Lecture Notes for Math 185 — Introduction to Complex Analysis

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Contents

Preface

Complex analysis is a highly integrative subject. It draws on ideas from real analysis, abstract algebra, linear algebra, and differential equations, and it recombines them into a theory that is at once elegant and surprisingly powerful. Beyond its classical applications, complex analysis also serves as a gateway to modern mathematics: many themes that recur throughout contemporary research—symmetry, singularities, geometry, and topology—first emerge here in a particularly transparent and workable form.

In this book, we try to write in a language that is accessible to undergraduates but still rigorous, and to use the most effective and powerful examples to let ideas and objects from modern mathematical research gradually permeate the text. For instance, we include homotopy, and we give complex-analytic proofs of results in algebraic topology about the fundamental group and covering spaces. Compared with the general formulations in algebraic topology, what we want to emphasize is the origin of these questions in complex analysis and the important role complex analysis plays in entering and shaping their solutions. As another example, we introduce Riemann surfaces in a natural way, and we aim to prepare students so that they can transition smoothly into Riemann surface theory. In our teaching experience, we have found that, when introduced in the right way, these ideas are entirely accessible to undergraduate students.

(more to be added ..)

CHAPTER 1

Complex numbers

1. The complex field

A **complex number** is an ordered pair $(x, y) \in \mathbb{R}^2$. We write (x, y) as

$$z = x + iy,$$

where i is the imaginary unit satisfying $i^2 = -1$. For the moment, you may regard i simply as a formal symbol used to record the second coordinate. We call $x = \Re z$ the **real part** and $y = \Im z$ the **imaginary part** of z. The set of all complex numbers is denoted by $\mathbb{C}$.

Each real number $x \in \mathbb{R}$ can be identified with the complex number $x + i0 \in \mathbb{C}$. In this way, we may view $\mathbb{R}$ as a subset of $\mathbb{C}$.

There are two binary operations on $\mathbb{C}$:

- **Addition:**
$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2);$$
- **Multiplication:**
$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

The additive identity is $0 = 0 + i0$, and the multiplicative identity is $1 = 1 + i0$. In particular,

$$i^2 = i \cdot i = -1.$$

Under the identification $\mathbb{C} \cong \mathbb{R}^2$, addition in $\mathbb{C}$ agrees with the usual vector addition in $\mathbb{R}^2$. However, the multiplication on $\mathbb{C}$ is *not* the coordinatewise multiplication on $\mathbb{R}^2$. For instance, the coordinatewise product

$$(x_1, y_1) * (x_2, y_2) = (x_1x_2,\ y_1y_2)$$

has zero divisors (e.g. $(1, 0) * (0, 1) = (0, 0)$), whereas the multiplication on $\mathbb{C}$ defined above has no zero divisors.

In fact, we have the following fundamental result, which serves as the foundation of complex analysis.

THEOREM 1.1. *$(\mathbb{C}, +, \cdot)$ is a field.*

REMARK 1.2. (1) From the point of view of modern algebra, $\mathbb{C}$ is a degree 2 algebraic extension of $\mathbb{R}$. Concretely,
$$\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1).$$

(2) $\mathbb{C}$ is *not* an ordered field: there is no total order on $\mathbb{C}$ that is compatible with the ring structure (equivalently, with addition and multiplication).

(3) We will prove in this course that $\mathbb{C}$ is **algebraically closed**; that is, every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$. This result is known as the **Fundamental Theorem of Algebra**.

EXAMPLE 1.3. Let $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$ with $z \neq 0$. Express z^{-1} in terms of x and y.

SOLUTION. Suppose $z^{-1} = a + ib$, where $a, b \in \mathbb{R}$. Then

$$zz^{-1} = (x + iy)(a + ib) = (xa - yb) + i(xb + ya) = 1$$

implies

$$xa - yb = 1, \quad xb + ya = 0.$$

Solving this system, we obtain

$$a = \frac{x}{x^2 + y^2}, \quad b = -\frac{y}{x^2 + y^2},$$

and

$$z^{-1} = \frac{x - iy}{x^2 + y^2}.$$

□

This motivates the following definitions.

For $z = x + iy$ with $x, y \in \mathbb{R}$, we define the **complex conjugate** of z by

$$\overline{z} = x - iy.$$

We also define the **modulus** (or **absolute value**) of z by

$$|z| = \sqrt{x^2 + y^2}.$$

With this notation, if $z \neq 0$ then its inverse can be written as

$$z^{-1} = \frac{\overline{z}}{|z|^2}.$$

The following properties are straightforward to verify and we leave details to the reader.

PROPOSITION 1.4. *For all $z, z_1, z_2 \in \mathbb{C}$, there are:*

(1) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$;
(2) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$;
(3) $z\overline{z} = |z|^2$;
(4) $|z| = |\overline{z}|$.

2. Geometry of $\mathbb{C}$

A complex number $z = x + iy$ corresponds to the point $(x, y) \in \mathbb{R}^2$, where $x = \Re z$ and $y = \Im z$. Through this identification, the Euclidean geometry of $\mathbb{R}^2$ is naturally encoded in $\mathbb{C}$.

For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the Euclidean distance between the corresponding points in $\mathbb{R}^2$ is

$$d\big((x_1, y_1), (x_2, y_2)\big) = |z_1 - z_2|.$$

In particular, the triangle inequality of d is equivalent to

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

In fact, complex conjugation allows us to recover both the dot product and the oriented area in $\mathbb{R}^2$. Indeed,

$$z_1\overline{z_2} = (x_1 + iy_1)(x_2 - iy_2) = (x_1x_2 + y_1y_2) - i(x_1y_2 - y_1x_2).$$

Therefore,

$$\Re(z_1\overline{z_2}) = x_1x_2 + y_1y_2 = \langle (x_1, y_1), (x_2, y_2) \rangle$$

which is the inner product, and

$$-\mathfrak{I}(z_1\overline{z_2}) = x_1 y_2 - y_1 x_2 = \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

which is the signed area of the parallelogram spanned by (x_1, y_1) and (x_2, y_2).

