CHAPTER 1

Preliminaries to Complex Analysis

1. Complex numbers and the complex plane

Set of complex numbers is the same as \( \mathbb{R}^2 \) and is denoted by
\[
\mathbb{C} = \{ z = x + iy \mid x, y \in \mathbb{R} \}.
\]
In \( z = x + it \), the \( x \) is called the real part and the \( y \) is called the imaginary part of \( z \), and write as
\[
x = \Re(z), \quad y = \Im(z).
\]

One can extend addition + and multiplication \( \cdot \) over \( \mathbb{R} \) to \( \mathbb{C} \) (as real part) via
\[
z + w = (\Re(z) + \Re(w)) + i(\Im(z) + \Im(w)),
\]
and
\[
z \cdot w = (\Re(z)\Re(w) - \Im(z)\Im(w)) + i(\Re(z)\Im(w) + \Im(z)\Re(w)).
\]

In particular, \( i^2 = i \cdot i = -1 \). It is easy to check \((\mathbb{C}, +, \cdot)\) is a field as a field extension of \( \mathbb{R} \).

It is much easier to express the multiplication using polar coordinates. For each \( r \geq 0 \), \( \theta \in \mathbb{R} \),
\[
 z = re^{i\theta}.
\]

Conversely, every nonzero complex number \( z \in \mathbb{C}^* \), there is a unique \( r > 0 \) and \( [\theta] \in \mathbb{R}/2\pi \) so that
\( z = re^{i\theta} \). Such \((r, \theta)\) is called a polar coordinate of \( z \).
Assume \( z_k = r_k e^{i\theta_k} \), \( k = 1, 2 \). Then
\[
z_1 \cdot z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}.
\]

The geometric meaning of \( r \) is the norm of \( z \), i.e., the distance between the origin and \( z \), as
\[
|z| := \sqrt{\Re(z)^2 + \Im(z)^2}.
\]

The geometric meaning of \( \theta \) is the argument of \( z \), i.e., the angle between the real axis and the radical line \( \overline{oz} \), as
\[
\tan \theta = \frac{y}{x}.
\]

Another operation over \( \mathbb{C} \) is the conjugation, which is the reflection about the real axis, i.e.,
\[
\bar{z} = \Re(z) - i\Im(z).
\]

It is easy to check
\[
|z|^2 = z\bar{z} = \bar{z}z, \quad \frac{1}{z} = \frac{\bar{z}}{|z|^2},
\]
and
\[
\Re(z) = \frac{z + \bar{z}}{2}, \quad \Im(z) = \frac{z - \bar{z}}{2i}.
\]

The norm \( |\cdot| \) introduces a natural distance
\[
d(z, w) = |z - w|.
\]
over \( \mathbb{C} \). As a metric space, \((\mathbb{C}, d)\) behaves exactly the same as \(\mathbb{R}^2\). We now review some basic properties for this metric space.

### 1.1. Sequences and convergence in \( \mathbb{C} \)

A sequence \( \{z_n\} \) is called convergent in \( \mathbb{C} \), if there is some \( w \in \mathbb{C} \) so that

\[
\lim_{n \to \infty} |z_n - w| = 0.
\]

Such \( w \), if exists, must be unique, and we denote such convergence as \( \lim_{n \to \infty} z_n = w \), or

\[
z_n \to w, \quad \text{as} \quad n \to \infty.
\]

**Lemma 1.1.** \( \lim_{n \to \infty} z_n = w \) if and only if

\[
\lim_{n \to \infty} \text{Re}(z_n) = \text{Re}(w), \quad \lim_{n \to \infty} \text{Im}(z_n) = \text{Im}(w).
\]

**Proof.** Details are left to you. \( \square \)

A sequence \( \{z_n\} \) is called a Cauchy sequence, if for any \( \varepsilon > 0 \), there exists some \( N > 0 \) so that any \( m, n > N \), \(|z_n - z_m| < \varepsilon\). Clearly, every convergent sequence is a Cauchy sequence. The converse is also true for \( \mathbb{C} \). In another word, we have

