}$ directly, we use $\lim_{t \to t_0} \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ to calculate the slope at $t = t_0$.

Example 2.3. For the cycloid on $[0, 2\pi]$, at what points the tangent horizontal? When is it vertical?

Solution: We first pick up all points that $\frac{dx}{dt} = 0$. For this, we solve

$$\frac{dx}{dt} = 1 - \cos t = 0$$

and get

$$t = 0 \text{ or } 2\pi.$$

Calculate the slopes at these two points, we get

$$\begin{aligned} k(0) = \lim_{t\to 0+} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} &= \lim_{t\to 0+} \frac{\sin t}{1-\cos t} \\ &= \lim_{t\to 0+} \frac{\cos t}{\sin t} \quad \text{Use L'Hospital Rule since it is of type} \frac{0}{0} \\ &= +\infty; \end{aligned}$$

and

$$\begin{aligned} k(2\pi) = \lim_{t\to 2\pi-} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} &= \lim_{t\to 2\pi-} \frac{\sin t}{1-\cos t} \\ &= \lim_{t\to 2\pi-} \frac{\cos t}{\sin t} \quad \text{Use L'Hospital Rule since it is of type} \frac{0}{0} \\ &= -\infty. \end{aligned}$$

Thus, we first know that tangents at these points are vertical.

Now we look at other points which are neither 0 nor 2π, whose slopes now can be expressed as

$$k = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1-\cos t}.$$

This slope can never be ∞ since the denominator is never 0. It is zero if and only if the numerator is zero, which is

$$\sin t = 0$$

and the corresponding t is π.

As a conclusion, the tangent is horizontal at $t = \pi$ and vertical at $t = 0$ and 2π.

$\square$

2.2. **Second derivatives.** Now we go further and see how we can calculate $y''(x)$ without going through the steps of eliminating parameter? Using the Chain rule again, we can derive the following formula

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}.$$

Example 2.4. Is the cycloid concave up or down in $[0, 2\pi]$?

Solution: We calculate

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} \\ &= \frac{\frac{d}{dt}(\frac{\sin t}{1-\cos t})}{1-\cos t} \\ &= -\frac{1}{(1-\cos t)^2} < 0. \end{aligned}$$

Hence we conclude that it is concave down. (You can also see this from its graph too.) □

Lecture 2 stopped here.

2.3. **Area.** Recall

- If can be represented as $y = f(x)$ as x goes from $x_0 = x(t_0)$ to $x_1 = x(t_1)$ and $y \geq 0$. The area between $y = 0$ and the curve on $x \in [x_0, x_1]$:

$$A(t_0, t_1) = \int_{t_0}^{t_1} y(t)x'(t)dt.$$

- If can be represented as $x = g(y)$ as y goes from $y_0 = y(t_0)$ to $y_1 = y(t_1)$ and $x \geq 0$. The area between $x = 0$ and the curve on $y \in [y_0, y_1]$:

$$A(t_0, t_1) = \int_{t_0}^{t_1} x(t)y'(t)dt.$$

Example 2.5. Area enclosed by the cycloid and x-axis in one period $[0, 2\pi]$.

Solution: We can consider it the area below the graph of cycloid and above x-axis though we don'e need to figure out which function this graph is of.

The area then can be calculated as

$$\begin{aligned} A(0, 2\pi) &= \int_0^{2\pi} y(t)x'(t)\,dt \\ &= \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\,dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt \\ &= 3\pi. \end{aligned}$$

□

2.4. **Arc length.** Given a parametric curve

$$r(t) = (x(t), y(t)),$$

we can approximate it by small line segments. Each line segment can be regarded as a slant side of a right triangle with two sides Δx and Δy and so it is length is

$$\Delta l = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Since both x and y are functions of t, we can use linear approximation to express Δx and Δy as

$$\Delta x \approx x'(t)\Delta t$$

and

$$\Delta y \approx y'(t)\Delta t.$$

Plug them into Δl, we get

$$\Delta l = \sqrt{x'(t)^2 + y'(t)^2}\,\Delta t.$$

Sum all such small line segments, and make the number of them go to infinity, one can prove that: If both $x(t)$ and $y(t)$ are smooth functions, the limit exists, and is the same as the definite integral

$$\int_{t_0}^{t^1} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

This process can be understood formally as follows: Use differential, write

$$dx = x'(t)\, dt, \quad dy = y'(t)\, dt,$$

we then get

$$\begin{aligned} dl &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{x'(t)^2 + y'(t)^2}\, dt. \end{aligned}$$

Sum all small line segments, we have the arc length

$$L(t_0, t_1) = \int_{t_0}^{t^1} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Example 2.6. Calculate the arc length of cycloid in one period.

Solution: Consider the period $t \in [0, 2\pi]$. Use the formula, the arc length

$$\begin{aligned} L(0, 2\pi) &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t\, dt} \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt \\ &= \int_0^{2\pi} \sqrt{4 \sin^2 \frac{t}{2}}\, dt \\ &= 2 \int_0^{2\pi} |\sin \frac{t}{2}|\, dt \\ &= 2 \int_0^{2\pi} \sin \frac{t}{2}\, dt \\ &= 8. \end{aligned}$$

Actually, for every period $[t_0, t_0 + 2\pi]$, the arc length is the same. See this by using substitution or directly from its geometric meaning. $\square$

3. Plane curves III – Polar coordinates

3.1. **Polar coordinates.** We introduce another way to represent points in $\mathbb{R}^2$. Instead of using x, y-coordinates, we can also use the distance from the point to origin ρ and the angle it is away from the positive x-axis θ.

Their relations are

$$\rho = \sqrt{x^2 + y^2}, \quad \tan \theta = y/x, \quad x \in \mathbb{R}, y \in \mathbb{R};$$

and conversely,

$$x = \rho \cos \theta, y = \rho \sin \theta, \quad \rho \in [0, \infty), \theta \in [0, 2\pi).$$

Lecture 3 stopped here.

Example 3.1. Find the polar coordinates for the following points:

Cartesian	$(1,1)$	$(1,-1)$	$(1,0)$	$(1,-0.01)$
Polar	$(\sqrt{2}, \frac{\pi}{4})$	$(\sqrt{2}, \frac{7\pi}{4})$	$(1,0)$	$(\approx 1, \approx 2\pi)$

Remark 3.2. From the last two points from the above example, you can see that this correspondence

$$(x, y) \mapsto (\rho, \theta)$$

is not a continuous map! (Why?) It is only continuous when exclude the half line: $x \geq 0, y = 0$.

3.2. **Polar curves.** We can regard (ρ, θ) as a new coordinate system just as Cartesian coordinate system by extending both ρ and θ to $(-\infty, +\infty)$. Then we can consider curves in polar coordinates, which are called polar curves. In general, they are of the forms

$$\rho = \rho(\theta).$$

Example 3.3. Sketch graphs of the following polar curves by plotting points:

(1) $\rho = 1 + \sin\theta$ ("cardioid");
(2) $\rho = \cos 2\theta$ ("four-leaved rose").

You can pay attention to the symmetries for these polar curves.

3.3. **Symmetries in polar coordinates.**

(1) If $\rho(\theta) = \rho(-\theta)$, it is symmetric about x-axis;
(2) If $\rho(\theta) = \rho(\pi - \theta)$, it is symmetric about y-axis;
(3) If $\rho(\theta) = \rho(\theta + \pi)$, it is symmetric about origin.

Example 3.4. (1) $\rho = 1 + \sin\theta$ ("cardioid") – Symmetric about y-axis;
(2) $\rho = \cos 2\theta$ ("four-leaved rose") – It has all these three types of symmetry.

4. Plane curves IV – Calculus in polar coordinates

Given a polar curve $\rho = \rho(\theta)$, we can get a corresponding parametric curve

$$x(\theta) = \rho(\theta)\cos\theta, \quad y(\theta) = \rho(\theta)\sin\theta,$$

with θ as parameter. Then we can use what we have learnt for parametric curves to study polar curves.

4.1. **Tangent.** With writing the polar curve $\rho = \rho(\theta)$ into a parametric curve

$$x(\theta) = \rho(\theta)\cos\theta, \quad y(\theta) = \rho(\theta)\sin\theta,$$

we can calculate the slope at θ as

$$k(\theta) = \frac{\frac{d}{d\theta}(\rho(\theta)\sin\theta)}{\frac{d}{d\theta}(\rho(\theta)\cos\theta)} = \frac{\rho'(\theta)\sin\theta + \rho(\theta)\cos\theta}{\rho'(\theta)\cos\theta - \rho(\theta)\sin\theta}.$$

Example 4.1. Prove that the tangent of the Cardioid at $\theta = \frac{3\pi}{2}$ vertical.

Proof. Since

For the cardioid, since
$$\rho'(\theta) = (1+\sin\theta)' = \cos\theta,$$
using the formula above, the slopes can be calculated as
$$k(\theta) = \frac{2\sin\theta\cos\theta + \cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}.$$
Notice this function is not defined for $\theta = \frac{3\pi}{2}$ since the denominator will be zero then.

Because of this, we calculate the slope at $\theta = \frac{3\pi}{2}$ using limit, which is,
$$\begin{aligned}
k(\frac{3\pi}{2}-) &= \lim_{\theta\to\frac{3\pi}{2}-} \frac{2\sin\theta\cos\theta + \cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta} \\
&= \lim_{\theta\to\frac{3\pi}{2}-} \frac{\sin 2\theta + \cos\theta}{\cos 2\theta - \sin\theta} \qquad \text{Type of } \frac{0}{0} \\
&= \lim_{\theta\to\frac{3\pi}{2}-} \frac{2\cos 2\theta - \sin\theta}{-2\sin 2\theta - \cos\theta} \qquad \text{L'Hospital rule} \\
&= +\infty.
\end{aligned}$$

Similarly, you can show for the $+$ part
$$\begin{aligned}
k(\frac{3\pi}{2}+) &= \lim_{\theta\to\frac{3\pi}{2}+} \frac{2\sin\theta\cos\theta + \cos\theta}{\cos^2\theta - \sin^2\theta - \sin\theta} \\
&= -\infty,
\end{aligned}$$
and all together prove that the tangent at $\theta = \frac{3\pi}{2}$ is vertical actually.

□

Lecture 4 stopped here.

4.2. **Arc length in polar coordinates.** Recall for parametric curves, the arc length from t_0 to t_1 for parameter t can be calculated as
$$L(t_0, t_1) = \int_{t_0}^{t^1} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$
Now, as we have already noticed that, a polar curve can be view as a parametric curve
$$x(\theta) = \rho(\theta)\cos\theta, \quad y(\theta) = \rho(\theta)\sin\theta,$$
with θ as parameter. Hence, we can calculate the arc length from $\theta = \theta_0$ to $\theta = \theta_1$ using the following formula:
$$\begin{aligned}
L(\theta_0, \theta_1) &= \int_{\theta_0}^{\theta_1} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta \\
&= \int_{\theta_0}^{\theta_1} \sqrt{(\rho(\theta)'\cos\theta - \rho(\theta)\sin\theta)^2 + (\rho(\theta)'\sin\theta + \rho(\theta)\cos\theta)^2}\, d\theta \\
&= \int_{\theta_0}^{\theta_1} \sqrt{\rho(\theta)'^2 + \rho(\theta)^2}\, d\theta.
\end{aligned}$$

Example 4.2. Calculate the arc length of the cardioid $\rho = 1 + \sin\theta$ in one period.

Solution: We calculate

$$\rho(\theta)' = (1+\sin\theta)' = \cos\theta,$$

and so from the formula, we have

$$\begin{aligned} L(0,2\pi) &= \int_0^{2\pi} \sqrt{\rho(\theta)'^2 + \rho(\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{\cos^2\theta + (1+\sin\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2+2\sin\theta}\, d\theta. \end{aligned}$$

(This part was skipped from lecture) The calculation for this definite integral is somehow tricky and the steps may go as this: We first figure out the anti-derivative for the function $\sqrt{1+\sin t}$ as

$$\begin{aligned} \int \sqrt{1+\sin t}\, dt &= \int \frac{\sqrt{1-\sin t}}{\sqrt{1-\sin t}} \sqrt{1+\sin t}\, dt \\ &= \int \frac{\sqrt{1-\sin^2 t}}{\sqrt{1-\sin t}}\, dt \\ &= \int \frac{\sqrt{\cos^2 t}}{\sqrt{1-\sin t}}\, dt \\ &= \int \frac{|\cos t|}{\sqrt{1-\sin t}}\, dt \end{aligned}$$

Let's assume $\cos t \geq 0$. Then we have

$$\begin{aligned} \int \sqrt{1+\sin t}\, dt &= \int \frac{\cos t}{\sqrt{1-\sin t}}\, dt \\ &= \int \frac{1}{\sqrt{1-x}}\, dx \quad \text{where } x = \sin t \\ &= -2\sqrt{1-x} + C \quad \text{where } C \text{ is any constant} \\ &= -2\sqrt{1-\sin t} + C \quad \text{where } C \text{ is any constant.} \end{aligned}$$

For the part $\cos t \leq 0$, similarly we have

$$\begin{aligned} \int \sqrt{1+\sin t}\, dt &= -\int \frac{\cos t}{\sqrt{1-\sin t}}\, dt \\ &= 2\sqrt{1-\sin t} + C \quad \text{where } C \text{ is any constant.} \end{aligned}$$

The we go back to the arc length of cardioid,

$$\begin{aligned} L(0,2\pi) &= \sqrt{2}\int_0^{2\pi} \sqrt{1+\sin t}\, dt \\ &= \sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{1+\sin t}\, dt + \sqrt{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{1+\sin t}\, dt + \sqrt{2}\int_{\frac{3\pi}{2}}^{2\pi} \sqrt{1+\sin t}\, dt \\ &= \sqrt{2}(-2\sqrt{1-\sin t})\big|_0^{\frac{\pi}{2}} + \sqrt{2}(2\sqrt{1-\sin t})\big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \sqrt{2}(-2\sqrt{1-\sin t})\big|_{\frac{3\pi}{2}}^{2\pi} \\ &= 2\sqrt{2} + 4 + (4 - 2\sqrt{2}) \\ &= 8. \end{aligned}$$

□

(We leave the calculation of areas bounded by polar curves to Chapter 15, where we are going to systematically introduce how to calculate areas using double integrals.)

5. Three dimensional spaces I – Vectors

5.1. **Affine space and vector space.**

Definition 5.1. A *vector* is an arrow with length in the 3-dimensional Euclidean space $\mathbb{R}^3$. The set of all vectors is called the *affine space* for the 3-dimensional Euclidean space $\mathbb{R}^3$.

Any vector $\vec{u}$ in the affine space has three ingredients:

- The initial point P_0;
- The end point P_1;
- The length (magnitude) $|\vec{u}|$ which is defined as the distance between P_0 and P_1.

Hence we can denote $\vec{u} = \overrightarrow{P_0P_1}$.

Now, if we fix a point P_0 is the 3-dimensional Euclidean space $\mathbb{R}^3$, we can consider all vectors whose initial points are P_0 and use V_{P_0} to denote the set of all such vectors. For such vectors, we can define two operations:

- Addition: For $\vec{u} = \overrightarrow{P_0P_1}$ and $\vec{v} = \overrightarrow{P_0P_2}$, the sum of them, $\vec{u} + \vec{v}$ is the vector with initial point P_0 and endpoint P as the opposite vertex of P_0 in the parallelogram spanned by $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$. Clearly, the sum $\vec{u} + \vec{v}$ is still in V_{P_0} (since its initial point is still P_0).
- Scalar multiplication: For $\vec{u} = \overrightarrow{P_0P_1}$ and $c \in \mathbb{R}$, the scalar multiplication $c \cdot \vec{u}$ is defined as follows:
  - If $c = 0$, then $c \cdot \vec{u} = \vec{0}_{P_0}$ which we denote the zero vector with initial point P_0;
  - If $c > 0$, then $c \cdot \vec{u}$ is the vector initial at P_0 and pointing to the same direction of $\vec{u}$ with length $c|\vec{u}|$;
  - If $c < 0$, then $c \cdot \vec{u}$ is the vector initial at P_0 and pointing to the opposite direction of $\vec{u}$ with length $-c|\vec{u}|$.