EXERCISE 2.1. Let $z_1, z_2 \neq 0$. Show that

$$\cos\theta = \frac{\mathfrak{R}(z_1\overline{z_2})}{|z_1||z_2|}, \qquad \sin\theta = \frac{-\mathfrak{I}(z_1\overline{z_2})}{|z_1||z_2|},$$

where θ is the oriented angle from z_1 to z_2.

REMARK 2.2 (The standard Hermitian form on $\mathbb{C}$ (*)). Consider $\mathbb{C}$ as a 1-dimensional vector space over $\mathbb{C}$. Define

$$h : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad h(z_1, z_2) = z_1\overline{z_2}.$$

Then h is a **Hermitian form**: it is $\mathbb{C}$-linear in the first variable, conjugate-linear in the second variable, and satisfies

$$h(z_2, z_1) = \overline{h(z_1, z_2)}.$$

Now identify $\mathbb{C}$ with $\mathbb{R}^2$ via

$$z = x + iy \longleftrightarrow v = \begin{pmatrix} x \\ y \end{pmatrix}.$$

For $z_k = x_k + iy_k$ corresponding to $v_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix} \in \mathbb{R}^2$, define the standard Euclidean inner product by

$$g(v_1, v_2) = x_1 x_2 + y_1 y_2,$$

and define the standard symplectic form by

$$\omega(v_1, v_2) = x_1 y_2 - y_1 x_2.$$

Then

$$h(z_1, z_2) = g(v_1, v_2) - i\,\omega(v_1, v_2).$$

Moreover, we may write

$$h(z_1, z_2) = v_1^{\mathsf{T}}[h]\,v_2, \qquad [h] = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = [g] - i[\omega],$$

where

$$[g] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad [\omega] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $[g]$ is the matrix of the Euclidean inner product g, namely

$$g(v_1, v_2) = v_1^{\mathsf{T}}[g]v_2,$$

and similarly $[\omega]$ represents the symplectic form ω via

$$\omega(v_1, v_2) = v_1^{\mathsf{T}}[\omega]v_2.$$

This viewpoint generalizes to higher dimensions in Kähler geometry, where complex geometry arises from the compatibility between a Riemannian geometry and a symplectic geometry.

3. Polar coordinates

Let $z = x + iy \in \mathbb{C}$ be a nonzero complex number. Its **modulus** is

$$r = |z| = \sqrt{x^2 + y^2} > 0.$$

Geometrically, r is the Euclidean distance from z to the origin.

Since $(x, y) \neq (0, 0)$, there exists an angle $\theta \in \mathbb{R}$ such that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Thus we may write

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

We call this the **polar form** of z. Any such θ is called an **argument** of z, and we write $\theta \in \arg(z)$. The argument is NOT unique: if $\theta \in \arg(z)$, then $\theta + 2\pi k \in \arg(z)$ for every $k \in \mathbb{Z}$.

REMARK 3.1 (Geometric vs. analytic definitions of sin and cos). In this section we view $\sin\theta$ and $\cos\theta$ geometrically, as the coordinates of the point on the unit circle at oriented angle θ (measured in radians). Alternatively, one may define sin and cos analytically by the power series

$$\cos\theta = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}, \qquad \sin\theta = \sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}.$$

These two definitions agree. One way to justify this is to note that both satisfy the same initial value problem

$$u' = -v, \qquad v' = u, \qquad u(0) = 1, \quad v(0) = 0,$$

and hence must coincide by the existence and uniqueness theorem for ODEs.

THEOREM 3.2 (Euler's formula). *For every $\theta \in \mathbb{R}$,*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

PROOF. This follows immediately from the power series expansions of e^z, $\sin z$, and $\cos z$. $\square$

REMARK 3.3. Euler's formula holds for all complex numbers too.

Using Euler's formula, the polar form can be written more compactly as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

We call $re^{i\theta}$ the **exponential form** of z.

THEOREM 3.4 (De Moivre's formula). *For every $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$,*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Equivalently,

$$(e^{i\theta})^n = e^{in\theta}.$$

PROOF. By Euler's formula,

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

$\square$

EXAMPLE 3.5. Using De Moivre's formula, we can express $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$. For example,

$$(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$$

implies, by expanding the left-hand side,

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta, \qquad \sin(2\theta) = 2\sin\theta\cos\theta.$$

PROPOSITION 3.6. *Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be nonzero complex numbers. Then*

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}, \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

In particular,

$$|z_1 z_2| = |z_1||z_2|, \qquad \arg(z_1 z_2) \equiv \theta_1 + \theta_2 \pmod{2\pi}.$$

EXAMPLE 3.7. Multiplication by i is the rotation counterclockwise by 90°, since $i = e^{i\pi/2}$.

EXERCISE 3.8. Find r and θ such that $z = re^{i\theta}$ for each of the following complex numbers.

(1) $z = 1$. Solution: $r = 1$ and $\theta = 2k\pi$, $k \in \mathbb{Z}$. In exponential form, $z = e^{i0}$.
(2) $z = i$. Solution: $r = 1$ and $\theta = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$. In exponential form, $z = e^{i\pi/2}$.
(3) $z = -1$. Solution: $r = 1$ and $\theta = \pi + 2k\pi$, $k \in \mathbb{Z}$. In exponential form, $z = e^{i\pi}$.
(4) $z = -i$. Solution: $r = 1$ and $\theta = -\frac{\pi}{2} + 2k\pi$ (equivalently $\frac{3\pi}{2} + 2k\pi$), $k \in \mathbb{Z}$. In exponential form, $z = e^{-i\pi/2}$.
(5) $z = 1 + i$. Solution: $r = \sqrt{2}$ and $\theta = \frac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$. In exponential form, $z = \sqrt{2}\, e^{i\pi/4}$.
(6) $z = -1 + i$. Solution: $r = \sqrt{2}$ and $\theta = \frac{3\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$. In exponential form, $z = \sqrt{2}\, e^{i3\pi/4}$.

4. Roots of unity

Let n be a positive integer. A complex number z satisfying

$$z^n = 1$$

is called an **nth root of unity**. Equivalently, the nth roots of unity are the complex solutions of the polynomial equation

$$z^n - 1 = 0.$$

Using the polar form, we can explicitly determine all nth roots of unity. Write $z = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$. Then

$$z^n = r^n e^{in\theta}.$$

Thus the equation $z^n = 1$ is equivalent to

$$r^n = 1, \qquad n\theta \equiv 0 \pmod{2\pi}.$$

Hence $r = 1$ and $\theta = \frac{2k\pi}{n}$ for some $k \in \mathbb{Z}$.