**Theorem 1.2.** \( \mathbb{C} \) is complete.

**Proof.** \( \{z_n\} \) is a Cauchy sequence if and only if \( \{\text{Re}(z_n)\}, \{\text{Im}(z_n)\} \) are both Cauchy sequences. Then the completeness of \( \mathbb{C} \) follows from completeness of \( \mathbb{R} \). \( \square \)

### 1.2. Open sets in \( \mathbb{C} \)

Denote by

\[
D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}, \quad r > 0
\]

the open disk of radius \( r \) centered at \( z_0 \). (All open disks form a topology base of \( \mathbb{C} \).) A closed disk is defined as

\[
\overline{D}_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| \leq r\}, \quad r > 0.
\]

We use \( \mathbb{D} \) to denote the unit (open) disk center at the origin.

For a subset \( \Omega \subset \mathbb{C} \), a point \( z \in \Omega \) is called an interior point, if there is some \( D_r(z) \subset \Omega \). We use \( \Omega^o \) to denote the set of interior points of \( \Omega \) and \( \Omega^c \) the interior of \( \Omega \). The set \( \Omega \) is open if and only if \( \Omega = \Omega^o \). A set \( \Omega \) is closed if \( \Omega^c \) is open. A set is closed if and only if it contains all limit points. For any set \( \Omega \), its closure \( \overline{\Omega} \) is defined as the union of itself with its limit points. It is closed. The boundary \( \partial \Omega \) is defined as

\[
\partial \Omega := \overline{\Omega} \setminus \Omega^o.
\]

The closed disk \( \overline{D}_r(z_0) \) is the closure of the open disk \( D_r(z_0) \), and

\[
\partial \overline{D}_r(z_0) = \partial D_r(z_0) = C_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| = r_0\}
\]

the circle of radius \( r \) centered at \( z_0 \).

**Lecture 1 stopped here.**

A subset \( \Omega \) in \( \mathbb{C} \) is called bounded, if there exists some \( M > 0 \) so that \( |z| \leq M \) for any \( z \in \Omega \). For a bounded set \( \Omega \), define its diameter as

\[
\text{diam}(\Omega) := \sup_{z,w \in \Omega} |z - w|,
\]

which is a finite number. Then the following statements are equivalent for a subset \( \Omega \) of \( \mathbb{C} \):
• Ω is compact;
• Ω is both closed and bounded;
• Ω is sequentially compact, i.e., every sequence in Ω has a convergent subsequence.

The following proposition is useful in proving the Goursat’s theorem later.

**Proposition 1.3.** Assume \( \Omega_1 \supset \Omega_2 \supset \ldots \) is a sequence of nested subsets of \( \mathbb{C} \), with each one nonempty, compact and

\[
\lim_{n \to \infty} \text{diam}(\Omega_n) = 0.
\]

Then their intersection contains a unique point \( z_0 \in \mathbb{C} \).

**Proof.** We first prove that \( \cap_n \Omega_n \) is not empty, i.e., it contains some point \( z_0 \in \mathbb{C} \). For this, take \( z_n \in \Omega_n \) for each \( n = 1, 2, \ldots \) and we obtain a Cauchy sequence \( \{ z_n \} \) in \( \mathbb{C} \) since \( \lim_{n \to \infty} \text{diam}(\Omega_n) = 0 \).

By the completeness of \( \mathbb{C} \), the sequence \( \{ z_n \} \) converges to some point \( z_0 \in \mathbb{C} \). The limit \( z_0 \) lives in each \( \Omega_n \) since each \( \Omega_n \) is (sequentially) compact. This proves \( z_0 \in \cap_n \Omega_n \).

Next we show \( z_0 \) is the only point in \( \cap_n \Omega_n \). Assume there is another point \( z' \in \cap_n \Omega_n \). Then their distance

\[
|z_0 - z'| \leq \text{diam}(\Omega_n), \quad n = 1, 2, \ldots
\]

take \( n \to \infty \), there must be \( |z_0 - z'| = 0 \) and then \( z' = z_0 \).

An open subset \( \Omega \) of \( \mathbb{C} \) is called connected, if there is no way to write \( \Omega \) as a union of two disjoint nonempty open sets in \( \mathbb{C} \). This is equivalent to say \( \Omega \) is path-connected, i.e., for any two points \( z_0, z_1 \in \Omega \), there is a continuous path in \( \Omega \) which connects \( z_0 \) and \( z_1 \).