Using this definition, we can prove (Exercise!)

Proposition 5.2. *Assume $\vec{u}$, $\vec{v}$, $\vec{w}$ are vectors in V_{P_0} (i.e., vectors with initial point P_0), $a, b, c \in \mathbb{R}$. Then*

(1) $\vec{0}_{P_0} + \vec{u} = \vec{u} = \vec{u} + \vec{0}_{P_0}$;
(2) $\vec{u} + (-\vec{u}) = \vec{0}_{P_0} = (-\vec{u}) + \vec{u}$;
(3) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$;
(4) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$;
(5) $0 \cdot \vec{u} = \vec{0}_{P_0}$;
(6) $1 \cdot \vec{u} = \vec{u}$;
(7) $a \cdot (\vec{u} + \vec{v}) = a \cdot \vec{u} + a \cdot \vec{v}$;
(8) $(a + b) \cdot \vec{u} = a \cdot \vec{u} + b \cdot \vec{v}$;
(9) $(ab) \cdot \vec{u} = a \cdot (b \cdot \vec{u})$.

Remark 5.3. Abstractly, any (abstract) space with operations addition and scalar multiplication, satisfying the above nine properties, is called a vector space (linear space).

In particular, we call V_{P_0} a *vector space*, for any P_0 in the 3-dimensional Euclidean space $\mathbb{R}^3$.

Definition 5.4. Given a vector $\vec{u}$ whose length is not zero, then we call the vector

$$\frac{1}{|\vec{u}|} \cdot \vec{u}$$

the unit vector of $\vec{u}$.

Example 5.5. Show the length of the unit vector is 1.

Proof. Using the definition of scalar multiplication, we calculate as follows.

$$|\frac{1}{|\vec{u}|} \cdot \vec{u}| = \frac{1}{|\vec{u}|}|\vec{u}| = 1.$$

□

5.2. **Cartesian coordinates and free vectors.** Now we consider the Cartesian coordinates for the 3-dimensional Euclidean space $\mathbb{R}^3$. Whenever the Cartesian coordinate system is chosen (for which we mean choose a point as origin and choose three coordinate axises x, y, z), every point in the 3-dimensional Euclidean space $\mathbb{R}^3$ can be represented by a triple of numbers (x, y, z).

Now for an arbitrary vector $\vec{u} = \overrightarrow{P_0P_1}$ with

$$P_0 = (x_0, y_0, z_0), \quad P_1 = (x_1, y_1, z_1),$$

we can get a new point, denoted by P whose Cartesian coordinate is given by

$$(x_1 - x_0, y_1 - y_0, z_1 - z_0).$$

Next we obtain a new vector denote by $\vec{u}_0$ as

$$\vec{u}_0 := \overrightarrow{OP}.$$

Obviously from construction, $\vec{u}_0 \in V_0$, i.e., the initial point of $\vec{u}_0$ is the origin. To summarize, this sets up a map from the affine space to the vector space V_0 which is given as

$$\vec{u} \mapsto \vec{u}_0.$$

We call $\vec{u}_0$ the *free vector* of $\vec{u}$.

Lemma 5.6. $|\vec{u}| = |\vec{u}_0|$. *Moreover, if* $\vec{u} = \overrightarrow{P_0P_1}$ *with*

$$P_0 = (x_0, y_0, z_0), \quad P_1 = (x_1, y_1, z_1),$$

the length is calculated as

$$|\vec{u}| = |\vec{u}_0| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

Conversely, if we are given $P_0 = (x_0, y_0, z_0)$ and a vector $\vec{u}_0 \in V_0$ whose endpoint is $P = (a, b, c)$, we can uniquely find the vector $\vec{u} =\in V_{P_0}$ whose free vector is $\vec{u}_0$. In fact, such $\vec{u}$ is given as $\vec{u} = \overrightarrow{P_0P_1}$ with $P_1 = (x_0 + a, y_0 + b, z_0 + c)$.

We have shown that

Lemma 5.7. *All free vectors form a vector space (linear space), and the set of free vectors is one-to-one correspondent to points in the* 3*-dimensional Euclidean space* $\mathbb{R}^3$.

From this, we know that one can use a triple (x,y,z) to denote both a point P in $\mathbb{R}^3$ and a free vector $\vec{u}_0 = \overrightarrow{OP}$. To simplify notations, we just write $\vec{u}_0 =< x,y,z >$ whose initial point is the origin and end point is (x,y,z) from now on. (I changed the notation to this according to your request from today's lecture.)

We have

Lemma 5.8. *For $\vec{u}_0 =< x_1, y_1, z_1 >$ and $\vec{v}_0 =< x_2, y_2, z_2 >$, we have*

- $\vec{u}_0 + \vec{v}_0 =< x_1 + x_2, y_1 + y_2, z_1 + z_2 >$;
- $c \cdot \vec{u}_0 =< cx_1, cx_2, cx_3 >$;
- $|\vec{u}_0| = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

Then as a corollary from this, we have

Corollary 5.9. *Given $P_0 = (x_0, y_0, z_0)$, $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, we can calculate*

- $\overrightarrow{P_0P_1} + \overrightarrow{P_0P_2} = \overrightarrow{P_0P}$ *with* $P = (x_1 + x_2 - x_0, y_1 + y_2 - y_0, z_1 + z_2 - z_0)$;
- $c \cdot \overrightarrow{P_0P_1} = \overrightarrow{P_0P}$ *with* $P = (x_0 + c(x_1 - x_0), y_0 + c(y_1 - y_0), z_0 + c(z_1 - z_0))$.

Lecture 5 stopped here.

Example 5.10. Assume under Cartesian coordinate, the points $P_0 = (1,2,3)$, $P_1 = (-1,3,0)$ and $P_2 = (1,0,-1)$. Denote by

$$\vec{u} = \overrightarrow{P_0P_1}, \quad \vec{v} = \overrightarrow{P_0P_2},$$

and

$$\vec{w} = \overrightarrow{P_0P} = 3\vec{u} + 2\vec{v}.$$

Figure out the Cartesian coordinate of P.

Solution: Consider the free vector associated to $\vec{u}$, which is

$$\vec{u}_0 =< -1-1, 3-2, 0-3 >=< -2, 1, -3 >$$

and the free vector associated to $\vec{v}$, which is

$$\vec{v}_0 =< 1-1, 0-2, -1-3 >=< 0, -2, -4 > .$$

Calculate

$$\begin{aligned} 3\vec{u}_0 + 2\vec{v}_0 &= 3 < -2, 1, -3 > +2 < 0, -2, -4 > \\ &= < -6, 3, -9 > + < 0, -4, -8 >=< -6, -1, -17 > . \end{aligned}$$

Hence the coordinate of $P = (x,y,z)$ satisfies

$$< x-1, y-2, z-3 >=< -6, -1, -17 >,$$

and so $P = (x,y,z) = (-5, 1, -14)$. □

(The numbers are different from the example I used in class because I forget the numbers I used...)

6. Three dimensional spaces II – Dot product

6.1. **Definition and basic properties.** Let's restrict ourselves to V_0, i.e., we only look at vectors whose initial point is the origin.

Definition 6.1. For two vectors

$$\vec{u} =< a_1, b_1, c_1 >, \quad \vec{v} =< a_2, b_2, c_2 >$$

in V_0, the dot product of them is defined as a real number by

$$\vec{u} \cdot \vec{v} = a_1 a_2 + b_1 b_2 + c_1 c_2.$$

You can quickly check the following properties:

Proposition 6.2. (1) $\vec{0} \cdot \vec{v} = 0$;
(2) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$;
(3) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$;
(4) *For* $c \in \mathbb{R}$, $(c \cdot \vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c \cdot \vec{v})$;
(5) $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

The last property indicates that $\vec{u} \cdot \vec{u} \geq 0$, and it is zero if and only if $\vec{u} =< 0, 0, 0 >$. (This property is called positivity of dot product.)

Lecture 6 stopped here.

Example 6.3. Given

$$\vec{u} =< 2, 3, 4 >, \quad \vec{v} =< 0, -1, -2 > .$$

Calculate

(1) $\vec{u} \cdot \vec{v}$,
(2) $(2\vec{u} - \vec{v}) \cdot \vec{v}$,
(3) $(2\vec{u} - \vec{v}) \cdot (2\vec{u} + \vec{v})$. Check $(2\vec{u} - \vec{v}) \cdot (2\vec{u} + \vec{v}) = 4|\vec{u}|^2 - |\vec{v}|^2$. (I did a slightly different example in lecture.)

6.2. **Geometric meaning of dot product.** Next we explain the geometric meaning of dot product. Assume $\vec{u}, \vec{v} \in V_0$, and let's θ denote the angle between them. Notice that θ can range from 0 to π. In fact, if neither $\vec{u}$ nor $\vec{v}$ is non-zero, then

(1) $\theta = 0$ if and only if $\vec{u}$ and $\vec{v}$ are in the same direction. In another word, the unit vector of $\vec{u}$ is the same as the unit vector of $\vec{v}$;
(2) $\theta = \pi$ if and only if $\vec{u}$ and $\vec{v}$ are in the opposite direction. In another word, the unit vector of $\vec{u}$ is the opposite of the unit vector of $\vec{v}$;
(3) $\theta = \frac{\pi}{2}$ if and only if $\vec{u}$ and $\vec{v}$ are perpendicular with each other.

Proposition 6.4.

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta.$$

Hence, if neither $\vec{u}$ nor $\vec{v}$ is non-zero, we have

$$|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|,$$

and it takes "$=$" if and only if when $\theta = 0$ or π.

Proof. Use the cosine formula for triangles, we can write the side opposite to the angle θ as

$$l^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta.$$

On the other hand, using picture, notice that

$$l = |\vec{u} - \vec{v}|.$$

So l^2 can also be calculated as

$$l^2 = |\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v}.$$

Hence we get

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta.$$

□

From this formula, we can see that

Corollary 6.5. $\vec{u} \cdot \vec{v} = 0$ *if and only if* $\vec{u}$ *and* $\vec{v}$ *are perpendicular to each other (including the special case that one or both them are zero vectors.)*

Definition 6.6. Two vectors in the affine space (which means they may have different initial points) are called perpendicular to each other, if their associated free vectors are perpendicular (which is equivalent to say that their dot product is zero). We write $\vec{u} \perp \vec{v}$.

(In fact, one should be careful to notice that this definition actually is independent of the choices of Cartesian coordinates. So the concept of "perpendicular" is a pure geometry concept about the Euclidean space $\mathbb{R}^3$.)

Example 6.7. Given four points $P_1 = (1, 2, 3)$, $P_2 = (-1, 3, 0)$, $P_3 = (1, 0, -1)$ and $P_4 = (2, 2, -1)$, show that

$$\overrightarrow{P_1P_2} \perp \overrightarrow{P_3P_4}.$$

Solution: The free vector associated to $\overrightarrow{P_1P_2}$ is $< -2, 1, -3 >$, and the free vector associated to $\overrightarrow{P_3P_4}$ is $< 1, 2, 0 >$. Because

$$< -2, 1, -3 > \cdot < 1, 2, 0 >= (-2) \cdot 1 + 1 \cdot 2 + (-3) \cdot 0 = 0,$$

so $\overrightarrow{P_1P_2} \perp \overrightarrow{P_3P_4}$. □

6.3. **Projection of a vector to another vector with the same initial point.** We consider two vectors $\vec{u}$ and $\vec{v}$ with the same initial point P_0. Then they will span a unique (whenever neither of them vanishes) plane. In this plane, we can draw a unique line through the endpoint of $\vec{u}$ and perpendicular with $\vec{v}$. Use P_1 to denote the intersection of this line and the line that $\vec{v}$ lives in. We call the vector $\overrightarrow{P_0P_1}$ the projection of $\vec{u}$ onto $\vec{v}$ and denote it by $\text{Proj}_{\vec{v}}\vec{u}$. Our question now is, how to calculate this projection.

Proposition 6.8.

$$\text{Proj}_{\vec{v}}\vec{u} = (\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2})|\vec{v}|.$$

Denote by

$$\vec{i} =< 1, 0, 0 >, \quad \vec{j} =< 0, 1, 0 >, \quad \vec{k} =< 0, 0, 1 > .$$

They are perpendicular to each other and all have unit lengths. We call $(\vec{i}, \vec{j}, \vec{k})$ an orthonormal basis (or frame) for $\mathbb{R}^3$.

Example 6.9. Assume $\vec{u} =< a, b, c >$. Prove that

$$\text{Proj}_{\vec{i}}\vec{u} = a\vec{i}, \quad \text{Proj}_{\vec{j}}\vec{u} = b\vec{j}, \quad \text{Proj}_{\vec{k}}\vec{u} = c\vec{k},$$

and further

$$\vec{u} = \text{Proj}_{\vec{i}}\vec{u} + \text{Proj}_{\vec{j}}\vec{u} + \text{Proj}_{\vec{k}}\vec{u}.$$

Example 6.10. Given $\vec{u} =< 1, 2, 3 >$, $\vec{v} =< 3, 2, 1 >$, calculate $\text{Proj}_{\vec{v}}\vec{u}$ and $\text{Proj}_{\vec{u}}\vec{v}$.

7. Three dimensional space III – Cross product

7.1. Preparation – Determinants of 2×2 and 3×3 matrices.

Definition 7.1. A 2×2 matrix is a collection of 4 numbers in two rows and two columns which looks like

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Its determinant, denoted by $\det A$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is defined as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Definition 7.2. A 3×3 matrix is a collection of 9 numbers in three rows and three columns which looks like

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Its determinant is defined as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

For example,

$$\det(\vec{i}, \vec{j}, \vec{k}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Lecture 7 stopped here.

7.2. **Definition of cross product and first properties.** Now we use another way to view the definition of determinant of 3×3 matrix. Consider three free vectors which are

$$\vec{u}_1 =< a_1, b_1, c_1 >, \vec{u}_2 =< a_2, b_2, c_2 >, \vec{u}_3 =< a_3, b_3, c_3 > .$$

Then we can regard the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

as a function of $\vec{u}$, $\vec{u}_2$ and $\vec{u}_3$, i.e., consider $\vec{u}$, $\vec{u}_2$ and $\vec{u}_3$ as three inputs and get a real number which is the determinant as output. Let's denote this function as $\det(\vec{u}_1, \vec{u}_2, \vec{u}_3))$ for later use.

Next define

$$\vec{u}_2 \times \vec{u_3} :=< \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} >,$$

and call it the cross product of $\vec{u}_2$ and $\vec{u_3}$. It follows that we can write

$$\det(\vec{u}_1, \vec{u}_2, \vec{u}_3)) = \vec{u}_1 \cdot (\vec{u}_2 \times \vec{u_3}).$$

By definition of the cross product, we can check the following properties:

Proposition 7.3. *For any vectors $\vec{u}$, $\vec{v}$, $\vec{u}_1$ and $\vec{u}_2$ and real numbers c_1 and c_2, we have*

(1) *(Anti-symmetry) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$; and it follows that $\vec{u} \times \vec{u} = \vec{0}$.*
(2) *(Linearity) $(c_1\vec{u}_1 + c_2\vec{u}_2) \times \vec{v} = c_1(\vec{u}_1 \times \vec{v}) + c_2(\vec{u}_2 \times \vec{v})$.*

Remark 7.4. The anti-symmetry indicates that the second entry also has linearity in a cross product.