Although θ has infinitely many choices, the values $e^{i\theta}$ produce only n distinct solutions. Therefore the nth roots of unity are

$$e^{i\frac{2k\pi}{n}}, \qquad k = 0, 1, \dots, n-1.$$

Denote by R_n the set of nth roots of unity. Then R_n is closed under complex multiplication, and hence forms a cyclic group of order n. In fact,

$$(\mathbb{Z}/n, +) \cong R_n,$$

via the isomorphism $k \mapsto e^{2\pi ik/n}$.

An element $\zeta \in \mathbb{R}_n$ is called a **primitive nth root of unity** if it generates R_n, i.e. if

$$R_n = \{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\}.$$

There are exactly $\varphi(n)$ primitive nth roots of unity, where φ denotes Euler's totient function.

EXAMPLE 4.1. Find all primitive 12th roots of unity.

SOLUTION. The 12th roots of unity are $\zeta^k = e^{2\pi ik/12}$ for $k = 0, 1, \dots, 11$. Such a root ζ^k is primitive if and only if $\gcd(k, 12) = 1$. Therefore the primitive 12th roots of unity correspond to

$$k \in \{1, 5, 7, 11\}.$$

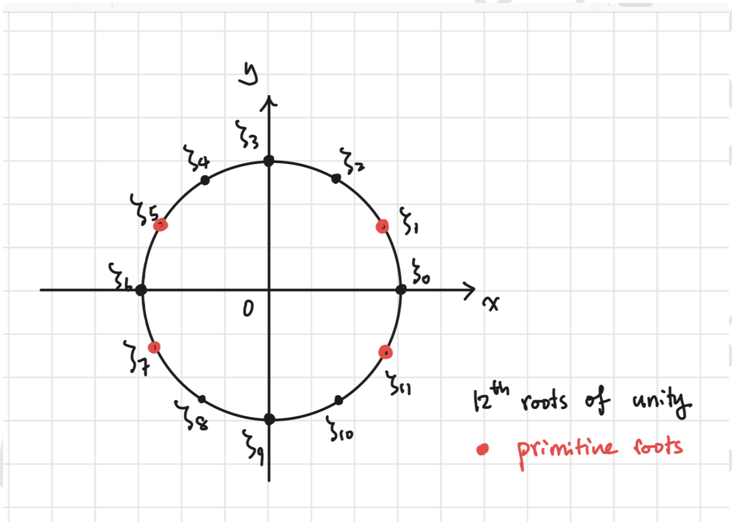
Hence the primitive 12th roots of unity are

$$e^{2\pi i/12}, \quad e^{10\pi i/12}, \quad e^{14\pi i/12}, \quad e^{22\pi i/12},$$

equivalently,

$$e^{\pi i/6}, \quad e^{5\pi i/6}, \quad e^{7\pi i/6}, \quad e^{11\pi i/6}.$$

□



Although it is difficult—and in general impossible—to find explicit formulas for all roots of an arbitrary polynomial equation, many special equations can still be solved directly.

EXAMPLE 4.2. Solve the equation $z^4 = i$.

SOLUTION. Write z in polar form $z = re^{i\theta}$ with $r > 0$ and $\theta \in \mathbb{R}$. Then

$$z^4 = r^4 e^{i4\theta}.$$

Since $i = e^{i(\pi/2+2\pi k)}$ for some $k \in \mathbb{Z}$, the equation $z^4 = i$ is equivalent to

$$r^4 e^{i4\theta} = e^{i(\pi/2+2\pi k)}.$$

Therefore,

$$r^4 = 1, \qquad 4\theta = \frac{\pi}{2} + 2\pi k.$$

Hence $r = 1$ and

$$\theta = \frac{\pi}{8} + \frac{k\pi}{2}, \qquad k = 0, 1, 2, 3.$$

So the solutions are

$$z = e^{i\pi/8},\ e^{i5\pi/8},\ e^{i9\pi/8},\ e^{i13\pi/8}.$$

$\square$

EXAMPLE 4.3. Solve the equation $1 + z + z^2 + z^3 = 0$.

SOLUTION. Factor the left-hand side as a geometric sum:

$$1 + z + z^2 + z^3 = \frac{z^4 - 1}{z - 1}, \qquad (z \neq 1).$$

Therefore

$$1 + z + z^2 + z^3 = 0 \quad \Longleftrightarrow \quad z^4 - 1 = 0 \ \text{ and } \ z \neq 1.$$

Hence z is a 4th root of unity other than 1, so the solutions are

$$z \in \{-1,\ i,\ -i\}.$$

$\square$

5. Euclidean geometry of $\mathbb{R}^2$ via $\mathbb{C}$

5.1. Lines and line segments. Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$.

- The **line segment** joining α and β is

$$\{(1-t)\alpha + t\beta | t \in [0,1]\}.$$

- The **perpendicular bisector** of the segment joining α and β is

$$\{z \in \mathbb{C} | |z - \alpha| = |z - \beta|\}.$$

  Equivalently, it is the locus of points that are equidistant from α and β.

EXERCISE 5.1. Describe the real axis and the imaginary axis using distances.

SOLUTION. The real axis consists of points equidistant from i and $-i$, hence it is

$$\{z \in \mathbb{C} | |z - i| = |z + i|\}.$$

Similarly, the imaginary axis consists of points equidistant from 1 and -1, hence it is

$$\{z \in \mathbb{C} | |z - 1| = |z + 1|\}.$$

$\square$

5.2. Circles. A circle with center $z_0 \in \mathbb{C}$ and radius $r > 0$ is given by

$$|z - z_0| = r.$$

We now give another useful description.

PROPOSITION 5.2 (Apollonius circles). *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$, and let $\lambda > 0$ with $\lambda \neq 1$. Then the locus*

$$|z - \alpha| = \lambda |z - \beta|$$

is a circle. Conversely, every circle can be written in this form for suitable α, β, λ.