We call \( \Omega \) a region in \( \mathbb{C} \) if it is both open and connected.

### 2. Functions on \( \mathbb{C} \)

#### 2.1. Continuous functions.

Assume \( \Omega \) is a subset of \( \mathbb{C} \) and \( z_0 \in \Omega \). A function \( f : \Omega \to \mathbb{C} \) is called continuous at \( z_0 \), if for any \( \varepsilon > 0 \), there exists some \( \delta > 0 \) so that

\[
|f(z) - f(z_0)| < \varepsilon, \quad \text{for any } |z - z_0| < \delta, z \in \Omega.
\]

The following equivalence is left as a homework problem: \( f \) is continuous at \( z_0 \) if and only if

\[
f(z_n) \to f(z_0) \text{ for any sequence } z_n \to z_0 \text{ as } n \to \infty.
\]

Clearly, \( f \) is continuous at \( z_0 \) if and only if both \( \text{Re}(f) \) and \( \text{Im}(f) \) are continuous at \( z_0 \).

We write \( f \in C^0(\Omega) \) if \( f \) is continuous on every \( z_0 \in \Omega \).

The definition of continuity and be generalized to functions defined over any metric spaces, for example, over \( \mathbb{C} \times \mathbb{C} \). Then it is easy to check addition, subtraction, multiplication, division, norm are all continuous functions on the corresponding natural domains.

The following property for continuous function defined over a compact domain is very useful.

**Theorem 2.1.** If \( \Omega \) is a compact subset in \( \mathbb{C} \) and \( f \in C^0(\Omega) \), then \( f \) is bounded and can obtain its maximum and minimum on \( \Omega \).
2.2. Holomorphic functions. Now there comes the key concept in complex analysis.

**Definition 2.2.** Assume $\Omega$ is an open subset in $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is called holomorphic at $z_0 \in \Omega$, if the function

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

is convergent as $h \to 0$ (with $z_0 + h \in \Omega$). The limit is called the derivative of $f$ at $z_0$ and is denoted by $f'(z_0)$.

If $f'(z_0)$ exists for every $z_0 \in \Omega$, then we say the function $f$ is a holomorphic function over $\Omega$. A holomorphic function $f$ defines a derivative function $f' : \Omega \to \mathbb{C}$. We are going to prove it is also holomorphic over $\Omega$ in later sections. A holomorphic function over $\mathbb{C}$ is called an entire function.

**Example 2.3.**

1. $f(z) = z$ is an entire function with $f'(z) = 1$.
2. Any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_0, a_1, \ldots, a_n \in \mathbb{C}, a_n \neq 0$ is an entire function with
   $$p'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}.$$
3. $f(z) = \frac{1}{z}$ is holomorphic over $\mathbb{C}^*$ with
   $$f'(z) = -\frac{1}{z^2}.$$
4. $f(z) = \bar{z}$ is NOT a holomorphic function.

   Consider $\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{\bar{h}}{h}$. It has limit 1 if $h$ converges to 0 along the real axis and has limit $-1$ if $h$ converges to 0 along the imaginary axis. Hence $\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$ has no limit as $h \to 0$.

Assume $f$ is holomorphic at $z_0 \in \Omega$, then we can write

$$f(z_0 + h) - f(z_0) = hf'(z_0) + \psi(h),$$

with $\psi(h) \to 0$ as $h \to 0$. Then it follows that $f$ must be continuous at $z_0$.

Similar to real functions, derivatives of complex functions has the following properties.