Example 7.5. Show that

$$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}.$$

To summarize, the two ordered inputs and the output follow the right-hand rule, which is actually a general property.

We go back to see the determinant of 3×3 matrix. From the definition of determinant, one can see immediately that

$$\det(\vec{u}, \vec{u}, \vec{v}) = 0 = \det(\vec{v}, \vec{u}, \vec{v}),$$

which can be rewritten as

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = 0 = \vec{v} \cdot (\vec{u} \times \vec{v}).$$

From the meaning of dot product we learned last time, this says that the cross product is both perpendicular to two inputs. Actually, you can also prove that, the output is followed the right-hand rule.

Example 7.6. Let $\vec{u} =< 1, 2, 3 >$ and $\vec{v} =< -1, 0, 2 >$. Calculate $\vec{u} \times \vec{v}$.

Solution: Introducing a formal symbol, we can calculate cross product as

$$\vec{u} \times \vec{v} =< 1, 2, 3 > \times < -1, 0, 2 >= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -1 & 0 & 2 \end{vmatrix} = 4\vec{i} - 5\vec{j} + 2\vec{k} =< 4, -5, 2 > .$$

$\square$

7.3. **Geometric meaning of cross product.**

Lemma 7.7. *For any two vectors $\vec{u}$ and $\vec{v}$, the length of cross product*

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta,$$

where θ is the angle between $\vec{u}$, $\vec{v}$ (we always take $\theta \in [0, \pi]$ and hence $\sin\theta \geq 0$).

Proof. We skip its proof and you can refer book P817 for it. □

Using the geometric meaning, we notice that

Corollary 7.8.

$$|\vec{u} \times \vec{v}| = \textit{the area of the parallelogram spanned by } \vec{u} \textit{ and } \vec{v}.$$

Further more,

Proposition 7.9.

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = \textit{the volume of the parallelepiped spanned by } \vec{u}, \vec{v} \textit{ and } \vec{w}.$$

In fact, if you check more carefully, you will find that

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \text{the volume of the parallelepiped spanned by } \vec{u}, \vec{v} \text{ and } \vec{w}$$

if $\vec{u}$, $\vec{v}$ and $\vec{w}$ follow the right-hand rule, and

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = -\text{the volume of the parallelepiped spanned by } \vec{u}, \vec{v} \text{ and } \vec{w}$$

if $\vec{u}$, $\vec{v}$ and $\vec{w}$ follow the left-hand rule.

It follows (or you can check using the definition of determinant),

Proposition 7.10.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v}).$$

Lecture 8 stopped here.

8. Three dimensional spaces IV – Lines and planes

8.1. **Lines.** From Euclidean geometry, we know that there are the following ways which can determine a unique line:

(1) Two distinct points;
(2) A point and a nonzero vector as direction;
(3) Two planes which are not parallel to each other.

Today we are going to answer how to write down the (parametric) equation of line for each case.

Let's start from the second case:

8.1.1. *A point and a nonzero vector as direction uniquely determine a line.* Assume we are given a point $P_0 = (x_0, y_0, z_0)$ and a nonzero vector $\vec{v} =< a, b, c >$, using what we learn for vectors, the line can be written as

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct,$$

which is called the system of parametric equations for a lines.

Conversely, given such a system of parametric equations, you should be able to read out the information that the line goes through the point (x_0, y_0, z_0) and parallel to the vector $< a, b, c >$.

Example 8.1. Write down the parametric equation for the line which goes through $(1, 2, 3)$ and parallel to $< 3, -1, 0 >$.

Solution: It is

$$x = 1 + 3t, \quad y = 2 - t, \quad z = 3.$$

□

8.1.2. *Two distinct points uniquely determine a line.* Given two distinct points $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$, we can get the free vector for $\overrightarrow{P_0P_1}$ which is

$$\vec{v} =< x_1 - x_0, y_1 - y_0, z_1 - z_0 > .$$

Then the line goes through P_0 and P_1 must be parallel to $\vec{v}$, which reduces this situation back to the previous case. Hence we can write the system of parametric equations for this line as

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t, \quad z = z_0 + (z_1 - z_0)t.$$

Example 8.2. Write down the parametric equation for the line which goes through the two distinct points $P_0 = (1, 2, 3)$ and $P_1 = (0, 1, -1)$.

Solution: Calculate the free vector of $\overrightarrow{P_0P_1}$ as

$$\vec{v} =< 0 - 1, 1 - 2, -1 - 3 >=< -1, -1, -4 > .$$

Hence the line is

$$x = 1 - t, \quad y = 2 - t, \quad z = 3 - 4t.$$

□

8.1.3. *Intersection of two non-parallel planes.* We postpone this case until we discuss planes.

8.2. **Planes.**

8.2.1. *A point and a vector uniquely determine a plane.* Given a point $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and a nonzero vector $\vec{n} =< a, b, c >$, then all points P such that $\overrightarrow{P_0P_1} \perp \vec{v}$ form a plane. Assume P has coordinate (x, y, z), then the condition $\overrightarrow{P_0P_1} \perp \vec{n}$ says that

$$< x - x_0, y - y_0, z - z_0 > \cdot \vec{n} = 0,$$

which is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

We call such $\vec{v}$ a normal vector. Notice that, given a plane, a normal vector is not unique but can be rescaled by any nonzero constant.

Example 8.3. Write down the equation for the plane which goes through the point $P_0 = (1, 2, 3)$ with $< -1, 0, 1 >$ as a normal vector.

Solution:

$$-(x-1)+(z-3)=0.$$

□

8.2.2. *General equations for planes.* In general, any equation of the form

$$ax+by+cz+d=0 \text{ with } <a,b,c> \neq \vec{0} \tag{1}$$

denotes a plane. Such plane has normal vector

$$\vec{n} =< a, b, c > .$$

To figure out a point it goes through, we can do the following steps. First, since $< a, b, c > \neq \vec{0}$, there at least one of a, b, c which is not zero. For example, if $a \neq 0$, then we take $y = 0$ and $z = 0$ and the equation (1) becomes $ax + d = 0$ and it follows $x = -\frac{d}{a}$. This says the plane goes through the point $(-\frac{d}{a}, 0, 0)$.

Example 8.4. Given a plane

$$y-z+5=0,$$

you should be able to read out that it has a normal vector

$$\vec{n} =< 0, 1, -1 > .$$

and goes through the point $(0, -5, 0)$. (Sure, you can find infinitely many such points.)

8.2.3. *Intersection of two planes.* Now, let's go back to answer the question that how to give the equation for the intersection line of two non-parallel planes.

Assume we have two planes

$$a_1x+b_1y+c_1z+d_1=0 \text{ and } a_2x+b_2y+c_2z+d_2=0.$$

(They are not parallel indicates that the normal vectors

$$\vec{n}_1 =< a_1, b_1, c_1 > \quad \text{and} \quad \vec{n}_2 =< a_2, b_2, c_2 >$$

has non-vanishing cross product.) The intersection line is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$, and hence parallel to

$$\vec{v} := \vec{n}_1 \times \vec{n}_2.$$

Then what we need to do left is only to find a point living on both planes. To do this, you may try to let x be zero and solve the equation of y and z. If you find a solution (y_0, z_0), then the point $(0, y_0, z_0)$ lives on both planes. If you find no solution, then try $y = 0$ and solve x and z; If this fails again, try $z = 0$. The condition that these two planes are not parallel ensures that there must be a solution at last.

Now, we reduce this problem back to the case of writing a line going through a point and being parallel to a vector.

Example 8.5. Write down the equation for the intersection line of two planes

$$\text{Plane I:} \quad 2x + 3y + z = 0$$

and

$$\text{Plane II:} \quad y - z + 4 = 0.$$

Solution:

$$\vec{n}_1 =< 2, 3, 1 >, \quad \vec{n}_2 =< 0, 1, -1 > .$$

Calculate

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 =< -4, 2, 2 > .$$

For a point on the intersection, take $x = 0$ and we get two equations

$$3y + z = 0, \quad y - z + 4 = 0.$$

Solve them, get

$$y = -1, \quad z = 3.$$

Hence the intersection line is

$$x = -4t, \quad y = -1 + 2t, \quad z = 3 + 2t.$$

□

8.2.4. *Two intersecting lines determine a unique plane.* Given two distinct lines which intersect at some point, then there exists a unique plane such that both lines live in it.

Example 8.6. Assume two lines

$$\text{Line I:} \quad x = 3t, y = 1 + 2t, z = 3 - t$$

and

$$\text{Line II:} \quad x = t, y = 1 + 3t, z = 3 - 5t.$$

Show they intersect with each other and figure out the equation of the plane spanned by these two lines.

Solution: Solve the equations

$$x = 3t = t, \quad y = 1 + 2t = 1 + 3t, \quad z = 3 - t = 3 - 5t,$$

and find that there is a unique solution that

$$t = 0, x = 0, y = 1, z = 3,$$

which says these two lines intersect at $(0, 1, 3)$. Since we want to get the equation for the plane that contains both lines, the normal vector of this plane should be perpendicular to both lines, and hence can be calculated by the cross product as

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 =< 3, 2, -1 > \times < 1, 3, -5 >=< -7, 14, 7 > .$$

Above all the plane is

$$-7(x - 0) + 14(y - 1) + 7(z - 3) = 0.$$

□

Lecture 9 stopped here.

8.3. **Positions of two lines.** For two distinct lines (notice two different parametric equations may represent the same line), there are three positions.

(1) Parallel to each other. For this case, you can see if this holds by checking if
$$\vec{v}_1 \times \vec{v}_2 = \vec{0}.$$
Two parallel lines always live in the same plane.

(2) Intersect with each other. For this case, the parametric equations have a unique solution which gives rise to the intersection point.

(3) Skew to each other. If two lines are neither parallel nor intersect, they are called skew.

8.4. **Distance from a point to a plane.** Assume $P_0 = (x_0, y_0, z_0)$ is a point in $\mathbb{R}^3$ and $ax + by + cz + d = 0$ is a plane. Notice that going through the point P_0, there exists a unique line which is perpendicular to the plane, and we can calculate out its parametric equation as
$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$$
It intersects with the plane at the some point which we denote by P. To get the coordinate of P, we plug the parametric equations of the line into the equation of plane and get
$$a(x_0 + at) + b(y_0 + bt) + c(z_0 + ct) + d = 0,$$
and further solve
$$t = -\frac{ax_0 + by_0 + cz_0 + d}{a^2 + b^2 + c^2}.$$
You could calculate out the coordinates of P by plug t into the equations of line. However, since we only want to calculate the distance from P_0 to P, we can directly look at
$$\begin{aligned}
\text{Distance} &= \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \\
&= \sqrt{(at)^2 + (bt)^2 + (ct)^2} \\
&= \sqrt{a^2 + b^2 + c^2} \cdot |t| \\
&= \sqrt{a^2 + b^2 + c^2} \cdot \left| -\frac{ax_0 + by_0 + cz_0 + d}{a^2 + b^2 + c^2} \right| \\
&= \sqrt{a^2 + b^2 + c^2} \cdot \frac{|ax_0 + by_0 + cz_0 + d|}{a^2 + b^2 + c^2} \\
&= \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.
\end{aligned}$$

Example 8.7. Find the distance from $(1, 2, 3)$ to the plane $x + 2y + 3 = 0$.

Solution: Using the formula, the distance is
$$\text{Distance} = \frac{|1 \cdot 1 + 2 \cdot 2 + 0 \cdot 3 + 3|}{\sqrt{1^2 + 2^2 + 0^2}} = \frac{8\sqrt{5}}{5}.$$
□

8.5. **An Excercise.** The following example is a combination of several types of problems in the section and the details are left to you.

Example 8.8. Given six points in $\mathbb{R}^3$ which are

$$P_1 = (1,1,1), P_2 = (-1,1,1), P_3 = (-1,-1,1), P_4 = (1,-1,1),$$
$$P_5 = (-1,-1,-1), P_6 = (1,1,-1).$$

(1) Write down the parametric equations of the line through P_1 and P_5 (let's name it line I), the one through P_2, P_4 (let's name it plane II) and the one through P_1, P_3 (let's name it plane III);
(2) Show line I and line II are skew. Find the intersection of line II and line III and write down the equation of the plane they live in.
(3) Write down the equation for the line segment connecting P_1 and P_5;
(4) Write down the equation of the plane through P_2, P_3, P_4 (let's name it plane I) and the one though P_1, P_5, P_6 (let's name it plane II);
(5) Show plane I and plane II intersect and write the equation for the intersection line.
(6) Show P_1 is in plane I.
(7) Calculate the distance from P_2 to plane II.

9. Three dimensional spaces V – Surfaces in $\mathbb{R}^3$

9.1. **Planes.** The simplest surfaces in $\mathbb{R}^3$ are planes whose equations are

$$ax + by + cz + d = 0, \quad < a, b, c > \neq \vec{0}.$$

You can sketch planes in $\mathbb{R}^3$ by examines its intersections with coordinate axises.

9.2. **Quadratic surface.** In general, a quadratic surface is determined by a quadratic equation

$$ax^2 + by^2 + cz^2 + dxy + eyz + fzx + hx + iy + jz + l = 0, \quad < a, b, c, d, e, f > \neq \vec{0}$$

where $a, b, c, d, e, f, g, h, i, j, k, l$ are all constants.

9.3. **Cylinders.** If the surface is given by an equation with at least one variable missing, for example, an equation with only x and y but no z given as

$$x - y^2 = 0$$

is a parabola cylinder.

For more about this part, refer the book Chapter 12. 6.

10. Space curve I – Parametric curves in $\mathbb{R}^3$

10.1. **Parametric curves.** With parameter t, a parametric curve in $\mathbb{R}^3$ can be written as

$$r(t) = (x(t), y(t), z(t)).$$

Since we can identify points in $\mathbb{R}^3$ with free vectors, we can also denote a parametric curves using

$$\vec{r}(t) =< x(t), y(t), z(t) > .$$

In this way, with t changing, the vector $\vec{r}(t)$ is moving with its endpoint $r(t) = (x(t), y(t), z(t))$ forming a space curve.

Example 10.1. Sketch the graph of parametric curve

$$\vec{r}(t) =< \cos t, \sin t, t >, \quad t \in \mathbb{R}.$$

This curve is called a helix.

In general, for a vector valued function $\vec{u}(t)$, we can define its derivative with respect to t as

$$\vec{u}(t)' = \frac{d}{dt}\vec{u}(t) = \lim_{\Delta t \to 0} \frac{\vec{u}(t+\Delta t) - \vec{u}(t)}{\Delta t}.$$

If $\vec{u}(t) =< x(t), y(t), z(t) >$, one can further see

$$\vec{u}(t)' = \frac{d}{dt}\vec{u}(t) =< x(t)', y(t)', z(t)' > .$$

Proposition 10.2. (1) $\frac{d}{dt}(a \cdot \vec{u}(t) + b \cdot \vec{v}(t)) = a \cdot \vec{u}(t)' + b \cdot \vec{v}(t)'$;
(2) $\frac{d}{dt}(f(t) \cdot \vec{u}(t)) = f(t)' \cdot \vec{u}(t) + f(t) \cdot \vec{u}(t)'$;
(3) $\frac{d}{dt}(\vec{u}(f(t))) = f(t)' \cdot \vec{u}(t)'$;
(4) $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}(t)' \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}(t)'$;
(5) $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}(t)' \times \vec{v}(t) + \vec{u}(t) \times \vec{v}(t)'$.