PROOF. Squaring both sides gives

$$|z-\alpha|^2 = \lambda^2|z-\beta|^2.$$

Writing $z = x+iy$, $\alpha = a+ib$, $\beta = c+id$ and expanding, then completing squares, we obtain an equation of the form

$$|z-z_0| = r,$$

where

$$z_0 = \frac{\alpha-\lambda^2\beta}{1-\lambda^2}, \qquad r = \frac{\lambda|\alpha-\beta|}{|1-\lambda^2|}.$$

This is a circle.

Conversely, given a circle $|z-z_0| = r$ with $r > 0$, choose any $\lambda > 0$ with $\lambda \neq 1$. Pick any $\beta \in \mathbb{C}$ satisfying $|\beta - z_0| = \frac{r}{\lambda}$, and define

$$\alpha = (1-\lambda^2)z_0 + \lambda^2\beta.$$

Then

$$|z-\alpha| = \lambda|z-\beta| \quad \Longleftrightarrow \quad |z-z_0| = r.$$

□

6. Topology of $\mathbb{C}$

By identifying $\mathbb{C}$ with $\mathbb{R}^2$, it becomes a metric space with the metric

$$d(z,w) = |z-w|.$$

With this metric, $\mathbb{C}$ is isometric to the Euclidean space $\mathbb{R}^2$.

In this section, we review some basic topology of $\mathbb{C}$. Most of the proofs will be skipped in our lectures, but they can be found in any standard text on mathematical analysis (for instance, *Principles of Mathematical Analysis* by Rudin is a great reference). You may safely skip the details of these proofs for the purpose of this course.

6.1. Open sets in $\mathbb{C}$. For $z_0 \in \mathbb{C}$ and $r > 0$, we denote the **open disk** centered at z_0 with radius r by

$$D(z_0;r) = \{z \in \mathbb{C} \mid |z-z_0| < r\}.$$

DEFINITION 6.1. Let $S \subseteq \mathbb{C}$. A point $z_0 \in S$ is called an **interior point** of S if there exists $r > 0$ such that

$$D(z_0;r) \subseteq S.$$

The set of all interior points of S is called the **interior** of S, and is denoted by S°.

DEFINITION 6.2. A subset $S \subseteq \mathbb{C}$ is called **open** if either $S = \emptyset$, or every point of S is an interior point of S (equivalently, $S = S^\circ$).

In particular, an open disk is open.

6.2. Closed sets in $\mathbb{C}$. There are two equivalent ways to define closed sets.

DEFINITION 6.3. A subset $S \subseteq \mathbb{C}$ is called **closed** if its complement $\mathbb{C} \setminus S$ is open.

From this definition, both $\emptyset$ and $\mathbb{C}$ are closed, since their complements are open.

For $z_0 \in \mathbb{C}$ and $r > 0$, we denote the **closed disk** centered at z_0 with radius r by

$$\overline{D(z_0; r)} = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}.$$

It is an exercise to show that $\overline{D(z_0; r)}$ is a closed set in $\mathbb{C}$.

Another way to define closed sets is via sequences. Recall that a sequence $(z_n)_{n \in \mathbb{Z}^+}$ is **convergent** to a point $w \in \mathbb{C}$ if

$$\lim_{n\to\infty} |z_n - w| = 0.$$

Equivalently, writing $z_n = x_n + iy_n$ and $w = x + iy$, we have $z_n \to w$ in $\mathbb{C}$ if and only if

$$x_n \to x \qquad \text{and} \qquad y_n \to y \quad \text{as } n \to \infty.$$

DEFINITION 6.4. A subset $S \subseteq \mathbb{C}$ is called **closed** if it contains the limits of all convergent sequences in S; that is, any sequence $(z_n)_{n\in\mathbb{Z}^+}$ in S with $z_n \to w$ in $\mathbb{C}$, we must have $w \in S$.

It is left as an exercise to the reader to prove that these two definitions are equivalent.

6.3. Compact sets in $\mathbb{C}$. We first recall the general definition of compactness in a metric space.

DEFINITION 6.5 (Compactness). Let (X, d) be a metric space. A subset $K \subseteq X$ is called **compact** if every open cover of K admits a finite subcover. That is, if $K \subseteq \bigcup_{\alpha\in A} U_\alpha$ where each U_α is open in X, then there exist $\alpha_1, \dots, \alpha_N \in A$ such that

$$K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_N}.$$

DEFINITION 6.6 (Sequential compactness). Let (X, d) be a metric space. A subset $K \subseteq X$ is called **sequentially compact** if every sequence (x_n) in K has a convergent subsequence, whose limit lies in K.

THEOREM 6.7. *In any metric space, a subset is compact if and only if it is sequentially compact.*

The proof is omitted here.

In the special case $X = \mathbb{C}$, there is a particularly simple characterization of compact sets.

THEOREM 6.8 (Heine–Borel theorem in $\mathbb{C}$). *A subset $K \subseteq \mathbb{C}$ is compact if and only if it is closed and bounded.*

6.4. $\mathbb{C}$ is complete.

DEFINITION 6.9 (Cauchy sequence). A sequence (z_n) in $\mathbb{C}$ is called a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$|z_n - z_m| < \varepsilon \qquad \text{for all } m, n \geq N.$$

Clearly, every convergent sequence is Cauchy.

Writing $z_n = x_n + iy_n$, we have

$$|z_n - z_m|^2 = (x_n - x_m)^2 + (y_n - y_m)^2.$$

It follows that (z_n) is a Cauchy sequence in $\mathbb{C}$ if and only if (x_n) and (y_n) are Cauchy sequences in $\mathbb{R}$.

DEFINITION 6.10 (Completeness). A metric space (X, d) is called **complete** if every Cauchy sequence in X converges to a limit in X.

THEOREM 6.11. *$\mathbb{C}$ is complete.*

PROOF. Let $z_n = x_n + iy_n$ be a Cauchy sequence in $\mathbb{C}$. Then (x_n) and (y_n) are Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exist $x, y \in \mathbb{R}$ such that $x_n \to x$ and $y_n \to y$. Let $z = x + iy$. Then $z_n \to z$ in $\mathbb{C}$, so $\mathbb{C}$ is complete. □

6.5. Connected sets. For a metric space (in particular, for $\mathbb{C}$), there are two common notions of connectedness: connected sets and path-connected sets. They are closely related but not always the same.