**Proposition 2.4.**

1. Assume $f, g$ are holomorphic functions over $\Omega$. Then $f \pm g$, $f \cdot g$, $f / g$
   at $g(z) \neq 0$ are all holomorphic with
   (a) $(f + g)' = f' + g'$;
   (b) $(f \cdot g)' = f'g + fg'$;
   (c) $(f / g)' = \frac{f'g - fg'}{g^2}$.
2. Assume $f$ is holomorphic over $\Omega$ and $g$ is holomorphic over $U$ with $U \supset f(\Omega)$. Then $g \circ f$ is
   holomorphic over $\Omega$ and
   $$\left(g \circ f\right)'(z) = g'(f(z)) \cdot f'(z), \quad z \in \Omega.$$

Lecture 2 stopped here.
2.3. Complex-valued functions as mappings. Any complex valued function

\[ f : \Omega \to \mathbb{C} \]

is equivalent to a pair of real valued functions

\[ u : \Omega \to \mathbb{R}, \quad v : \Omega \to \mathbb{R} \]

with

\[ u = \text{Re}(f), \quad v = \text{Im}(f), \quad \text{i.e., } f = u + iv. \]

Equivalently, we can regard \( u, v \) as real-valued functions defined over the open subset \( \Omega \) in \( \mathbb{R}^2 \), and consider the mapping

\[ F = (u, v) : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2. \]

We can ask what properties \( F \) has if \( f \) is a holomorphic function.

**Theorem 2.5.** Assume \( \Omega \) is an open subset of \( \mathbb{C} \). The function \( f \) is holomorphic over \( \Omega \) if and only if \( F \) is differentiable over \( \Omega \) and the partial derivatives

\[
\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}
\]

satisfy the Cauchy–Riemann equations (C–R equations)

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]

Recall from real analysis (e.g., Rudin Definition 9.11) that, the function \( F \) is differentiable over \( \Omega \subset \mathbb{R}^2 \) if for every \((x_0, y_0) \in \Omega\), there is a \( 2 \times 2 \) matrix \( J_F \) so that

\[
\lim_{h= (h_1, h_2) \to (0,0)} \frac{|F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0) - J \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}|}{|h|} = 0.
\]

It is not hard to check that the matrix \( J_F \) must be of the form

\[
J_F := \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}
\]

if it exists. (Why?) It is called the Jacobian matrix of the map \( F \).

The Jacobian matrix \( J_F : \mathbb{R}^2 \to \mathbb{R}^2 \) is the linearization of the map \( F \) (assuming \( F \) is differentiable) and the Cauchy–Riemann equations force the determinant of the Jacobian matrix is nonnegative for holomorphic function since

\[
\text{det } J_F = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \geq 0.
\]

**Example 2.6.** We know the function \( f(z) = \bar{z} \) is not holomorphic, and its Jacobian matrix is

\[
J_F := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]

with determinant \(-1\).

Now let’s prove Theorem 2.5.
PROOF. (1) "⇒". Take any \((x_0, y_0) \in \Omega\) and denote by \(z_0 = x_0 + iy_0\). Since \(f\) is holomorphic at \(z_0\), the limit
\[
\frac{f(z_0 + h) - f(z_0)}{h} = \frac{(u(z_0 + h) - u(z_0)) + i(v(z_0 + h) - v(z_0))}{h}
\]
eexists as \(h \to 0\). In particular, take \(h \in \mathbb{R}\), and look at the real part only, we obtain that
\[
\lim_{h \to 0, h \in \mathbb{R}} \frac{u(z_0 + h) - u(z_0)}{h}
\]
eexists and this shows \(\frac{du}{dx}(x_0, y_0)\) exists. Similarly, the other three partial derivatives also exist.

At the same time, above calculation also shows that
\[
f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{1}{i} \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right).
\]
Hence it follows
\[
\text{Re}(f') = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{Im}(f') = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},
\]
which are the Cauchy–Riemann equations.

To see \(F\) is differentiable at \(z_0\), we rewrite
\[
|F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0) - J(x_0, y_0) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}| \leq |h|
\]
\[
= \frac{|f(z_0 + h) - f(z_0) - f'(z_0)h|}{|h|}
\]
using the Cauchy–Riemann equations, which converges to zero as \(h \to 0\) and we are done.

"⇐". This follows from (2.1) under the assumption of differentiability.

\(\square\)

We remark that the assumption \(\Omega\) is open in \(\mathbb{C}\) is crucial. (See HW1 problem 12 from the book.)