Corollary 10.3. *If $|\vec{u}|$ is constant, then $\vec{u}$ is perpendicular to $\vec{u}'$.*

Proof. If $|\vec{u}|$ is constant, then $|\vec{u}|^2 = \vec{u} \cdot \vec{u}$ is also a constant, hence vanishes after taking derivative.
Hence

$$\frac{d}{dt}(\vec{u} \cdot \vec{u}) = 2\vec{u}' \cdot \vec{u} = 0,$$

which means that $\vec{u}'$ is perpendicular to $\vec{u}$.

□

Now we go back to see our parametric curves $\vec{r}(t)$. The first order derivative $\vec{r}'(t)$ is called the velocity of this parametric curve and usually denoted by

$$\vec{v}(t) = \vec{r}'(t).$$

The length of the velocity vector is called the speed of this parametric curve, and usually denoted by

$$v(t) = |\vec{v}(t)| = |\vec{r}'(t)|.$$

The second order derivative $\vec{r}''(t)$ is called the acceleration of this parametric curve and usually denoted by

$$\vec{a}(t) = \vec{r}''(t) = \vec{v}'(t).$$

Example 10.4. Calculate $\vec{v}$, v, $\vec{a}$ for the helix

$$\vec{r}(t) =< \cos t, \sin t, t > .$$

Solution:

$$\begin{aligned}
\vec{v}(t) &= \vec{r}'(t) =< x'(t), y'(t), z'(t) >=< -\sin t, \cos t, 1 > \\
v(t) &= |\vec{v}(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{2} \\
\vec{a}(t) &= \vec{r}''(t) =< x''(t), y''(t), z''(t) >=< -\cos t, -\sin t, 0 > .
\end{aligned}$$

□

10.2. **Arc length.** Similarly to the parametric curves in plane, the arc length of a parametric curve

$$\vec{r}(t) =< x(t), y(t), z(t) >$$

from t_0 to t_1 can be calculated as

$$L(t_0, t_1) = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \int_{t_0}^{t_1} |\vec{v}(t)|\, dt.$$

Example 10.5. Calculate the arc length of the helix

$$\vec{r}(t) =< \cos t, \sin t, t >$$

between $P_0 = (1, 0, 0)$ to $P_1 = (1, 0, 2\pi)$.

Solution: The point P_0 corresponds to $t_0 = 0$ and the point P_1 corresponds to $t_1 = 2\pi$. Hence the arc length between P_0 and P_1 can be calculated as

$$L(0, 2\pi) = \int_0^{2\pi} |\vec{v}(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

□

Lecture 10 stopped here.

11. Space curve II – Intrinsic information for a space curve: the Frenet coordinates and arc length parameter

In this section, we introduce an (intrinsic) coordinate system attached to every point of a space curve, which is called the Frenet coordinates. Later using this coordinate system, we are going to introduce some new functions, such as curvature and torsion to characterize more about the properties of a space curve.

11.1. **The unit tangent $\vec{T}$.** As the name, the unit tangent is defined as the unit vector of the velocity, i.e.,

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|}.$$

This vector is always tangent to the curve and pointing to the direction that the parameter goes.

11.2. **The principal unit normal $\vec{N}$.** The principal unit normal (unit normal) is defined as the unit vector of $\frac{d}{dt}\vec{T}(t)$, i.e.,

$$\vec{N}(t) = \frac{\frac{d}{dt}\vec{T}(t)}{|\frac{d}{dt}\vec{T}(t)|}.$$

Notice that since $\vec{T}$ is unit vector, it follows $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$ for every t.

By the definition of $\vec{N}$, it always points to the direction where the curve bend in. For example, for a circle, the unit normal vector is pointing to the center of the circle.

11.3. **The unit binormal $\vec{B}$.** We introduce another vector to complete the coordinate system by defining

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

From the definition, we see

(1) $\vec{B}$ is perpendicular to both $\vec{T}$ and $\vec{N}$, and $(\vec{T}, \vec{N}, \vec{B})$ follows the right hand rule.
(2) $\vec{B}$ is a unit vector since

$$|\vec{B}| = |\vec{T}(t) \times \vec{N}(t)| = |\vec{T}(t)||\vec{N}(t)| \sin\frac{\pi}{2} = 1.$$

We call $\vec{B}(t)$ the unit binormal, and $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ the Frenet coordinate system along the curve.

Example 11.1. Calculate the Frenet coordinate system $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ for the helix

$$\vec{r}(t) =< \cos t, \sin t, t > .$$

Solution: Using the results we got before,

$$\vec{v}(t) =< -\sin t, \cos t, 1 >, \quad v(t) = \sqrt{2},$$

it follows

$$\vec{T}(t) = \frac{1}{\sqrt{2}} < -\sin t, \cos t, 1 >=< -\frac{\sqrt{2}}{2}\sin t, \frac{\sqrt{2}}{2}\cos t, \frac{\sqrt{2}}{2} > .$$

To calculate $\vec{N}$, take derivative to $\vec{T}$ and get

$$\frac{d}{dt}\vec{T}(t) =< -\frac{\sqrt{2}}{2}\cos t, -\frac{\sqrt{2}}{2}\sin t, 0 >,$$

whose length is

$$|\frac{d}{dt}\vec{T}(t)| = \frac{\sqrt{2}}{2}.$$

Hence the unit normal

$$\vec{N}(t) = \frac{\frac{d}{dt}\vec{T}(t)}{|\frac{d}{dt}\vec{T}(t)|} =< -\cos t, -\sin t, 0 > .$$

Next, the unit binormal is given by the cross product

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) =< \frac{\sqrt{2}}{2}\sin t, \frac{\sqrt{2}}{2}\cos t, \frac{\sqrt{2}}{2} > .$$

$\square$

Lecture 11 stopped here.

11.4. **The power of Frenet coordinate system and why it is intrinsic.** The Frenet coordinate system is an intrinsic coordinate system attached to the space curve. Take C a space curve in $\mathbb{R}^3$, then we have many different ways to parametrize it. For example, if

$$\vec{r}_1(t) =< x(t), y(t), z(t) >, \quad t \in [0,1]$$

is a parametric curve for C, then

$$\vec{r}_2(s) = \vec{r}_1(2s) =< x(2s), y(2s), z(2s) >, \quad s \in [0, \frac{1}{2}]$$

is another parametric curve which denotes the same curve C.

Actually, any

$$\vec{\rho}(s) = \vec{r}(f(s)), \quad s \in [a,b]$$

with $f(s) : [a,b] \to [0,1]$ as a bijective map is a parametric curve for C. Usually, we say $\vec{\rho}$ is a reparametrization of $\vec{r}$ using f. Moreover, if $f'(t) > 0$, we say the reparametrization preserves orientations. We restrict ourselves by looking at orientation-preserving reparametrizations for now.

Assume P is a point on C with $\overrightarrow{OP} = \vec{r}(t_0)$. Then under the parametric curve

$$\vec{\rho}(s) = \vec{r}(f(s)),$$

we have $\overrightarrow{OP} = \vec{\rho}(s_0)$, where $f(s_0) = t_0$.

Proposition 11.2. *The Frenet coordinate system for $\vec{\rho}(s)$ at s_0 is the same as the Frenet coordinate system for $\vec{r}(t)$ at t_0. In another word, the Frenet coordinate system is independent of the choice of parametrizaitons (up to orientations).*

Remark 11.3. If you take an orientation revering reparametrization, then $\vec{N}$ stays the same but both $\vec{T}$ and $\vec{B}$ will reverse to opposite.

Hence everything later defined from the Frenet coordinate system is also intrinsic.

11.5. **The arc length parameter.** Since we have seen, there are many different ways to parametrize the same curve. Now, we are going to introduce a canonical parametrization, which is called the parametrization using the arc length parameter.

Definition 11.4. Assume $\vec{r}(t)$ is a parametric curve. We call t is an arc length parameter, if

$$|\vec{v}(t)| \equiv 1.$$

It immediately follows that, if t is an arc length parameter, then the arc length between t_0 and t_1 can be simply calculated as

$$L(t_0, t_1) = t_1 - t_0,$$

and that is why it is named the arc length parameter.

Given a parametric curve $\vec{r}(t)$, one can always use substitute $t = t(s)$ to make

$$\vec{\rho}(s) = \vec{r}(t(s))$$

have s as an arc length parameter by choose such $t = t(s)$ with

$$t'(s) = \frac{1}{|\vec{v}(t)|}. \tag{2}$$

Then from the chain rule,

$$|\frac{d}{ds}\vec{\rho}(s)| = t'(s)|\vec{v}(t)| = 1.$$

To construct the function $t = t(s)$ satisfying (2), we can instead construct its inverse function $s = s(t)$. From the chain rule, we should have now

$$s'(t) = \frac{1}{t'(s)} = |\vec{v}(t)|.$$

Hence

$$s(t) = \int_0^t |\vec{v}(x)|\, dx$$

is the arc length parameter we want.

Example 11.5. Does the following parametric helix

$$\vec{r}(t) =< \cos t, \sin t, t >, \quad t \in [0, 2\pi]$$

have t as arc length parameter? If now, reparametrize it into a parametric curve with arc length parameter.

Proof. Because

$$|\vec{v}(t)| = \sqrt{2} \neq 1,$$

the original parametric curve

$$\vec{r}(t) =< \cos t, \sin t, t >, \quad t \in [0, 2\pi]$$

is not a one with arc length parameter.

We can take $s = s(t)$ by solving

$$s'(t) = |\vec{v}(t)| = \sqrt{2},$$

whose solution is $s(t) = \sqrt{2}t$ to get the arc length parameter.

Then substitute t using the inverse function of $s(t)$, i.e., $t = \frac{1}{\sqrt{2}}s$, we get the new parametric curve

$$\vec{\rho}(s) = \vec{r}(\frac{1}{\sqrt{2}}s) =< \cos\frac{1}{\sqrt{2}}s, \sin\frac{1}{\sqrt{2}}s, \frac{1}{\sqrt{2}}s > .$$

The domain for t now needs to be changed to the corresponding domain for s using $s = \sqrt{2}t$ as

$$s \in [0, 2\sqrt{2}\pi].$$

□

You can check that, with the initial point fixed, e.g. by setting $s_0 = 0$, the parametric curve with arc length parameter is unique, and that is why we say such parameter is canonical.

Using this, given any curve C, we can now use the parametric curve $\vec{r}(s)$ with s as the arc length parameter to represent it.

12. Space curve III – The curvature and the torsion

12.1. **Definition.** Assume C is a space curve and $\vec{r}(s)$ is its arc length parameter representation.

Definition 12.1. The curvature of C at a point $p = \vec{r}(s_0)$ is defined as

$$\kappa(s_0) = |\frac{d}{ds}\vec{T}(s_0)| = \frac{d}{ds}\vec{T}(s_0) \cdot \vec{N}(s_0).$$

The torsion of C at a point $p = \vec{r}(s_0)$ is defined as

$$\tau(s_0) = -\frac{d}{ds}\vec{B}(s_0) \cdot \vec{N}(s_0).$$

(The sign for torsion here is purely a convention issue.)

Example 12.2. Calculate the curvature and torsion for the helix with arc length parameter

$$\vec{r}(s) =< \cos\frac{1}{\sqrt{2}}s, \sin\frac{1}{\sqrt{2}}s, \frac{1}{\sqrt{2}}s >$$

at the point $(1, 0, 0)$ which corresponds to $\vec{r}(0)$.

Solution: We calculate out

$$\begin{aligned}
\vec{T}(s) &= \frac{1}{\sqrt{2}} < -\sin\frac{1}{\sqrt{2}}s, \cos\frac{1}{\sqrt{2}}s, 1 > \\
\vec{N}(s) &= < -\cos\frac{1}{\sqrt{2}}s, -\sin\frac{1}{\sqrt{2}}s, 0 > \\
\vec{B}(s) &= \frac{1}{\sqrt{2}} < \sin\frac{1}{\sqrt{2}}s, \cos\frac{1}{\sqrt{2}}s, 1 > .
\end{aligned}$$

Hence the curvature

$$\kappa(s) = |\frac{d}{ds}\vec{T}(s)| = |\frac{1}{2} < -\cos\frac{1}{\sqrt{2}}s, -\sin\frac{1}{\sqrt{2}}s, 0 > | = \frac{1}{2},$$

and the torsion

$$\tau(s) = -\frac{1}{2} < \cos\frac{1}{\sqrt{2}}s, -\sin\frac{1}{\sqrt{2}}s, 0 > \cdot < -\cos\frac{1}{\sqrt{2}}s, -\sin\frac{1}{\sqrt{2}}s, 0 >= \frac{1}{2}.$$

Since they are constants now, which are also the curvature and torsion at the point $(1, 0, 0)$. □

12.2. **Calculate curvature and torsion directly from arbitrary parametric equations.** In fact, with the help of the chain rule, one doesn't need to always first obtain the arc length parametrization to calculate the curvature and torsion.

Proposition 12.3. *Assume $\vec{r}(t)$ is a parametric curve. Then the curvature at $\vec{r}(t)$ can be calculated as*

$$\kappa(t) = \frac{|\frac{d}{dt}\vec{T}(t)|}{|\vec{v}(t)|}.$$

Similarly, you can also prove that

Proposition 12.4. *Assume $\vec{r}(t)$ is a parametric curve. Then the torsion at $\vec{r}(t)$ can be calculated as*

$$\tau(t) = -\frac{1}{|\vec{v}(t)|}\frac{d}{dt}\vec{B}(t) \cdot \vec{N}(t).$$

Example 12.5. Using the parametric equation

$$\vec{r}(t) =< \cos t, \sin t, t >$$

to directly calculate the curvature and torsion at the point $(1, 0, 0)$.

Solution: Direct calculation shows that $\kappa = \frac{1}{2}$ and $\tau = \frac{1}{2}$ again using this way. $\square$

This following examples give you some intuition about the geometric meaning of curvature and torsion of a space curve whose details are left to you to figure out.

Example 12.6. (1) The circle of radius R has constant curvature $\frac{1}{R}$;

(2) Plane curves have vanishing torsions;

(3) The helix

$$\vec{r}_c(t) =< \cos t, \sin t, ct >$$

has curvature $\frac{1}{1+c^2}$ and torsion $\frac{c}{1+c^2}$, where c is a nonnegative constant.

12.3. **More about the Frenet coordinates.** With introducing curvature and torsion, in fact, we can catch all information of a space curve, which can be seen as follows.

Assume $\vec{r}(s)$ is a space curve with arc length parameter s. First, notice that from the definition for curvature, we can rewrite

$$\frac{d}{ds}\vec{T}(s) = \kappa\vec{N}(s).$$

The change of $\vec{B}$, $\frac{d}{ds}\vec{B}(s)$, actually is parallel to $\vec{N}$ too. The reason is

(1) It is perpendicular to $\vec{B}$ since $\vec{B}$ is a unit vector. It follows it lives in the plane spanned by $\vec{T}$ and $\vec{N}$;

(2) It is perpendicular to $\vec{T}$ too:

$$\begin{aligned}\frac{d}{ds}\vec{B}\cdot\vec{T} &= \frac{d}{ds}(\vec{T}\times\vec{N})\cdot\vec{T}\\ &= (\frac{d}{ds}\vec{T}\times\vec{N})\cdot\vec{T} + (\vec{T}\times\frac{d}{ds}\vec{N})\cdot\vec{T}\\ &= (\kappa\vec{N}\times\vec{N})\cdot\vec{T} + (\vec{T}\times\frac{d}{ds}\vec{N})\cdot\vec{T}\\ &= 0 + 0 = 0.\end{aligned}$$

Hence, it is parallel to $\vec{N}$.