DEFINITION 6.12 (Connected set). A subset $S \subseteq \mathbb{C}$ is called **disconnected** if there exist two nonempty open sets $U, V \subseteq \mathbb{C}$ such that

$$S \subseteq U \cup V, \qquad S \cap U \neq \emptyset, \qquad S \cap V \neq \emptyset, \qquad \text{and} \qquad (S \cap U) \cap (S \cap V) = \emptyset.$$

Equivalently, S is disconnected if it can be written as a union of two nonempty disjoint sets that are open in the subspace topology on S. A set S is called **connected** if it is not disconnected.

DEFINITION 6.13 (Path-connected set). A subset $S \subseteq \mathbb{C}$ is called **path-connected** if for any $z, w \in S$, there exists a continuous map

$$\gamma : [0, 1] \to S$$

such that $\gamma(0) = z$ and $\gamma(1) = w$.

PROPOSITION 6.14. *Every path-connected set is connected.*

We omit the proof.

In general, the converse is false: there exist sets in $\mathbb{R}^2$ (hence also in $\mathbb{C}$) that are connected but not path-connected. A classical example is the *topologist's sine curve*.

However, for open sets the two notions coincide.

THEOREM 6.15. *Let $U \subseteq \mathbb{C}$ be open. Then U is connected if and only if U is path-connected.*

We omit the proof.

DEFINITION 6.16 (Domain). An open and connected subset $\Omega \subseteq \mathbb{C}$ is called a **domain**.

Both descriptions (connectedness and path-connectedness) will be useful in this course.

6.6. $\mathbb{C}$-valued functions. In this course, we will mostly consider complex-valued functions defined on a domain in $\mathbb{C}$. That is, let $\Omega \subseteq \mathbb{C}$ be a domain (an open and connected set), and let

$$f : \Omega \to \mathbb{C}$$

be a function.

A $\mathbb{C}$-valued function is equivalent to a pair of $\mathbb{R}$-valued functions, namely its real part and imaginary part. Concretely, for each $z \in \Omega$, we may write

$$f(z) = u(z) + iv(z),$$

where

$$u = \mathfrak{R}(f) : \Omega \to \mathbb{R}, \qquad v = \mathfrak{I}(f) : \Omega \to \mathbb{R}.$$

Under the identification $\mathbb{C} \cong \mathbb{R}^2$, the function f can be viewed as a map

$$f : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (u(x, y),\ v(x, y)).$$

Many basic properties of f (such as limits and continuity) can be studied by analyzing u and v.

We briefly recall the notion of continuity for complex-valued functions.

Let Ω be a nonempty open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is a function.

DEFINITION 6.17 (Continuity via limits). The function f is **continuous at** $z_0 \in \Omega$ if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

We say that f is **continuous on** Ω if it is continuous at every point of Ω.

PROPOSITION 6.18 (Continuity via open sets). *A function $f : \Omega \to \mathbb{C}$ is* ***continuous*** *if for every open set $U \subseteq \mathbb{C}$, the preimage*

$$f^{-1}(U) = \{z \in \Omega | f(z) \in U\}$$

is open in Ω.

As an immediate consequence, the composition of continuous functions is continuous.

Now write $f = u + iv$, where $u, v : \Omega \to \mathbb{R}$.

PROPOSITION 6.19. *The function f is continuous at $z_0 \in \Omega$ if and only if u and v are continuous at z_0. In particular, f is continuous on Ω if and only if u and v are continuous on Ω.*

EXAMPLE 6.20. Both functions $f(z) = z$ and $g(z) = \overline{z}$ are continuous on $\mathbb{C}$. More generally, let $\Phi \in \mathbb{C}[x, y]$ be a polynomial in two variables. Then the function

$$f(z) = \Phi(z, \overline{z})$$

is continuous on $\mathbb{C}$.

EXAMPLE 6.21. Define $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by

$$f(z) = \frac{|z|^2}{z}.$$

Then f is continuous on $\mathbb{C} \setminus \{0\}$. Moreover, f admits a continuous extension to $\mathbb{C}$ by setting $f(0) = 0$.

On the other hand, the function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$g(z) = \frac{|z|}{z}$$

cannot be continuously extended to $z = 0$, hence it is discontinuous at 0.

PROOF. For $z \neq 0$, we have

$$\frac{|z|^2}{z} = \frac{z\bar{z}}{z} = \bar{z},$$

so $f(z) = \bar{z}$ on $\mathbb{C} \setminus \{0\}$, and therefore

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \bar{z} = 0.$$

Thus defining $f(0) = 0$ makes f continuous on all of $\mathbb{C}$.

For g, write $z = re^{i\theta}$ with $r > 0$. Then

$$g(z) = \frac{|z|}{z} = \frac{r}{re^{i\theta}} = e^{-i\theta}.$$

Along the positive real axis ($\theta = 0$), $g(z) \to 1$, while along the positive imaginary axis ($\theta = \frac{\pi}{2}$), $g(z) \to -i$. Hence $\lim_{z\to 0} g(z)$ does not exist. $\square$

The following properties of continuous functions are useful.

THEOREM 6.22. *The image of a (sequentially) compact set under a continuous function is (sequentially) compact.*

It has the following consequence.

COROLLARY 6.23. *A real valued continuous function f on a compact set $K \subseteq \mathbb{C}$ attains its maximum and minimum on K.*

THEOREM 6.24. *The image of a (path-)connected set under a continuous function is (path-)connected.*

For real-valued functions, this implies the Intermediate Value Theorem.

CHAPTER 2

Holomorphic Functions

1. Definition of holomorphic functions

DEFINITION 1.1. Assume Ω is an open subset in $\mathbb{C}$. A function $f : \Omega \to \mathbb{C}$ is called **complex differentiable** at $z_0 \in \Omega$, if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit, denoted by $f'(z_0)$, is called the **derivative** of f at z_0.

If $f'(z_0)$ exists for every $z_0 \in \Omega$, then we say the function f is a **holomorphic function** over Ω. A holomorphic function is also called a **complex differentiable** function. (We will see later that holomorphic functions are analytic, so it is also called analytic functions.)