EXAMPLE 2.7. Assume \(f = u + iv\) is holomorphic over \(\Omega\) (with \(u, v\) smooth). Then \(u, v\) must be harmonic function, i.e., \(\Delta u = 0 = \Delta v\). This follows from the C–R equations and the equality of mixed partial derivatives:
\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial}{\partial y} \frac{\partial v}{\partial x} = 0.
\]

We can introduce some useful notations. Define
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]
Then the C–R equations are equivalent to
\[
\frac{\partial f}{\partial \bar{z}} = 0,
\]
and when \(f\) is holomorphic,
\[
f' = \frac{\partial f}{\partial z}.
\]

Lecture 3 stopped here.
3. Power series

We have seen that any polynomial
\[ p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_0, a_1, a_2, \ldots, a_n \in \mathbb{C}, a_n \neq 0 \]
is an entire function, i.e., a holomorphic function over \( \mathbb{C} \).

A power series can be considered as a generalization of a polynomial by allowing infinitely many nonzero terms, i.e.,
\[ \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, n = 0, 1, 2, \ldots. \]

Then we have to answer the question that when a power series is convergent, and further, if it is holomorphic over its convergent domain.

**Example 3.1.**

(1) Consider the power series
\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots. \]
The same as the real variable case, this series is convergent everywhere and this defines the exponential function
\[ e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}. \]
We will see later it is holomorphic over \( \mathbb{C} \).

(2) Consider the following power series which is called the geometric series
\[ \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots. \]
It can be explicitly calculated out as follows: Look at partial sums
\[ S_N(z) = \sum_{n=0}^{N} z^n = 1 + z + z^2 + z^3 + \cdots + z^N. \]
By multiplying \( z \), we get
\[ zS_N(z) = z \sum_{n=0}^{N} z^n = z + z^2 + z^3 + \cdots + z^{N+1}, \]
and so
\[ zS_N(z) - S_N(z) = z^{N+1} - 1, \quad S_N = \frac{1 - z^{N+1}}{1 - z} \]
whenever \( z \neq 1 \). (Obviously, the series is divergent at \( z = 1 \).)

From the explicit expression of the partial sum, it becomes apparent that this series is convergent (and also holomorphic) if \( |z| < 1 \) and it is divergent if \( |z| > 1 \).

Now we give a complete answer for the convergence question which works for any power series.
First, define
\[ R := \limsup_{n \to \infty} \sqrt[n]{|a_n|}. \]
It is a real number in \( [0, +\infty] \) and is called the convergence radius of the power series \( \sum_{n=0}^{\infty} a_n z^n \).

**Example 3.2.**

(1) The exponential function \( \sum_{n=0}^{\infty} \frac{z^n}{n!} \) has convergent radius \( R = +\infty \).

(2) The geometric series \( \sum_{n=0}^{\infty} z^n \) has convergent radius \( R = 1 \).

The open disk \( D_R \) is called the convergent disk and the reason is the following theorem.

**Theorem 3.3.** The power series \( \sum_{n=0}^{\infty} a_n z^n \)

(1) is absolutely convergent if \( |z| < R \);
(2) It is divergent if $|z| > R$.

**Proof.** $R = 0$ or $+\infty$ cases are left to you and we assume here $0 < R < +\infty$.

(1) Take any $z_0$ with $|z_0| < R$, we prove now the series is convergent at $z_0$.

First, $|z_0| < R$ implies that $|z_0| \limsup_{n \to \infty} \sqrt[n]{|a_n|} < 1$. We insert some $r$ so that

$$|z_0| \limsup_{n \to \infty} \sqrt[n]{|a_n|} < r < 1,$$

and further some small $\varepsilon > 0$ so that

$$|z_0| \limsup_{n \to \infty} \sqrt[n]{|a_n|} + \varepsilon < r < 1,$$

Then there exists some $N$ (may depend on $\varepsilon$) so that every $n > N$,

$$\sqrt[n]{|a_n|} < \limsup_{n \to \infty} \sqrt[n]{|a_n|} + \varepsilon.$$

It follows

$$|z_0| \sqrt[n]{|a_n|} < |z_0| \limsup_{n \to \infty} \sqrt[n]{|a_n|} + \varepsilon < |z_0| \sqrt[n]{|a_n|} + \varepsilon < r < 1$$

and

$$|a_n| |z_0|^n < r^n.$$

The series $\sum n^r$ is convergent since $0 < r < 1$. Then the comparison theorem implies that $\sum a_n z_0^n$ must be absolutely convergent.