Then from the definition of torsion, we have now, the change of $\vec{B}$, $\frac{d}{ds}\vec{B}(s)$ is

$$\frac{d}{ds}\vec{B}(s) = -\tau\vec{N}(s).$$

With the expressions of $\frac{d}{ds}\vec{T}(s)$ and $\frac{d}{ds}\vec{B}(s)$, the change of $\vec{N}$, $\frac{d}{ds}\vec{N}(s)$ can also be obtained:

$$\begin{aligned}\frac{d}{ds}\vec{N} &= \frac{d}{ds}(\vec{B}\times\vec{T})\\ &= \frac{d}{ds}\vec{B}\times\vec{T} + \vec{B}\times\frac{d}{ds}\vec{T}\\ &= -\tau\vec{N}\times\vec{T} + \kappa\vec{B}\times\vec{N}\\ &= -\kappa\vec{T} + \tau\vec{B}.\end{aligned}$$

We can summarize the above derivation into the following important formula

Theorem 12.7 (Fundamental Theorem of Space Curves – Frenet-Serret Formula)**.**

$$\frac{d}{ds}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{bmatrix}.$$

We get an anti-symmetric matrix from this formula.

(I don't require you completely understand the derivation for this formula, but do hope you can feel some beauty of math from it.)

Lecture 12 stopped here.

13. Functions of multivariables I – Definition

All the contents in this part have strict analogues in single variable functions.

13.1. Functions of multivariables.

13.1.1. *Map.* Consider a map as a machine that whenever one input two numbers, which can be considered as one point in $\mathbb{R}^2$, into it one can get a number as output. Such a map, denoted by $f(x, y)$, is called a function of two variables, where x, y are two inputs, and the value $f(x, y)$ is the output.

Example 13.1. $f(x, y) = x^2 + xy + y^2$.

- Input $x = 0, y = 0$, get $f(0, 0) = 0^2 + 0 \cdot 0 + 0^2 = 0$;
- Input $x = -1, y = 0$, get $f(-1, 0) = (-1)^2 + (-1) \cdot 0 + 0^2 = 1$;
- Input $x = 0, y = 1$, get $f(0, 1) = 0^2 + 0 \cdot 1 + 1^2 = 1$;
- Input $x = 1, y = 1$, get $f(1, 1) = 1^2 + 1 \cdot 1 + 1^2 = 3$;
- ....

However, no matter how many points input, one can not exhaust all points. Hence this way can only provide an intuition but never provide any property of a function rigorously.

13.1.2. *Domain.* For a function f, sometimes, one can not input all points in $\mathbb{R}^2$ but only a subset of it. This subset is called the domain of this function f. In our course, we are interested in the following types of functions and their combinations, whose natural domains are listed below"

(1) Polynomial functions. E.g., $f(x, y) = x^3 + 4x^2y + 3xy^2 - y^3$. Natural domain is $\mathbb{R}^2$;
(2) Rational functions are fractions of two polynomial functions. E.g.,

$$f(x, y) = \frac{x^3 + 4x^2y + 3xy^2 - y^3}{1 - x^2 - y^2}.$$

Natural domain consists of the points which make the denominator nonvanishing. For example, for such f, the natural domain is

$$\{(x, y)|1 - x^2 - y^2 \neq 0\} = \{(x, y)|x^2 + y^2 \neq 1\}.$$

(3) Exponential functions. E.g., $f(x, y) = e^x e^y$ has natural domain $\mathbb{R}^2$.
(4) Logarithmic functions. E.g., $\ln(x + y)$ has natural domain

$$\{(x, y)|x + y > 0\}.$$

(5) Trigonometric functions. E.g., $\sin(x + y)$ has natural domain $\mathbb{R}^2$.

(6) Inverse trigonometric functions. E.g., $\arcsin(x+y)$ has natural domain

$$\{(x,y)| -1 \leq x+y \leq 1\}.$$

(7) ....

Example 13.2. What is the natural domain of the following functions:

(1) $f(x,y) = \frac{x+y}{\ln(x^2+y^2)}$
(2) $f(x,y) = \frac{x}{\sqrt{x^2+y^2+1}-1}$
(3) $f(x,y) = \sqrt{9-x^2-y^2}$.

Solutions: (1) First, from the natural domain of log function, we know $(x,y) \neq (0,0)$. Then as a denominator for a fraction, $\ln(x^2+y^2)$ can not be zero, and hence, $x^2+y^2 \neq 1$. Above all, the natural domain is

$$\{(x,y)|(x,y) \neq (0,0), x^2+y^2 \neq 1\}.$$

(2) $\{(x,y)|(x,y) \neq (0,0)\}$.
(3) $\{(x,y)|x^2+y^2 \leq 9\}$.

$\square$

13.1.3. *Range.* The set consists of all $f(x,y)$ for (x,y) in the domain is called the range of f. Some functions defined on natural domains has natural ranges. For example, because the single variable function $sinx$ has range $[-1,1]$, the function $\sin(x+2y)$ also has range $[-1,1]$ since $x+2y$ has range $\mathbb{R}$.

13.2. **Graph.** Given a function $f(x,y)$, then surface $z = f(x,y)$ in $\mathbb{R}^3$ is called its graph.

Example 13.3. Sketch the graph of function $f(x,y) = x^2+y^2$.

Example 13.4. Sketch the graph of function $f(x,y) = \sqrt{x^2+y^2}$.

13.3. **Level Curve.** Given a function $f(x,y)$ and a constant k in its range, the equation

$$f(x,y) = k$$

is a curve in $\mathbb{R}^2$, which is called the level curve of level k. All the points (x,y,k) with $f(x,y) = k$ consists a curve on the graph $z = f(x,y)$ whose z-coordinate is constant k.

Remark 13.5. When we consider a function of three variables, like $f(x,y,z)$, then we can replace the concept of level curve by level surface.

p- We can catch the general information of a surface $z = f(x,y)$ using the contour map.

13.4. **Transformation of functions.** Lecture 13 stopped here.

14. Functions of multivariable II – Limits and continuity

14.1. **Limit.**

14.1.1. *Definition.* Given a function $f(x,y)$ with domain D. Assume (x_0,y_0) is a point in $\mathbb{R}^2$ which may not be in D. As for single variable functions, we can consider

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y),$$

whose strict definition is given before.

Definition 14.1. We say the limit

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y),$$

exists, if there is some real number L make the following true:

For any $\epsilon > 0$, there exists some $\delta > 0$ such that for any

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} \leq \delta,$$

have

$$|f(x,y) - L| \leq \epsilon.$$

If this holds, we say the limit is L and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

(One can prove if the limit exists, it must be unique.)

Conversely, if there is no such L, we say the limit does not exist.

14.1.2. *Show a limit DNE.* Using this definition, one can prove that

Proposition 14.2. *If along two different paths $r_1(t) = (x_1(t), y_1(t))$ and $r_2(t) = (x_2(t), y_2(t))$ with*

$$r_1(t) \to (x_0,y_0) \quad \textit{and} \quad r_2(t) \to (x_0,y_0), \quad \textit{as } t \to 0,$$

the limits

$$\lim_{t\to 0} f((x_1(t), y_1(t)) \neq \lim_{t\to 0} f((x_2(t), y_2(t)),$$

then the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

This statement is very useful when we try to prove a limit doesn't exist.

Example 14.3. Show $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+3y^2}$ does not exist.

Solution: Take different paths $y = kx$ for different k. Then as $x \to 0$, $(x,kx) \to (0,0)$. Calculate

$$\lim_{x\to 0} \frac{x(kx)}{x^2 + 3(kx)^2} = \lim_{x\to 0} \frac{kx^2}{(1+3k^2)x^2} = \frac{k}{1+3k^2}$$

which depends on k. Hence using Prop 14.2, we see the limit DNE.

□

Example 14.4. Show $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4}$ does not exist.

Solution: For this problem, the linear functions $y = kx$ can not do the job. Instead, we need to use $x = ky^2$. As $y \to 0$, $(ky^2, y) \to (0,0)$. Calculate

$$\lim_{y\to 0} \frac{(ky^2)y^2}{(ky^2)^2 + 3y^4} = \lim_{x\to 0} \frac{ky^4}{(k^2+3)y^4} = \frac{k}{k^2+3}$$

which depends on k. Hence using Prop 14.2, we see the limit DNE. □

14.1.3. *Calculate a limit.* Notice, we can never use Prop 14.2 to show a limit exists! To argue a limit exists and calculate limit, the following ways can do the job.

Proposition 14.5. *If* $\lim_{(x,y)\to(x_0,y_0)} f_1(x,y) = L_1$ *and* $\lim_{(x,y)\to(x_0,y_0)} f_2(x,y) = L_2$*, then*

(1) $\lim_{(x,y)\to(x_0,y_0)}(f_1(x,y) + f_2(x,y))$ *exists and is* $L_1 + L_2$*;*
(2) $\lim_{(x,y)\to(x_0,y_0)}(f_1(x,y) \cdot f_2(x,y))$ *exists and is* $L_1 \cdot L_2$*;*
(3) *if moreover,* $L_2 \neq 0$*,* $\lim_{(x,y)\to(x_0,y_0)} \frac{f_1(x,y)}{f_2(x,y)}$ *exists and is* $\frac{L_1}{L_2}$*.*

Proposition 14.6. *If* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0$ *and* $g(x,y)$ *is bounded by some positive number, i.e., there exists some* $C > 0$ *such that* $|g(x,y)| \leq C$ *for any* (x,y)*, then*

$$\lim_{(x,y)\to(x_0,y_0)}(f(x,y) \cdot g(x,y)) = 0.$$

Example 14.7. Calculate $\lim_{(x,y)\to(0,0)} \frac{x^3(\sin(xy)+1)}{x^2+y^2}$.

Solution: Consider the function $\frac{x^3(\sin(xy)+1)}{x^2+y^2}$ as the product of the function

$$f(x,y) = x$$

and

$$g(x,y) = \frac{x^2(\sin(xy)+1)}{x^2+y^2}.$$

The reason is

$$\lim_{(x,y)\to(0,0)} x = 0$$

and

$$|\frac{x^2(\sin(xy)+1)}{x^2+y^2}| = |\frac{x^2}{x^2+y^2}||(\sin(xy)+1)| \leq |\frac{x^2+y^2}{x^2+y^2}|(|\sin(xy)| + |1|) \leq 1 \cdot 2 = 2.$$

Using Prop 14.6, we get

$$\lim_{(x,y)\to(0,0)} \frac{x^3(\sin(xy)+1)}{x^2+y^2} = 0.$$

□

Besides the methods introduced above, the polar coordinates is another powerful way of calculating limits.

Proposition 14.8. *The limit* $\lim_{(x,y)\to(0,0)} f(x,y)$ *exists if and only if the following limit exists and independent of* $\theta \in [0, 2\pi)$*,*

$$\lim_{\rho\to 0} f(\rho\cos\theta, \rho\sin\theta).$$

Moreover, the two limits are the same.

Example 14.9. Calculate

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2).$$

Solution: Consider the problem under polar coordinates:

$$\lim_{\rho\to 0}\rho^2\ln\rho^2.$$

This limit can be calculates as

$$\begin{aligned}\lim_{\rho\to 0}\rho^2\ln\rho^2 &= \lim_{\rho\to 0}\frac{2\ln\rho}{\frac{1}{\rho^2}} && \text{type of } \frac{\infty}{\infty}\\ &= \lim_{\rho\to 0}\frac{2\frac{1}{\rho}}{-2\frac{1}{\rho^3}} && \text{L'Hospital}\\ &= \lim_{\rho\to 0}(-\rho^2)\\ &= 0,\end{aligned}$$

which is independent of θ. Hence using 14.8, we know the original limit exists and is 0. □

14.2. **Continuity.**

Definition 14.10. Assume $f(x,y)$ is a function with domain D and (x_0,y_0) is a point in D. Call f is continuous at (x_0,y_0), if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Remark 14.11. We can define it through ϵ,δ language too, which is: We say $f(x,y)$ is continuous at (x_0,y_0), if for any $\epsilon>0$, there exists some $\delta>0$ such that for any

$$\sqrt{(x-x_0)^2+(y-y_0)^2}\le\delta,$$

have

$$|f(x,y)-f(x_0,y_0)|\le\epsilon.$$

Proposition 14.12. *The functions considered in Section 13.1.2. are continuous in their natural domains.*

Example 14.13. If a function f is defined as follows

$$f(x,y)=\begin{cases}\frac{x^2+y^2+x^3}{x^2+y^2} & (x,y)\neq(0,0)\\ k & (x,y)=(0,0).\end{cases}$$

Determine the constant k to make f be a continuous function on $\mathbb{R}^2$.

Solution: If $(x,y)\neq(0,0)$, the function f is of the form $\frac{x^2+y^2+x^3}{x^2+y^2}$, which is continuous on $\mathbb{R}^2-\{(0,0)\}$. Hence, we only need to make f continuous at $(0,0)$ too.

To get this, we need

$$\lim_{(x,y)\to(0,0)} f(x,y)=f(0,0).$$

The LHS can be calculates as

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x^2+y^2+x^3}{x^2+y^2}=\lim_{(x,y)\to(0,0)}(1+\frac{x^3}{x^2+y^2})=1.$$

The RHS is given as k. Thus, f continuous at $(0,0)$ if and only if $k=1$. □

Lecture 14 stopped here.

15. Functions of multivariable III – Partial derivatives

15.1. **Definition of partial derivatives.** Assume $f(x, y)$ is a function of two variables and (x_0, y_0) is a point in its domain. We are going to define the partial derivatives of f at this point.

Definition 15.1. The partial derivative of f with respect to x at (x_0, y_0) is defined as

$$\frac{\partial f}{\partial x}|_{(x_0,y_0)} = f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

and the partial derivative of f with respect to y at (x_0, y_0) is defined as

$$\frac{\partial f}{\partial y}|_{(x_0,y_0)} = f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

To calculate the partial derivatives, you can just think it as derivatives by looking at the other variable as constant.

Example 15.2. Calculate the partial derivatives for the function

$$f(x, y) = x^3 + x^2y + x\sin(xy) + y^3$$

at $(1, 0)$.

Solution: We calculate the partial derivatives using two different ways and you can compare them.

Method 1. To calculate $\frac{\partial f}{\partial x}|_{(1,0)}$, we can fix $y = 0$ and consider the function of x as

$$f(x, 0) = x^3.$$

Then using definition

$$\frac{\partial f}{\partial x}|_{(1,0)} = \frac{d}{dx}|_{x=1}f(x, 0) = 3x^2|_{x=1} = 3.$$

Similarly, to calculate $\frac{\partial f}{\partial y}|_{(1,0)}$, we can fix $x = 1$ and consider the function of y as

$$f(1, y) = 1 + y + \sin y + y^3.$$

Then using definition

$$\frac{\partial f}{\partial y}|_{(1,0)} = \frac{d}{dy}|_{y=0}f(1, y) = (1 + \cos y + 3y^2)|_{y=0} = 2.$$

Method 2. We can first get the function $\frac{\partial f}{\partial x}$ and then plug the point $(1, 0)$ into this function to get $\frac{\partial f}{\partial x}|_{(1,0)}$.

To calculate $\frac{\partial f}{\partial x}$, we need to regard y as constant and do the algebraic calculation to x, which is

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy + \sin(xy) + x(y\cos(xy)).$$

Plugging the point $(1, 0)$, we get

$$\frac{\partial f}{\partial x}|_{(1,0)} = (3x^2 + 2xy + \sin(xy) + x(y\cos(xy)))|_{(1,0)} = 3.$$

Similarly, you can do it to y and will be left to you as an exercise.