A holomorphic function over $\mathbb{C}$ is called an **entire function**.

For an open set Ω, we will use $\mathcal{O}(\Omega)$ to denote the set of holomorphic functions on Ω.

EXAMPLE 1.2. (1) Constant functions are holomorphic functions.

(2) $f(z) = z$ is an entire function with $f'(z) = 1$. (Check by definition.)

(3) $f(z) = z^2$ is an entire function with $f'(z) = 2z$. (Check by definition.)

(4) Any polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbb{C}[z]$, where $a_0, a_1, \cdots, a_n \in \mathbb{C}, a_n \neq 0$, is an entire function with

$$f'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}.$$

(5) $f(z) = \frac{1}{z}$ is holomorphic over $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ with

$$f'(z) = -\frac{1}{z^2}.$$

(6) NEITHER $f(z) = \Re(z)$ nor $f(z) = \Im(z)$ is complex differentiable at any point $z_0 \in \mathbb{C}$.

(7) $f(z) = \bar{z}$ is NOT complex differentiable at any point $z_0 \in \mathbb{C}$.

Assume f is holomorphic at $z_0 \in \Omega$. Then we can write

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + \psi(z - z_0)(z - z_0),$$

where $\psi(h) \to 0$ as $h \to 0$. In particular, f is continuous at z_0.

Similar to real functions, derivatives of complex functions have the following properties.

PROPOSITION 1.3. (1) *Assume f and g are complex differentiable at $z_0 \in \Omega$. Then $f \pm g$ and $f \cdot g$ are complex differentiable at z_0. Moreover, if $g(z_0) \neq 0$, then f/g is also complex differentiable at z_0. In each case,*

(a) $(f+g)'(z_0) = f'(z_0) + g'(z_0)$*;*

(b) $(f \cdot g)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$*;*

(c) $(f/g)'(z_0) = \dfrac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$.

(2) *(Chain rule) Assume f is complex differentiable at $z_0 \in \Omega$. Let $U \subset \mathbb{C}$ be an open set containing $f(z_0)$ such that $f(z) \in U$ for all z in a neighborhood of z_0. If $g : U \to \mathbb{C}$ is complex differentiable at $f(z_0)$, then $g \circ f$ is complex differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0).$$

EXAMPLE 1.4. Calculate $f'(z)$ for $f(z) = \frac{z^2+iz+1}{z-i}$.

REMARK 1.5. The set of holomorphic functions on an open set Ω, denoted by $\mathcal{O}(\Omega)$, is a (commutative) algebra over $\mathbb{C}$: if $f, g \in \mathcal{O}(\Omega)$ and $c \in \mathbb{C}$, then $f + g$, fg, and cf are in $\mathcal{O}(\Omega)$, and the constant function 1 is in $\mathcal{O}(\Omega)$ which is the unity of the algebra.

2. Cauchy–Riemann equations

We now relate complex differentiability to conditions on the real and imaginary parts of f. We will see that complex differentiability imposes a system of partial differential equations on these two functions.

Write

$$f(z) = u(z) + iv(z),$$

where u and v are real-valued functions. Write

$$z = x + iy, \qquad z_0 = x_0 + iy_0.$$

Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\big(u(z) - u(z_0)\big) + i\big(v(z) - v(z_0)\big)}{z - z_0}.$$

Multiplying the numerator and denominator by $\overline{z - z_0}$, we obtain

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\big(u(z) - u(z_0)\big)\overline{(z - z_0)} + i\big(v(z) - v(z_0)\big)\overline{(z - z_0)}}{|z - z_0|^2}.$$

Since $\overline{(z - z_0)} = (x - x_0) - i(y - y_0)$, separating the real and imaginary parts gives

$$\Re\left(\frac{f(z) - f(z_0)}{z - z_0}\right) = \frac{\big(u(z) - u(z_0)\big)(x - x_0) + \big(v(z) - v(z_0)\big)(y - y_0)}{|z - z_0|^2},$$

$$\Im\left(\frac{f(z) - f(z_0)}{z - z_0}\right) = \frac{-\big(u(z) - u(z_0)\big)(y - y_0) + \big(v(z) - v(z_0)\big)(x - x_0)}{|z - z_0|^2}.$$

In particular, if $f'(z_0)$ exists, then the above expressions must have a limit as $z \to z_0$, which is the same as

$$x \to x_0, \quad y \to y_0.$$

Now we consider two paths:

Approaching along the real axis. Let $z \to z_0$ with $y = y_0$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i\,\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}.$$

Letting $x \to x_0$ shows that the partial derivatives $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ exist and

$$f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0).$$

Approaching along the imaginary axis. Let $z \to z_0$ with $x = x_0$. Then

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)} + i\,\frac{v(x_0,y)-v(x_0,y_0)}{i(y-y_0)} = \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0} - i\,\frac{u(x_0,y)-u(x_0,y_0)}{y-y_0}.$$

Letting $y \to y_0$ shows that the partial derivatives $u_y(x_0, y_0)$ and $v_y(x_0, y_0)$ exist and

$$f'(z_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0).$$

Comparing the two expressions for $f'(z_0)$, we conclude that

$$u_x(x_0, y_0) = v_y(x_0, y_0), \qquad u_y(x_0, y_0) = -\, v_x(x_0, y_0). \tag{2.1}$$

The system (2.1) is called the **Cauchy–Riemann equations**.

The above arguments show that if $f'(z_0)$ exists, then the partial derivatives of u and v exist at (x_0, y_0), and the Cauchy–Riemann equations hold at (x_0, y_0).

In fact, we have a stronger result.