(2) This is assigned as a homework problem.

Over the boundary $|z| = R$, the series can be either convergent or divergent.

**Example 3.4.** (Homework problem). Consider the following there series with convergent radius $R = 1$

1. $\sum n z^n$. It is divergent on $|z| = 1$.
2. $\sum \frac{n^r}{n!}$. It is convergent on $|z| = 1$.
3. $\sum \frac{z^n}{n}$. It is convergent on $|z| = 1$ except at $z = 1$.

Next, we answer the question about if the power series is holomorphic.

Assume the power series $\sum_{n=0}^{\infty} a_n z^n$ has convergent radius $R$, and denote by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D_R.$$ 

By differentiate it term by term, we obtain another power series $\sum_{n=1}^{\infty} a_n n z^{n-1}$. It is easy to check this power series also has $R$ as its convergent radius, hence defines a function

$$g(z) = \sum_{n=1}^{\infty} a_n n z^{n-1}, \quad z \in D_R.$$ 

We have the following theorem.

**Theorem 3.5.** The function $f$ is holomorphic over $D_R$ with $f' = g$.

**Lecture 4 stopped here.**
PROOF. We assume $0 < R < \infty$. The cases $R = 0$ and $R = +\infty$ are left you to handle.

Denote by
\[ S_N(z) := \sum_{n=0}^{N} a_n z^n, \quad E_N(z) := \sum_{n=N+1}^{\infty} a_n z^n; \]
and
\[ \tilde{S}_N(z) := \sum_{n=1}^{N} a_n nz^{n-1}, \quad \tilde{E}_N(z) := \sum_{n=N+1}^{\infty} a_n nz^{n-1}. \]

Obviously from definition,
\[ f = S_N + E_N, \quad g = \tilde{S}_N + \tilde{E}_N, \quad S'_N = \tilde{S}_N. \]

We show now for any $z_0 \in D_R$, $f'(z_0)$ exists and it is $g(z_0)$.

For this, we first pick some fix some $0 < r < R$ so that $z_0, z_0 + h \in D_r$, and write
\[
\frac{f(z_0 + h) - f(z_0)}{h} = \frac{S_N(z_0 + h) - S_N(z_0)}{h} - \tilde{S}_N
\]
\[
+ \frac{E_N(z_0 + h) - E_N(z_0)}{h}
\]
\[
+ (-\tilde{E}_N).
\]

For each fixed $N$, the term $|(3.1)|$ is small as $h$ is close to 0; the term $|(3.3)|$ is small for large $N$. We focus on estimating $|(3.2)|$.

For it, we look at an estimate of the general term as
\[
\left| \frac{a_n(z_0 + h)^n - a_n z_0^n}{h} \right| = \left| \frac{a_n(z_0 + h - z_0)((z_0 + h)^n - z_0^n + (z_0 + h - z_0)z_0^{n-1} + \cdots + (z_0 + h - z_0)z_0^{n-2} + z_0^{n-1})}{h} \right|
\]
\[
\leq |a_n|nr^{n-1}.
\]

Then since the series $\sum |a_n|nr^{n-1}$ is convergent, the term $|(3.2)|$ converges to zero as $N \to \infty$.

Now we wrap up the proof using $\epsilon - \delta$ arguments: For any $\epsilon > 0$, there we can pick $N$ large enough so that
\[ |(3.2)| < \frac{\epsilon}{3}, \quad |(3.3)| < \frac{\epsilon}{3}, \]
and then for such $N$, take $\delta > 0$ small so that
\[ |(3.1)| < \frac{\epsilon}{3} \quad \text{for any } 0 < |h| < \delta. \]

By this way, for any $0 < |h| < \delta$, $\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| \leq \epsilon$, and this shows
\[ \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = g(z_0). \]

An immediate corollary from this theorem as

**Corollary 3.6.** Assume the power series $\sum_{n=0}^{\infty} a_n z^n$ has $R$ as its convergence radius. Then it is infinitely order complex differentiable in the convergence disk $D_R$, and the complex derivatives can be calculated term by term.
The power series can be generalized to ones with any \( z_0 \in \mathbb{C} \) as center. To be more precise, a power series with \( z_0 \in \mathbb{C} \) as center is defined as
\[
\sum_{n=0}^{\infty} a_n (z - z_0)^n.
\]
Then the above results can be stated for this case by shifting the convergence disk from \( D_R \) to \( D_R(z_0) \).