□

The second method here already indicates that, from the function f, we can get two new functions, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Example 15.3. Calculate partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function

$$f(x,y) = 4x^3 + 2x^2y - xy^2 + y^3.$$

Solution: Regard y and x as constant respectively, we get

$$\frac{\partial f}{\partial x} = 12x^2 + 4xy - y^2$$

and

$$\frac{\partial f}{\partial y} = 2x^2 - 2xy + 3y^2.$$

$\square$

Remark 15.4. For a function f, both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the point (x_0, y_0), can not ensure the function is continuous at this point.

For example, consider a function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

We can calculate the partial derivatives at $(0,0)$ as

$$\begin{aligned} \frac{\partial f}{\partial x}|_{(0,0)} &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h\cdot 0}{h^2+0^2} - 0}{h} = 0. \\ \frac{\partial f}{\partial y}|_{(0,0)} &= \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{0\cdot h}{0^2+h^2} - 0}{h} = 0. \end{aligned}$$

Hence both partial derivatives exist at $(0,0)$.

However, we know $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ DNE, and hence this function f is not continuous at $(0,0)$.

15.2. **Implicit differentiation.** Some functions are not given in explicit form but are given in some implicit way. For example, the equation

$$x^2yz + e^z + 3 = 0$$

determines a function $z = f(x,y)$. In general, we have

Theorem 15.5 (Implicit function theorem)**.** *Given an equation*

$$F(x,y,z) = 0$$

with continuous partial derivatives. If at a point (x_0, y_0, z_0) satisfying the equation, we have

$$\frac{\partial F}{\partial z}|_{(x_0,y_0,z_0)} \neq 0,$$

then near this point, there exists some function $z = f(x,y)$ such that

$$F(x,y,f(x,y)) \equiv 0.$$

We leave the derivation of the partial derivatives for f to the section of chain rule, but only from concrete example now how to calculate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ without solving z.

Example 15.6. Assume $z = f(x, y)$ is determined by the equation

$$x^2yz + e^z + 3 = 0.$$

Calculate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ without solving the equation.

Solution: Since we assume that $z = f(x, y)$ solves the equation, we have

$$x^2yf(x, y) + e^{f(x,y)} + 3 \equiv 0.$$

Take partial derivatives to x for both sides, we get

$$2xyz + x^2y\frac{\partial f}{\partial x} + e^z\frac{\partial f}{\partial x} = 0.$$

We solve from it

$$\frac{\partial f}{\partial x} = -\frac{2xyz}{x^2y + e^z}.$$

Similarly,

$$\frac{\partial f}{\partial y} = -\frac{x^2z}{x^2y + e^z}$$

□

15.3. **Higher order partial derivatives and the Clairaut's theorem.** After we get the partial derivatives for the function f, we can regard $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as two new functions, and further take partial derivatives to x and to y. Denote by

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= f_{xx} := \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) \\
\frac{\partial^2 f}{\partial yx} &= f_{xy} := \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) \\
\frac{\partial^2 f}{\partial xy} &= f_{yx} := \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) \\
\frac{\partial^2 f}{\partial y^2} &= f_{yy} := \frac{\partial}{\partial y}(\frac{\partial f}{\partial y})
\end{aligned}$$

and call them second derivatives of the function f.

Example 15.7. Calculate second derivatives of

$$f(x, y) = 4x^3 + 2x^2y - xy^2 + y^3.$$

Solution:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= f_{xx} := \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = 24x + 4y \\
\frac{\partial^2 f}{\partial y\partial x} &= f_{xy} := \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = 4x - 2y \\
\frac{\partial^2 f}{\partial x\partial y} &= f_{yx} := \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = 4x - 2y \\
\frac{\partial^2 f}{\partial y^2} &= f_{yy} := \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = -2x + 6y.
\end{aligned}$$

□

We notice that in the previous example, the two mixed second partial derivatives are equal to each other, i.e.,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Actually, this is NOT a coincidence. In general we have,

Theorem 15.8 (Clairaut's theorem)**.** *If a function f has continuous second partial derivatives at any given point in the interior of its domain, then*

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Example 15.9. If $f(x, y)$ has arbitrary orders of partial derivatives, then at most how many different third order partial derivatives can one get? (The answer is 4. Why?)

To end this section, we introduce a notation which is useful in math and physics you are going to learn in future.

Definition 15.10. (1) Given a function $f(x, y)$ of two variables, define

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

(2) Given a function $f(x, y, z)$ of three variables, define

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

If a function f has $\Delta f = 0$ on its domain D, then f is called harmonic on D.

Example 15.11. Direct calculation can check that the function

$$f(x, y) = \arctan \frac{y}{x}, \quad (x, y) \in (0, 1) \times (0, 1)$$

is harmonic on $(0, 1) \times (0, 1)$.

Lecture 15 stopped here.

16. Functions of multivariable IV – Tangent plane and linear approximation

16.1. **Equation for tangent plane.** For the definition of partial derivatives, we know that they are be regarded as the slopes of the intersection curves. To be concrete,

$$\begin{aligned} \frac{\partial f}{\partial x}|_{(x_0,y_0)} &= \text{the slope of the line } f(x, y_0) \text{ at } x_0 \\ \frac{\partial f}{\partial y}|_{(x_0,y_0)} &= \text{the slope of the line } f(x_0, y) \text{ at } y_0. \end{aligned}$$

Remark 16.1. The intersection curve of the graph of $f(x, y_0)$ can be written as a parametric curve

$$\vec{r}_1(x) =< x, y_0, f(x, y_0) >,$$

and the intersection curve of the graph of $f(x_0, y)$ can be written as a parametric curve

$$\vec{r}_2(y) =< x_0, y, f(x_0, y) > .$$

Denote by $\vec{v}_1$ and $\vec{v}_2$ for their velocities respectively. Then

$$\vec{v}_1(x_0, y_0) = \frac{d}{dx}|_{x=x_0}\vec{r}_1(x) =< 1, 0, \frac{\partial f}{\partial x}|_{(x_0,y_0)} >$$

and

$$\vec{v}_2(x_0, y_0) = \frac{d}{dy}|_{y=y_0}\vec{r}_2(y) =< 0, 1, \frac{\partial f}{\partial y}|_{(x_0,y_0)} > .$$

The tangent plane of $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ should be both perpendicular to $\vec{v}_1(x_0, y_0)$ and to $\vec{v}_2(x_0, y_0)$. Hence we can use the cross product of these two vector as the normal vector of the tangent plane. The normal vector is

$$\vec{v}_1(x_0, y_0) \times \vec{v}_2(x_0, y_0) =< -\frac{\partial f}{\partial x}|_{(x_0,y_0)}, -\frac{\partial f}{\partial y}|_{(x_0,y_0)}, 1 > .$$

For the above remark, we conclude that

Proposition 16.2. *The equation for the tangent plane of $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is*

$$-\frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) - \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0) + (z - z_0) = 0.$$

Example 16.3. Write down the equation of the tangent plane of the surface $z = x^3+x^2y+x\sin(xy)+y^3$ at $(1, 0)$.

Solution: From 15.2, we know

$$\frac{\partial z}{\partial x}|_{(1,0)} = 3, \quad \frac{\partial z}{\partial y}|_{(1,0)} = 2.$$

Moreover, $z(1, 0) = 1$. Hence the tangent plane is

$$-3(x - 1) - 2(y - 0) + (z - 1) = 0.$$

□

16.2. **Linear approximation.** If we rewrite the equation for the tangent plane into

$$z = L_{(x_0,y_0)}(x, y) := f(x_0, y_0) + \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0),$$

then we can regard it as a linear equation $z = L_{(x_0,y_0)}(x, y)$ and it is called the linearization of $z = f(x, y)$ at (x_0, y_0). When x is close to x_0, y is close to y_0, we have $f(x, y)$ is close to $L_{(x_0,y_0)}(x, y)$.

Example 16.4. Use the linear approximation to calculate the approximate value of $\sqrt{3.1^2 + 3.9^2}$.

Solution: Consider a function $f(x, y) = \sqrt{x^2 + y^2}$. Then what we want to calculate is $f(3.1, 3.9)$. Notice that $(3.1, 3.9)$ is close to $(3, 4)$, and $f(3, 4) = 5$ is easy to calculate, we can use the linear approximation of $f(x, y)$ at $(3, 4)$ to get the approximate value of $\sqrt{3.1^2 + 3.9^2}$.

To do this, calculate the partial derivatives at $(3, 4)$ at

$$\frac{\partial f}{\partial x}|_{(3,4)} = \frac{x}{\sqrt{x^2 + y^2}}|_{(3,4)} = 0.6, \frac{\partial f}{\partial y}|_{(3,4)} = \frac{y}{\sqrt{x^2 + y^2}}|_{(3,4)} = 0.8.$$

Hence the linear approximation is

$$\begin{aligned} L(x, y) &= f(3, 4) + \frac{\partial f}{\partial x}|_{(3,4)} \cdot (3.1 - 3) + \frac{\partial f}{\partial y}|_{(3,4)} \cdot (3.9 - 4) \\ &= 5 + 0.6 \cdot 0.1 + 0.8 \cdot (-0.1) \\ &= 4.98. \end{aligned}$$

Above all,

$$\sqrt{3.1^2 + 3.9^2} \approx 4.98.$$

□

Lecture 16 stopped here.

16.3. **Differentiability.**

Definition 16.5. If a function $f(x, y)$ at a point (x_0, y_0) can be written as

$$f(x, y) = L_{(x_0,y_0)}(x, y) + h_1(x, y) \cdot (x - x_0) + h_2(x, y) \cdot (y - y_0)$$

with

$$\lim_{(x,y)\to(x_0,y_0)} h_1(x, y) = 0, \qquad \lim_{(x,y)\to(x_0,y_0)} h_2(x, y) = 0,$$

then we call f is differentiable at the point (x_0, y_0).

Remark 16.6. Usually, people write such error terms $h_1(x, y) \cdot (x - x_0) + h_2(x, y) \cdot (y - y_0)$ as $o(\Delta x, \Delta y)$.

Definition 16.7. If f is differential everywhere in some open domain D, then we use

$$df := \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

to denote its differential.

The existence of the first order partial derivatives is NOT enough to ensure that f is differentiable.

Theorem 16.8. *If f_x and f_y exist and are continuous everywhere in the interior of the domain D, then f is differentiable in the interior of D.*

Remark 16.9. This can be proved using the mean value theorem but not required to know. The reverse statement of this theorem actually is NOT true. There are some counter examples but are quite subtle so we skip.

Lecture 17 stopped here.

17. Functions of multivariables V – The Chain rule

17.1. **The chain rule.** Recall, for a function of one variable $f(x)$, when x is also a function of another variable t, we can get a new function by composing f and x as

$$g(t) = f(x(t)).$$

To calculate the derivative of g with respect to t, we have the chain rule for one variable function:

$$\frac{dg}{dt} = \frac{df}{dx}\Big|_{x(t)} \cdot \frac{dx}{dt}.$$

Now consider a function of two variables $f(x, y)$, when x and y are functions of another variable t, we can similarly get a new function of t which is

$$g(t) = f(x(t), y(t)).$$

For the current case, we also have the chain rule.

Theorem 17.1.
$$\frac{dg}{dt} = \frac{\partial f}{\partial x}|_{(x(t),y(t))} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}|_{(x(t),y(t))} \cdot \frac{dy}{dt}.$$

Proof. □

Example 17.2. Calculate the derivative of the function $g(t) = t^3 + t^2e^t$ using the chain rule.

Solution: Direct calculation can give the derivative $g'(t) = 3t^2 + 2te^t + t^2e^2$. However, we can also calculate it using the chain rule we just learnt. We give two different ways of doing the chain rule.

(1) Consider $x(t) = t^3$ and $y(t) = t^2e^t$. Take $f(x, y) = x + y$ and then
$$g(t) = f(x(t), y(t)).$$
Using the chain rule
$$\begin{aligned} g'(t) &= \frac{\partial f}{\partial x}|_{(x(t),y(t))} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}|_{(x(t),y(t))} \cdot \frac{dy}{dt} \\ &= 1 \cdot (3t^2) + 1 \cdot (2te^t + t^2e^2) \\ &= 3t^2 + 2te^t + t^2e^2. \end{aligned}$$

(2) Write $g(t) = t^2(t + e^t)$ and then consider $x(t) = t^2$ and $y(t) = t + e^t$. Take $f(x, y) = xy$ and then
$$g(t) = f(x(t), y(t)).$$
Using the chain rule
$$\begin{aligned} g'(t) &= \frac{\partial f}{\partial x}|_{(x(t),y(t))} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}|_{(x(t),y(t))} \cdot \frac{dy}{dt} \\ &= y|_{(t^2,t+e^t)} \cdot (2t) + x|_{(t^2,t+e^t)} \cdot (1 + e^t) \\ &= (t + e^t) \cdot 2t + t^2(1 + e^t) \\ &= 3t^2 + 2te^t + t^2e^2. \end{aligned}$$

□

Now we generalize the chain rule a little bit. Instead of taking x and y are functions of one variable t, let's consider they are of two variable functions
$$x = x(s, t), \quad y = y(s, t).$$
Then we get a function of two variables
$$g(s, t) = f(x(s, t), y(s, t)).$$
By looking at the other variable as a constant, we get the chain rule for this case as follows.

Theorem 17.3.
$$\begin{aligned} \frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x}|_{(x(s,t),y(s,t))} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}|_{(x(s,t),y(s,t))} \cdot \frac{\partial y}{\partial s} \\ \frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x}|_{(x(s,t),y(s,t))} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}|_{(x(s,t),y(s,t))} \cdot \frac{\partial y}{\partial t}. \end{aligned}$$

Example 17.4. Use the chain rule to calculate the partial derivatives of
$$g(s, t) = (s + t)^2e^{s+t}.$$

Solution: Consider

$$x(s,t) = (s+t)^2, \quad y(s,t) = e^{s+t},$$

and $f(x,y) = xy$. Then $g(s,t) = f(x(s,t), y(s,t))$. Calculate

$$\begin{aligned}
\frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x}|_{(x(s,t),y(s,t))} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}|_{(x(s,t),y(s,t))} \cdot \frac{\partial y}{\partial s} \\
&= e^{s+t} \cdot 2(s+t) + (s+t)^2 \cdot e^{s+t},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x}|_{(x(s,t),y(s,t))} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}|_{(x(s,t),y(s,t))} \cdot \frac{\partial y}{\partial t} \\
&= e^{s+t} \cdot 2(s+t) + (s+t)^2 \cdot e^{s+t}.
\end{aligned}$$

□

Example 17.5. Assume $g(s,t) = f(s^2 - t^2, st)$. Use the following information

$$f(-3,2) = 3, \quad \frac{\partial f}{\partial x}|_{(-3,2)} = 4, \quad \frac{\partial f}{\partial y}|_{(-3,2)} = -1.$$

to write down the equation for the linearization of $g(s,t)$ at the point $(1,2)$.