THEOREM 2.1. *Assume Ω is an open subset of $\mathbb{C}$, where we also regard it as an open set $\mathbb{R}^2$. The function $f = u + iv$ is complex differentiable at $z_0 = x_0 + iy_0 \in \Omega$ if and only if $F = (u, v)^T$ is differentiable at $(x_0, y_0) \in \Omega$ and the partial derivatives*

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

satisfy the Cauchy–Riemann equation at (x_0, y_0), i.e.,

$$\frac{\partial u}{\partial x}\Big|_{(x_0,y_0)} = \frac{\partial v}{\partial y}\Big|_{(x_0,y_0)}, \qquad \frac{\partial u}{\partial y}\Big|_{(x_0,y_0)} = -\frac{\partial v}{\partial x}\Big|_{(x_0,y_0)}.$$

Recall from real analysis (e.g., Rudin Definition 9.11) that, the function F is **differentiable** at $(x_0, y_0) \in \Omega$, if there is a 2×2 matrix J so that

$$\lim_{x\to x_0, y\to y_0} \frac{|F(x,y) - F(x_0,y_0) - J\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

By taking $y \equiv y_0$ and $x \equiv= x_0$ respectively, one get the matrix J must be of the form

$$J(F)|_{(x_0,y_0)} := \begin{bmatrix} \frac{\partial u}{\partial x}|_{(x_0,y_0)} & \frac{\partial u}{\partial y}|_{(x_0,y_0)} \\ \frac{\partial v}{\partial x}|_{(x_0,y_0)} & \frac{\partial v}{\partial y}|_{(x_0,y_0)} \end{bmatrix}$$

if it exists. (This is left as an exercise for the reader).

The matrix $J(F)$ is called the **Jacobian matrix** of the map F.

The differentiability of F at (x_0, y_0) is stronger than the existence of partial derivatives at this point.

EXAMPLE 2.2. Consider the complex valued function $f(z) = \sqrt{|\Re z||\Im(z)|}$, $z \in \mathbb{C}$. Direct calculation shows that all partial derivatives of $u = \sqrt{|x||y|}$ exist at $(0, 0)$ and are zero. Hence the Cauchy–Riemann equation holds at $(0, 0)$. However, the function f is not holomorphic at $0 \in \mathbb{C}$.

Correspondingly, the vector valued function F with respect f to is not real differentiable at $(0, 0)$ though partial derivatives exist.

Now let's prove Theorem 2.1.

PROOF. (1) "$\Rightarrow$". Since we have derived above that partial derivatives of u, v at (x_0, y_0) exist and satisfy the Cauchy–Riemann equation, we only need to show $F = (u, v)^T$ is differentiable at (x_0, y_0). For this, notice that when f is holomorphic at z_0, denote by

$$h_1 = x - x_0, \quad h_2 = y - y_0, \quad h = h_1 + ih_2,$$

there follows

$$\Re(f'(z_0)h) = \Re((\frac{\partial u}{\partial x}|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}|_{(x_0,y_0)})(h_1 + ih_2)) = \frac{\partial u}{\partial x}|_{(x_0,y_0)}h_1 - \frac{\partial v}{\partial x}|_{(x_0,y_0)}h_2$$

and by the Cauchy–Riemann equation, this is

$$\frac{\partial u}{\partial x}|_{(x_0,y_0)}h_1 + \frac{\partial u}{\partial y}|_{(x_0,y_0)}h_2$$

which is exactly the first row of $J(F)|_{(x_0,y_0)}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$.

Similarly, $\Im(f'(z_0)h)$ can be written as

$$\frac{\partial v}{\partial x}|_{(x_0,y_0)}h_1 + \frac{\partial v}{\partial y}|_{(x_0,y_0)}h_2$$

which is exactly the second row of $J(F)|_{(x_0,y_0)}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$.

This shows that in fact

$$\begin{aligned} &\frac{|F(x_0 + h_1, y_0 + h_2) - F(x_0, y_0) - J(F)|_{(x_0,y_0)}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}|}{|h|} \\ = &\frac{|f(z_0 + h) - f(z_0) - f'(z_0)h|}{|h|}, \end{aligned} \tag{2.2}$$

and hence F is differentiable at (x_0, y_0).

"$\Leftarrow$". Assuming F is differentiable at (x_0, y_0) and satisfies the Cauchy–Riemann equation at (x_0, y_0), then we can write

$$J(F)|_{(x_0,y_0)}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \Re((\frac{\partial u}{\partial x}|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}|_{(x_0,y_0)})(h_1 + ih_2)) \\ \Im((\frac{\partial u}{\partial x}|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}|_{(x_0,y_0)})(h_1 + ih_2)). \end{bmatrix}$$

Using the Cauchy–Riemann equations, we can express the limit as

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0) - (\frac{\partial u}{\partial x}|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}|_{(x_0,y_0)})h}{h} = 0$$

which is equivalent to $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists and is $\frac{\partial u}{\partial x}|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}|_{(x_0,y_0)}$.

This proves f is holomorphic at z_0 and in fact,

$$f'(z_0) = \frac{\partial u}{\partial x}|_{(x_0,y_0)} + i\frac{\partial v}{\partial x}|_{(x_0,y_0)}.$$

$\square$

Since every C^1 function (i.e., a function whose partial derivatives are continuous) is automatically real differentiable, to check whether a function $f : \Omega \to \mathbb{C}$ is holomorphic it suffices to verify that it satisfies the Cauchy–Riemann equations.

EXAMPLE 2.3. Consider the following functions $f = u+iv$, where u and v are real-valued functions. We use the Cauchy–Riemann equations to determine whether f is holomorphic.

(1) $u(x, y) = x^2 - y^2$, $\quad v(x, y) = -2xy$.

Here
$$u_x = 2x, \quad u_y = -2y, \quad v_x = -2y, \quad v_y = -2x.$$
Thus $u_x = v_y$ forces $2x = -2x$, which only holds when $x = 0$. Hence the Cauchy–Riemann equations do not hold on any open set, so f is not holomorphic anywhere. (Notice: We say that f is holomorphic at a point z_0 only if f is complex differentiable at every point in some open neighborhood of z_0.)

(2) $u(x, y) = x^2 - y^2$, $\quad v(x, y) = 2xy$.

Here
$$u_x = 2x, \quad u_y = -2y, \quad v_x = 2y, \quad v_y = 2x,$$
so the Cauchy–Riemann equations hold everywhere. Therefore f is entire. In fact,
$$f(z) = (x^2 - y^2) + i(2xy) = (x + iy)^2 = z^2, \qquad f'(z) = 2z.$$

(3) $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$, $\quad v(x, y) = \arctan\frac{y}{x}$, where $x > 0$.