We end this section by introducing the following definition.

**Definition 3.7.** Assume \( \Omega \) is an open subset in \( \mathbb{C} \). A function \( f : \Omega \to \mathbb{C} \) is called analytic at \( z_0 \in \Omega \), if there exists an open disk \( D_R(z_0) \subseteq \Omega \) so that \( f \) can be written as a power series centered at \( z_0 \) over \( D_R(z_0) \). If \( f \) is analytic at every point \( z_0 \in \Omega \), we call \( f \) is analytic over \( \Omega \).

We just proved that if \( f \) is analytic over \( \Omega \), then \( f \) must be holomorphic over \( \Omega \). In fact, the reverse statement is also true which will be proved next chapter.

**4. Integration along a curve**

By a parametrized curve, we mean a map
\[
z : [a, b] \to \mathbb{C}.
\]
We need the following terminology for parametrized curves:

A parametrized curve \( z : [a, b] \to \mathbb{C} \)
- is called smooth, if
  1. derivative \( z' \) exists and continuous on \([a, b]\), i.e., \( z \in C^1([a, b])\);
  2. \( z'(t) \) is nowhere vanishing;
- is called piecewise smooth, if there exists a splitting
  \[
a = a_0 < a_1 < a_2 < \cdots < a_n = b
\]
  so that \( z_{[a_i, a_{i+1}]} : [a_i, a_{i+1}] \to \mathbb{C} \) is smooth, \( i = 0, 1, \cdots, n - 1 \);
- is called simple, if it is injective over \((a, b)\);
- is called a loop, if \( z(a) = a(b) \).

Two parametrized smooth curves
\[
z_k : [a_k, b_k] \to \mathbb{C}, \quad k = 1, 2,
\]
are called equivalent, if there exists a \( C^1 \) bijective map \( \phi : [a_1, b_1] \to [a_2, b_2] \) so that
\[
z_2 \circ \phi = z_1, \quad \text{and} \quad \phi'(t) > 0.
\]
It is an equivalence relation on the set of smooth parametrized curves. A smooth curve is an equivalence class of smooth parametrized curves.

**Example 4.1.** The circle \( C_R(z_0) \) can be parametrized by
\[
z : [0, 2\pi] \to \mathbb{C}, \quad z(t) = z_0 + R e^{it}
\]
and can be also parametrized by
\[
z^- : [0, 2\pi] \to \mathbb{C}, \quad z(t) = z_0 + R e^{-it}.
\]
These two parametrized curves are NOT equivalent. The first one has positive orientation and the second one has negative orientation. We use \( C_R^+(z_0) \) to describe the equivalence class of positive oriented
smooth parametrizations and use $C^\infty_R(z_0)$ to describe the equivalence class of negative oriented smooth parametrizations.

Now assume $\gamma \subset \mathbb{C}$ is a curve with a smooth parametrization $z : [a, b] \to \mathbb{C}$. $f$ is a continuous function defined over an open set $\Omega$ which contains $\gamma$. The integration of $f$ over $\gamma$ is defined as

$$\int_\gamma f(z) dz := \int_a^b f(z(t)) \cdot z'(t) dt.$$

**Lemma 4.2.** The integration $\int_\gamma f(z) dz$ is independent of smooth parametrization.

**Proof.** This immediately follows from substitution rule for definition integrations. \qed

**Example 4.3.**

1. $\int_{C^+} \frac{dz}{z} = \int_0^{2\pi} \frac{d(Re^{it})}{Re^{it}} = \int_0^{2\pi} \frac{Re^{it}}{Re^{it}} dt = 0$;
2. $\int_{C^+} -z \frac{dz}{z} = \int_0^{2\pi} \frac{1}{Re^{it}} d(Re^{it}) = \int_0^{2\pi} idt = 2\pi i$.

Lecture 5 stopped here.