Solution: We derive the linearization of $g(s,t)$ at the point $(1,2)$, we need to know the two partial derivatives $\frac{\partial g}{\partial s}$ and $\frac{\partial g}{\partial t}$ at $(1,2)$. Denote by

$$x(s,t) = s^2 - t^2, \quad y(s,t) = st, \quad g(s,t) = f(x(s,t), y(s,t)).$$

Using the chain rule,

$$\begin{aligned}
\frac{\partial g}{\partial s}|_{(1,2)} &= \frac{\partial f}{\partial x}|_{(x(1,2),y(1,2))} \cdot \frac{\partial x}{\partial s}|_{(1,2)} + \frac{\partial f}{\partial y}|_{(x(1,2),y(1,2))} \cdot \frac{\partial y}{\partial s}|_{(1,2)} \\
&= \frac{\partial f}{\partial x}|_{(-3,2)} \cdot (2 \cdot 1) + \frac{\partial f}{\partial y}|_{(-3,2)} \cdot 2 \\
&= 4 \cdot 2 + (-1) \cdot 2 = 6,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial g}{\partial t}|_{(1,2)} &= \frac{\partial f}{\partial x}|_{(x(1,2),y(1,2))} \cdot \frac{\partial x}{\partial t}|_{(1,2)} + \frac{\partial f}{\partial y}|_{(x(1,2),y(1,2))} \cdot \frac{\partial y}{\partial t}|_{(1,2)} \\
&= \frac{\partial f}{\partial x}|_{(-3,2)} \cdot (-2 \cdot 2) + \frac{\partial f}{\partial y}|_{(-3,2)} \cdot 1 \\
&= 4 \cdot (-4) + (-1) \cdot 1 = -17.
\end{aligned}$$

Hence the linearization at $(1,2)$ is

$$L(x,y) = f(-3,2) + 6(s-1) - 17(t-2) = 3 + 6(s-1) - 17(t-2).$$

□

17.2. **General formula for implicit differentiation.** Consider an equation $F(x, y, z) = 0$ and assume that $\frac{\partial F}{\partial z} \neq 0$ everywhere. Then using the implicit function theorem, we know that there exists a solution $z = f(x, y)$ for the equation, i.e.,

$$F(x, y, f(x, y)) \equiv 0.$$

Using the chain rule, we calculate the partial derivatives for both siders and get

$$\begin{aligned}\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial f}{\partial x} &= 0\\ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial f}{\partial y} &= 0.\end{aligned}$$

From them we solve

$$\frac{\partial f}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial f}{\partial y} = -\frac{F_y}{F_z}.$$

Lecture 18 stopped here.

18. Functions of multivariables VI – Directional derivatives

18.1. **Directional derivatives.** Recall the partial derivatives are defined as

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h\to 0}\frac{f(x + h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y}(x, y) = \lim_{h\to 0}\frac{f(x, y + h) - f(x, y)}{h},$$

and we can rewrite them using vector $\vec{r} =< x, y >$ as

$$\frac{\partial f}{\partial x}(\vec{r}) = \lim_{h\to 0}\frac{f(\vec{r} + h\cdot\vec{i}) - f(\vec{r})}{h}, \quad \frac{\partial f}{\partial y}(\vec{r}) = \lim_{h\to 0}\frac{f(\vec{r} + h\cdot\vec{j}) - f(\vec{r})}{h}.$$

The latter enlightens us to consider more general situations as followings. Take any unit vector $\vec{u}$, we can consider the limit

$$D_{\vec{u}}f(\vec{r}) := \lim_{h\to 0}\frac{f(\vec{r} + h\cdot\vec{u}) - f(\vec{r})}{h}.$$

This is called the directional derivative of f at (x, y) in the direction $\vec{u}$. (Caution: the vector $\vec{u}$ here must be a UNIT vector!)

In particular,

$$D_{\vec{i}}f = \frac{\partial f}{\partial x}, \quad D_{\vec{j}}f = \frac{\partial f}{\partial y}.$$

Example 18.1. Use definition to calculate the directional derivative for the function $f(x, y) = 3x+4y$ in the direction $\vec{u} =< \frac{3}{5}, \frac{4}{5} >$.

Solution: By definition,

$$\begin{aligned}D_{\vec{u}}f &= \lim_{h\to 0}\frac{f(\vec{r} + h\cdot\vec{u}) - f(\vec{r})}{h}\\ &= \lim_{h\to 0}\frac{f(x + \frac{3}{5}h, y + \frac{4}{5}h) - f(x, y)}{h}\\ &= \lim_{h\to 0}\frac{(3(x + \frac{3}{5}h) + 4(y + \frac{4}{5}h)) - (3x + 4y)}{h}\\ &= \lim_{h\to 0}\frac{\frac{9}{5}h + \frac{16}{5}h}{h}\\ &= \frac{9}{5} + \frac{16}{5} = 5.\end{aligned}$$

□

The method of using definition to calculate the directional derivative is ok for this linear function. However, you can imagine in general, the calculation for limit could be very complicated and we need to figure out a new way.

Let's first introduce a new symbol which is called the gradient vector of a function.

Definition 18.2. Assume $f(x, y)$ is function of two variables x and y, then its gradient vector is defined as

$$\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > .$$

Example 18.3. Calculate the gradient vector for the function

$$f(x, y) = x^3 y - 3x \cos y.$$

Solution: $\nabla f(x, y) =< 3x^2 y - 3 \cos y, x^3 + 3x \sin y >$. □

Proposition 18.4. *Assume $\vec{u}$ is a unit vector and f is differentiable. Then*

$$D_{\vec{u}} f(x, y) = \nabla f \cdot \vec{u}.$$

Proof. Assume $\vec{u} =< a, b >$ with $a^2 + b^2 = 1$. Because f is differentiable, we can write

$$f(x + ha, y + hb) - f(x, y) = \frac{\partial f}{\partial x}(ha) + \frac{\partial f}{\partial t}(hb) + h_1(x, y) \cdot (ha) + h_2(x, y) \cdot (hb),$$

where for any (x, y),

$$h_1(x, y) \to 0, h_2(x, y) \to 0, \qquad \text{as } h \to 0.$$

Hence the directional derivative can be calculated as

$$\begin{aligned} D_{\vec{u}} f(x, y) &= \lim_{h \to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(ha) + \frac{\partial f}{\partial t}(hb) + h_1(x, y) \cdot (ha) + h_2(x, y) \cdot (hb)}{h} \\ &= \lim_{h \to 0} (\frac{\partial f}{\partial x} a + \frac{\partial f}{\partial t} b + h_1(x, y) a + h_2(x, y) b) \\ &= \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial t} b \\ &= \nabla f \cdot \vec{u}. \end{aligned}$$

□

Example 18.5. Calculate the directional derivative of

$$f(x, y) = x^3 y - 3x \cos y$$

in the direction of the vector $\vec{v} =< 3, -4 >$.

Solution: Because vector $\vec{v}$ is NOT a unit vector, we need to first calculate its direction

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{< 3, -4 >}{| < 3, -4 > |} =< \frac{3}{5}, -\frac{4}{5} > .$$

Then

$$\begin{aligned} D_{\vec{u}}f &= \nabla f(x,y)\cdot\vec{u} \\ &= <3x^2y-3\cos y, x^3+3x\sin y>\cdot<\frac{3}{5},-\frac{4}{5}> \\ &= \frac{9}{5}x^2y-\frac{4}{5}x^3-\frac{9}{5}\cos y-\frac{12}{5}x\sin y. \end{aligned}$$

□

18.2. **Maximizing the directional derivative.** Because $D_{\vec{u}}f = \nabla f(x,y)\cdot\vec{u}$, it follows

$$D_{\vec{u}}f = \nabla f(x,y)\cdot\vec{u} = |\nabla f|\cos\theta,$$

where θ is the angle between the gradient vector and the direction vector $\vec{u}$. Hence it follows that

- If the direction vector $\vec{u}$ is the same as the direction of ∇f, i.e., $\theta = 0$, then $D_{\vec{u}}f$ can obtain its maxima which is $|\nabla f|$;
- If the direction vector $\vec{u}$ is opposite to the direction of ∇f, i.e., $\theta = \pi$, then $D_{\vec{u}}f$ can obtain its minima which is $-|\nabla f|$.

Example 18.6. In which direction can the directional derivative of the function

$$f(x,y) = x^3y - 3x\cos y$$

obtain its maxima and in which direction can it obtain its minima at the point $(1,0)$? Calculate out the maxima and minima.

Solution: Calculate

$$\nabla f(1,0) =<-3,1>,$$

whose unit vector is

$$\vec{u} = \frac{\nabla f(1,0)}{|\nabla f(1,0)|} =<-\frac{3}{\sqrt{10}},\frac{1}{\sqrt{10}}>.$$

Hence when the directional derivative $D_{\vec{u}}f(1,0)$ is the maxima whose value is $\sqrt{10}$ and the directional derivative $D_{-\vec{u}}f(1,0)$ is the maxima whose value is $-\sqrt{10}$. □

Lecture 19 stopped here.

18.3. **Geometric meaning of the gradient vector.**

18.3.1. 2-*dimensional case.* Consider the graph of function $f(x,y)$ which is the surface $z = f(x,y)$. Then the point (x_0,y_0,z_0) where $z_0 = f(x_0,y_0))$ is on the surface. The equation $z_0 = f(x,y)$ determines a level curve of this surface.

Proposition 18.7. *The tangent line of the level curve $f(x,y) = z_0$ at the point (x_0,y_0) is*

$$\nabla f(x_0,y_0)\cdot<x-x_0,y-y_0>=0.$$

The proof is left to you as an exercise.

Example 18.8. Write down the equation for the tangent line of the curve $x^2y+e^x\cos y = e$ at $(1,0)$.

Proof. First check the point $(1, 0)$ is on this curve. We can regard it as the level curve of the function

$$f(x, y) = x^2 y + e^x \cos y$$

at the level e. Calculate the gradient of this function at the point $(1, 0)$ as

$$\nabla f(1, 0) =< 2xy + e^x \cos y, x^2 - e^x \sin y > |_{(1,0)} =< e, 1 > .$$

Hence the equation for the tangent line is

$$< e, 1 > \cdot < x - 1, y - 0 >= 0,$$

which can be simplified to

$$e(x - 1) + y = 0.$$

□

18.3.2. 3-*dimensional case.* The 3-dimensional case can be regarded as an analogue. Consider the graph of function $f(x, y, z)$ which is the 3-dimensional solid $w = f(x, y, z)$. Then the point (x_0, y_0, z_0, w_0) where $w_0 = f(x_0, y_0, z_0))$ is on this solid. The equation $w_0 = f(x, y, z)$ determines a level surface of this solid.

Proposition 18.9. *The tangent plane of the level surface* $f(x, y, z) = w_0$ *at the point* (x_0, y_0, z_0) *is*

$$\nabla f(x_0, y_0, z_0) \cdot < x - x_0, y - y_0, z - z_0 >= 0.$$

The proof is also left to you as an exercise.

Example 18.10. Write down the equation for the tangent plane of the surface $x^2 + y^2 + z^2 = R^2$ at $(\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}, -\frac{R}{\sqrt{3}})$.

Proof. First check the point $(\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}, -\frac{R}{\sqrt{3}})$ is on the surface. We can regard it as the level surface of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

at the level R^2. Calculate the gradient of this function at the point $(\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}, -\frac{R}{\sqrt{3}})$ as

$$\nabla f(\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}, -\frac{R}{\sqrt{3}}) =< 2x, 2y, 2z > |_{(\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}, -\frac{R}{\sqrt{3}})} =< \frac{2R}{\sqrt{3}}, \frac{2R}{\sqrt{3}}, -\frac{2R}{\sqrt{3}}) > .$$

Hence the equation for the tangent line is

$$< \frac{2R}{\sqrt{3}}, \frac{2R}{\sqrt{3}}, -\frac{2R}{\sqrt{3}}) > \cdot < x - \frac{R}{\sqrt{3}}, y - \frac{R}{\sqrt{3}}, z + \frac{R}{\sqrt{3}} >= 0,$$

which can be simplified to

$$\frac{2R}{\sqrt{3}}(x - \frac{R}{\sqrt{3}}) + \frac{2R}{\sqrt{3}}(y - \frac{R}{\sqrt{3}}) - \frac{2R}{\sqrt{3}}(z + \frac{R}{\sqrt{3}}) = 0,$$

and further to

$$x + y - z - R\sqrt{3} = 0.$$

□

19. FUNCTIONS OF MULTIVARIABLES VII – MAXIMUM AND MINIMUM VALUES

19.1. **Absolute (global) maximum and minimum values.** Consider a function $f(x, y)$ of two variables first. Assume f is defined on the domain D.

Definition 19.1. For a point $(x_0, y_0) \in D$,

(1) if $f(x_0, y_0) \geq f(x, y)$ for any $(x, y) \in D$, then we call (x_0, y_0) an absolute (also call global) maximum point. The value $f(x_0, y_0)$ is the absolute maximum value;
(2) if $f(x_0, y_0) \leq f(x, y)$ for any $(x, y) \in D$, then we call (x_0, y_0) an absolute (also call global) minimum point. The value $f(x_0, y_0)$ is the absolute minimum value.

Maximum and minimal points are also called extremal points and their values are called extrema.

Example 19.2. Using the definition, we can directly check

(1) $(0, 0)$ is a (actually the unique) global maximum point of the function
$$f(x, y) = 2 - x^2 - y^2$$
over $\mathbb{R}^2$.
(2) $(0, 0)$ is a (actually the unique) global minimum point of the function
$$f(x, y) = x^2 + y^2$$
over $\mathbb{R}^2$.

The following theorem is an important fact and its proof is not easy.

Theorem 19.3. *If the domain D is closed and bounded, and the function f is continuous on D, then there must exist an absolute (global) maximum point and an absolute minimum point.*

Using this theorem, for example, any continuous function defined on the closed unique disk must have an absolute (global) maximum point and an absolute minimum point. (Caution: the condition that the domain D is closed and bounded is very important. $\mathbb{R}^2$ is not satisfied because it is not bounded. The open disk $x^2 + y^2 < 1$ is not satisfied because it is not closed.)

Lecture 21 stopped here.

19.2. **Local maximum and minimum values.**

Definition 19.4. For a point $(x_0, y_0) \in D$,

(1) if there exists a small region of (x_0, y_0) in D such that $f(x_0, y_0) \geq f(x, y)$ for any (x, y) in this small region, then we call (x_0, y_0) an local maximum point. The value $f(x_0, y_0)$ is the local maximum value;
(2) if there exists a small region of (x_0, y_0) in D such that $f(x_0, y_0) \leq f(x, y)$ for any (x, y) in this small region, then we call (x_0, y_0) an local minimum point. The value $f(x_0, y_0)$ is the local minimum value.

Because if a point (x_0, y_0) is a local extreme point, then it must also be a extreme point on for the graph of functions
$$z = f(x_0, y) \quad \text{and } z = f(x, y_0).$$

From the one-variable calculus class, we know an extreme point of a one-variable function must be a critical point. Hence we introduce a similar concept of critical points for function of two variables,

Definition 19.5. Over an open region region D, a point $(x_0, y_0) \in D$ is called a critical point of the function $f(x, y)$, if

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0, \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

(In another word, the gradient of f at this point vanishes: $\nabla f(x_0, y_0) = 0$.)

With this definition, we can conclude that

Proposition 19.6. *If $(x_0, y_0) \in D$ is a local extreme point of $f(x, y)$, then (x_0, y_0) is a critical point, i.e., $\nabla f(x_0, y_0) = 0$.*

Example 19.7. Find the critical points of the function $f(x, y) = 3x^3 - 4xy^2 + y$

Solution: Calculate

$$\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >=< 9x^2 - 4y^2, -4xy + 1 >= 0,$$

which means

$$9x^2 - 4y^2 = 0 \quad \text{and} \quad -4xy + 1 = 0.$$

The first equation indicates that

$$3x = \pm 2y.$$

If $3x = 2y$, plugging into the second equation, we can get two points

$$(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{4}) \quad (-\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{4}).$$

If $3x = -2y$, plugging into the second equation we get $4x^2 + 1 = 0$, and there is no solution for this case.

Above all, there are two critical points, which are $(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{4})$ and $(-\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{4})$

□

To detect more properties for a critical point, we now explain the following second derivative test. To describe the second derivatives, we define a matrix as follows and call it the Hessian matrix at the point (x_0, y_0).

Definition 19.8. Define the Hessian for f at (x_0, y_0) is the following 2×2 matrix

$$H(x_0, y_0) := \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) \\ \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{bmatrix}.$$

Theorem 19.9 (Second derivative test)**.** *Assume $f(x, y)$ is defined on an open domain D and assume $(x_0, y_0) \in D$ is a critical point. Then we have the following statements:*

- *If* $\det H(x_0, y_0) > 0$ *and*
  - $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$*, then* (x_0, y_0) *is a local minimal point;*
  - $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$*, then* (x_0, y_0) *is a local maximal point.*
- *If* $\det H(x_0, y_0) < 0$*, then* (x_0, y_0) *is a saddle point.*

Example 19.10. Using the second derivative test to analyze the types of the critical points of the following functions.

(1) $f(x, y) = x^2 + 3y^2 + 3$;
(2) $f(x, y) = 4 - x^2 - 2y^2$;
(3) $f(x, y) = 2xy$.

Solution: (1) $\nabla f =< 2x, 6y >= 0$ and so there is only one critical point $(0, 0)$. The Hessian at $(0, 0)$ is

$$H(0,0) := \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial y \partial x}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}.$$

The determinant is 12 which is positive and the first entry 2 is also positive. Hence $(0, 0)$ is a local minimal.

(2) $\nabla f =< -2x, -4y >= 0$ and so there is only one critical point $(0, 0)$. The Hessian at $(0, 0)$ is

$$H(0,0) := \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial y \partial x}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}.$$

The determinant is 8 which is positive and the first entry -2 is negative. Hence $(0, 0)$ is a local maximal.

(3) $\nabla f =< 2y, 2x >= 0$ and so there is only one critical point $(0, 0)$. The Hessian at $(0, 0)$ is

$$H(0,0) := \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial y \partial x}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

The determinant is -4 which is negative. Hence $(0, 0)$ is a saddle point.

□

19.3. **Detect global max/min over a closed domain.** To find global maximum/minimal points over a closed domain, we just need to compare values of all critical points and critical points over boundary.

Example 19.11. Find the global maximum and minimal points of the function

$$f(x, y) = x^2 - 2xy + 2y$$

on the rectangle $D = \{(x, y)|0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Solution: Refer book page 966 Example 7 for this one. □

19.4. **Lagrange Multipliers.** In general, the Lagrange multiplier is used to find extremal points of the function over some domain D with some constraints. Lecture 21 stopped here.

20. Integration I – Double integrals

20.1. **Definition.** Two ingredients for double integrals:

(1) Integration domain: a 2-dimensional region D in $\mathbb{R}^2$;
(2) Integration functions: a function of two variables, $f(x, y)$.

We enclose D into a fixed rectangle R in $\mathbb{R}^2$ and then divide R into $m \times n$ equal small rectangles $R_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then we consider a new function F on R with is the same as f on $D \subset R$ and 0 on $R - D$.

The double integral $\iint_D f(x,y)\,dxdy$ is defined as the limit of the sum of $f(x_i,y_j)A_{ij}$, where (x_i,y_j) is the middle point of each small rectangle R_{ij} and A_{ij} is the area of each small rectangle R_{ij}. To be concrete,

$$\iint_D f(x,y)\,dxdy := \lim_{m,n\to\infty} \Sigma_{1\leq i\leq m, 1\leq j\leq n} f(x_i,y_j)A_{ij}.$$

Theorem 20.1. *This limit exists if f is continuous and D has piecewise smooth boundary.*

In our class, we always assume f is continuous and D has piecewise smooth boundary.

Remark 20.2. It doesn't matter to the value of the limit that how we choose (x_i,y_j). For example, they can be chosen as left-down end point or anything else.

Direct properties from definition:

(1) The geometric meaning:

$$\iint_D 1\,dxdy = \text{Area}(D).$$

(2) Linear w.r.t integration functions:

$$\iint_D (c\cdot f(x,y) + d\cdot g(x,y))\,dxdy = c\iint_D f(x,y)\,dxdy + d\iint_D g(x,y)\,dxdy.$$

Here c and d are some constants.

(3) Splitting for integration domain: If $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$ or the intersection is a curve or some points, then

$$\iint_D f(x,y)\,dxdy = \iint_{D_1} f(x,y)\,dxdy + \iint_{D_2} f(x,y)\,dxdy.$$

20.2. **Calculate double integrals.** If the region is rectangle $[a,b]\times[c,d]$, we have the following Fubini's theorem:

Theorem 20.3 (Fubini's)**.**

$$\int_c^d \left(\int_a^b f(x,y)\,dx\right)dy = \iint_{[a,b]\times[c,d]} f(x,y)\,dxdy = \int_a^b \left(\int_c^d f(x,y)\,dy\right)dx.$$

Using Fubini's theorem for rectangle domains, one can prove the following two useful formulas for the calculation of double integrals:

Proposition 20.4. (1) *If*

$$D = \{a(y) \leq x \leq b(y), c \leq y \leq d\},$$

then

$$\iint_D f(x,y)\,dxdy = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\,dx\right)dy.$$

(2) *If*

$$D = \{a \le x \le b, c(x) \le y \le d(x)\},$$

then

$$\iint_D f(x,y)\,dxdy = \int_a^b (\int_{c(y)}^{d(y)} f(x,y)\,dy)\,dx.$$

This formulas sometimes can also help one to calculate some iterated integrations.

21. Integration II – Calculate double integrals in polar coordinates

Recall every point (x, y) in Cartesian coordinates can also be represented as (r, θ) in polar coordinates as

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

Conversely, given a polar coordinate point (r, θ), one can get a point (x, y) in Cartesian coordinates as

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Hence for a function $f(x, y)$ of two variables, one can write it as a new function

$$F(r,\theta) := f(r\cos\theta, r\sin\theta).$$

The integral domain D will also change to a new region in polar coordinates corresponding, and let's denote it by R. Sometimes, the region D could be very complicated but it turns to be very simple one R as using polar coordinates. This is the most situation that one considers polar coordinates.

After changing D to R and f to F, one also need to change $dxdy$ to $rdrd\theta$, and we get the formula:

$$\iint_D f(x,y)\,dxdy = \iint_R F(r,\theta) r\,drd\theta.$$

The reason that the r appears can be understood as follows: The small area for each small rectangle in polar coordinates is

$$\frac{1}{2}\Delta\theta(r+\Delta r)^2 - \frac{1}{2}\Delta\theta r^2 = r\Delta r\Delta\theta + \frac{1}{2}\Delta\theta(\Delta r)^2.$$

The first term $r\Delta r\Delta\theta$ will appear as $r\,drd\theta$ in the double integral, and the second term will vanish in the limit process.

22. Integration III – Triple integrals

22.1. **Definition.** We can do the very similar things as we did for double integrals: For triple integrals, there are two ingredients:

(1) Integration domain: a 3-dimensional region V in $\mathbb{R}^3$;
(2) Integration functions: a function of three variables, $f(x, y, z)$.

We enclose V into a fixed rectangular box B in $\mathbb{R}^3$ and then divide B into $m \times n \times l$ equal small rectangular boxes $B_{i,j,k}$ for $1 \le i \le m$, $1 \le j \le n$ and $1 \le k \le l$. Then we consider a new function F on B with is the same as f on $V \subset B$ and 0 on $B - V$.

The triple integral $\iiint_V f(x,y,z)\,dxdydz$ is defined as the limit of the sum of $f(x_i,y_j,z_k)v_{ijk}$, where (x_i,y_j,z_k) is the middle point of each small rectangular box B_{ijk} and v_{ijk} is the volume of each small rectangular box B_{ijk}. To be concrete,

$$\iiint_V f(x,y,z)\,dxdydz := \lim_{m,n,l\to\infty} \Sigma_{1\le i\le m, 1\le j\le n, 1\le k\le l} f(x_i,y_j,z_k)v_{ijk}.$$

Theorem 22.1. *This limit exists if f is continuous and V has piecewise smooth boundary.*

In our class, we always assume f is continuous and V has piecewise smooth boundary.

Remark 22.2. It doesn't matter to the value of the limit that how we choose (x_i,y_j,z_k).

Direct properties from definition:

(1) The geometric meaning:

$$\iint_V 1\,dxdydz = \text{Volume}(V).$$

(2) Linear w.r.t integration functions:

$$\iiint_V (c\cdot f(x,y,z)+d\cdot g(x,y,z))\,dxdydz = c\iiint_V f(x,y,z)\,dxdydz + d\iiint_V g(x,y,z)\,dxdydz.$$

Here c and d are some constants.

(3) Splitting for integration domain: If $V = V_1\cup V_2$ and $V_1\cap V_2=\emptyset$ or the intersection is of dimension less than 3, then

$$\iiint_V f(x,y,z)\,dxdydz = \iiint_{V_1} f(x,y,z)\,dxdydz + \iiint_{V_2} f(x,y,z)\,dxdydz.$$

22.2. **Calculate triple integrals.** We name two ways of calculating triple integrals as "Bread Pieces" and "Potato Chips".

- "Bread Pieces". Let's take pieces over z as example. The other two cases, over x and y, are very similar and you should be able to figure out by yourself.

  For this case, the integration domain V can be written as

  $$V = \{(x,y,z)\in\mathbb{R}^3 | a\le z\le b, (x,y)\in D(z)\}.$$

  Here $D(z)$ denotes for some region in $\mathbb{R}^2$ which varies as z varies.

  Then we can regard every $D(z)$ as a bread piece, and to do the triple integral over V can be realized by summing all pieces for z goes from a to b. To be concrete,

  Proposition 22.3. *For any function f defined on V,*

  $$\iiint_V f(x,y,z)\,dxdydz = \int_a^b (\iint_{D(z)} f(x,y,z)\,dxdy)\,dz.$$

- "Potato Chips". Still, let's take chips over xy-plane as example. The other two cases, over xz and yz-planes, are very similar and you should be able to figure out by yourself.

  For this case, the integration domain V can be written as

  $$V = \{(x,y,z)\in\mathbb{R}^3 | a(x,y)\le z\le b(x,y), (x,y)\in D\}.$$

  Here D is a region in xy-plane and $a(x,y)\le b(x,y)$ are two functions over D.

Then we can regard every $a(x,y) \le z \le b(x,y)$ as a potato chip, and to do the triple integral over V can be realized by summing all chips for $(x,y) \in D$. To be concrete,

Proposition 22.4. *For any function f defined on V,*

$$\iiint_V f(x,y,z)\,dxdydz = \iint_D (\int_{a(x,y)}^{b(x,y)} f(x,y,z)\,dz)\,dxdy.$$

Example 22.5. Assume V is the region in $\mathbb{R}^3$ which is enclosed by the surface $z = 2 - x^2 - y^2$ and xy-plane. Use two ways, the "Bread Pieces" and the "Potato Chips", to express the triple integral

$$\iiint_V f(x,y,z)\,dxdydz,$$

where f is a smooth function defined on V. In particular, for $f(x,y,z) = x^2$, calculate out this triple integral in two ways.

Solution: First step is to sketch the graph of the region V in the Cartesian coordinates for $\mathbb{R}^3$. Then we calculate the triple integral in two ways:

- "Bread Pieces". We can consider V as stacks of bread pieces along z from 0 to 2 (why?). Let's denote each piece by $D(z)$ which is the region bounded by the intersection of the surface $z = 2 - x^2 - y^2$ and the z plane. To be precise,

$$D(z) = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \le 2 - z\}.$$

Hence the triple integral can be expressed as

$$\begin{aligned}\iiint_V f(x,y,z)\,dxdydz &= \int_0^2 (\iint_{D(z)} f(x,y,z)\,dxdy)\,dz \\ &= \int_0^2 (\iint_{x^2+y^2\le 2-z} f(x,y,z)\,dxdy)\,dz\end{aligned}$$

You could use polar coordinate to calculate the double integrals (for each z) $\iint_{x^2+y^2\le 2-z} f(x,y,z)\,dxdy$ as follows:

$$\begin{aligned}&\iint_{x^2+y^2\le 2-z} f(x,y,z)\,dxdy \\ =& \iint_{0\le r\le\sqrt{2-z},0\le\theta\le 2\pi} f(r\cos\theta, r\sin\theta, z)r\,drd\theta \\ =& \int_0^{2\pi} (\int_0^{\sqrt{2-z}} f(r\cos\theta, r\sin\theta, z)r\,dr)\,d\theta.\end{aligned}$$

For example, if $f(x,y,z)=x^2$,

$$\begin{aligned}
&\iint_{x^2+y^2\le 2-z} x^2\,dxdy\\
=&\iint_{0\le r\le\sqrt{2-z},0\le\theta\le 2\pi}(r\cos\theta)^2 r\,drd\theta\\
=&\int_0^{2\pi}(\int_0^{\sqrt{2-z}}(r\cos\theta)^2 r\,dr)\,d\theta\\
=&\int_0^{2\pi}(\int_0^{\sqrt{2-z}} r^3\cos^2\theta\,dr)\,d\theta\\
=&\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^{\sqrt{2-z}} r^3\,dr\\
=&\frac{\pi}{4}(2-z)^2.
\end{aligned}$$

Then the triple integral for this case $f(x,y,z)=x^2$ is

$$\iiint_V x^2\,dxdydz=\int_0^2\frac{\pi}{4}(2-z)^2\,dz=\frac{2}{3}\pi.$$

- "Potato Chips". We can regard V as bunches of potato chips over the disk

$$D=\{(x,y)\in\mathbb{R}^2|x^2+y^2\le 2\},$$

and each chip is from $z\in[0,2-x^2-y^2]$. (Why?)

We can then express the triple integral as

$$\begin{aligned}
\iiint_V f(x,y,z)\,dxdydz &= \iint_D(\int_0^{2-x^2-y^2} f(x,y,z)\,dz)\,dxdy\\
&= \iint_{x^2+y^2\le 2}(\int_0^{2-x^2-y^2} f(x,y,z)\,dz)\,dxdy.
\end{aligned}$$

You could also express the double integral in polar coordinates, which is

$$\begin{aligned}
&\iint_{x^2+y^2\le 2}(\int_0^{2-x^2-y^2} f(x,y,z)\,dz)\,dxdy\\
=&\iint_{0\le r\le\sqrt{2},0\le\theta\le 2\pi}(\int_0^{2-r^2} f(r\cos\theta,r\sin\theta,z)\,dz)r\,drd\theta.
\end{aligned}$$

For example, if $f(x,y,z)=x^2$, we can calculate the triple integral as

$$\begin{aligned}
\iiint_V x^2\,dxdydz &= \iint_D(\int_0^{2-x^2-y^2} x^2\,dz)\,dxdy\\
&= \iint_{0\le r\le\sqrt{2},0\le\theta\le 2\pi}(\int_0^{2-r^2}(r\cos^2\theta)^2\,dz)r\,drd\theta\\
&= \iint_{0\le r\le\sqrt{2},0\le\theta\le 2\pi}(2-r^2)(r\cos^2\theta)^2 r\,drd\theta\\
&= \int_0^{\sqrt{2}}(2-r^2)r^3\,dr\int_0^{2\pi}\cos^2\theta\,d\theta\\
&= \frac{2}{3}\pi.
\end{aligned}$$

□