On the domain $\{x > 0\}$ one finds
$$u_x = \frac{x}{x^2 + y^2}, \quad u_y = \frac{y}{x^2 + y^2}, \quad v_x = -\frac{y}{x^2 + y^2}, \quad v_y = \frac{x}{x^2 + y^2}.$$
Hence the Cauchy–Riemann equations hold on $\{x > 0\}$, and f is holomorphic there. This is the principal branch of the logarithm that we will explain more in later chapters.

(4) $u(x, y) = \frac{x}{x^2+y^2}$, $\quad v(x, y) = \frac{y}{x^2+y^2}$.

On $\mathbb{C} \setminus \{0\}$,
$$u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad u_y = -\frac{2xy}{(x^2 + y^2)^2}, \quad v_x = -\frac{2xy}{(x^2 + y^2)^2}, \quad v_y = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$
Thus $u_x \neq v_y$ in general, so the Cauchy–Riemann equations do not hold on any open set. Therefore f is not holomorphic anywhere. (Indeed $u + iv = \dfrac{1}{\bar{z}}$.)

(5) $u(x, y) = e^x \cos y$, $\quad v(x, y) = e^x \sin y$.

Here
$$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = e^x \sin y, \quad v_y = e^x \cos y,$$
so the Cauchy–Riemann equations hold everywhere. Therefore f is entire. In fact,
$$f(z) = e^x(\cos y + i \sin y) = e^{x+iy} = e^z, \qquad f'(z) = e^z.$$

3. Complex notation of partial derivatives

Assume f is a C^1 function on an open set in $\mathbb{C}$. We introduce the operators
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$
Then
$$\frac{\partial z}{\partial z} = 1, \qquad \frac{\partial \bar{z}}{\partial z} = 0, \qquad \frac{\partial z}{\partial \bar{z}} = 0, \qquad \frac{\partial \bar{z}}{\partial \bar{z}} = 1.$$
Moreover, among all operators of the form $A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$ with $A, B \in \mathbb{C}$, the operator $\frac{\partial}{\partial z}$ is uniquely determined by the first two identities, and $\frac{\partial}{\partial \bar{z}}$ is uniquely determined by the last two identities.

Using this notation, the Cauchy–Riemann equations can be written compactly as
$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Indeed, write $f = u + iv$ with u, v real-valued. Then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\Big(u_x + v_y + i(v_x - u_y)\Big),$$

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\Big(u_x - v_y + i(v_x + u_y)\Big).$$

Therefore $\frac{\partial f}{\partial \bar z} = 0$ is equivalent to

$$u_x = v_y, \qquad u_y = -v_x,$$

which are exactly the Cauchy–Riemann equations. Moreover, when the Cauchy–Riemann equations hold, we have

$$\frac{\partial f}{\partial z} = u_x + iv_x = v_y - iu_y,$$

so $\frac{\partial f}{\partial z}$ agrees with the complex derivative $f'(z)$.

As a corollary, we have the following consequence: Let $\Phi \in \mathbb{C}[z, w]$ be a polynomial in two variables, and define

$$f(z) = \Phi(z, \bar z), \qquad z \in \mathbb{C}.$$

Then f is holomorphic on $\mathbb{C}$ if and only if Φ is independent of w, i.e. $\Phi(z, w) = P(z)$ for some $P \in \mathbb{C}[z]$. Equivalently, the expression $\Phi(z, \bar z)$ contains no $\bar z$-term.

A similar statement holds for rational functions (on any open set where the expression is defined).

In particular, linear fractional maps provide an important class of holomorphic rational functions.

EXAMPLE 3.1 (Linear fractional maps). Let

$$T(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

Then T is holomorphic on its domain $\mathbb{C} \setminus \{-d/c\}$ (if $c \neq 0$), and on all of $\mathbb{C}$ if $c = 0$. A direct computation gives

$$T'(z) = \frac{(az+b)'(cz+d) - (az+b)(cz+d)'}{(cz+d)^2} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}.$$

In particular, $T'(z) \neq 0$ everywhere on its domain.

Such a map is determined by the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}),$$

and we write $T = T_A$. Note that scaling A by a nonzero complex number does not change the map:

$$T_{\lambda A}(z) = T_A(z) \qquad (\lambda \in \mathbb{C}^\times).$$

Therefore linear fractional maps are naturally parametrized by equivalence classes of matrices in $GL_2(\mathbb{C})$ modulo scalar multiples. Restricting to matrices with determinant 1, this gives the group

$$PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\},$$

which acts on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

A linear fractional map is also called a **Möbius transformation**.

The following proposition is left to the reader as an exercise using the Cauchy–Riemann equations.

PROPOSITION 3.2. *Assume $f = u + iv$ is complex differentiable at $z_0 = x_0 + iy_0$. Then the Jacobian matrix*

$$J(x_0, y_0) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{(x_0,y_0)}$$

has nonnegative determinant. In fact,

$$\det J(x_0, y_0) = u_x v_y - u_y v_x = \left|f'(z_0)\right|^2 \geq 0.$$

4. Harmonic functions

Let $\Omega \subseteq \mathbb{R}^2 \simeq \mathbb{C}$ be an open set. For a C^2 function $u \in C^2(\Omega)$, we define the Laplacian

$$\Delta u \;=\; u_{xx} + u_{yy}.$$

DEFINITION 4.1. A function $u \in C^2(\Omega)$ is called **harmonic** on Ω if

$$\Delta u = 0 \quad \text{on } \Omega.$$

EXAMPLE 4.2. (1) Consider $f(z) = z^2$. Writing $z = x + iy$, we have

$$z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy).$$

Thus

$$u(x, y) = x^2 - y^2, \qquad v(x, y) = 2xy.$$

A direct computation shows $\Delta u = 0$ and $\Delta v = 0$.

(2) Consider $f(z) = z^3$. Writing $z = x + iy$, we have

$$z^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

Thus

$$u(x, y) = x^3 - 3xy^2, \qquad v(x, y) = 3x^2y - y^3,$$

and again one checks that $\Delta u = 0$ and $\Delta v = 0$.

THEOREM 4.3. *Let f be holomorphic on Ω and write $f = u + iv$, where u and v are real-valued. Then u and v are harmonic on Ω.*

We are not yet ready to give a complete proof of this result. However, assuming u and v are C^2, the conclusion follows easily.

A PARTIAL PROOF. Assume u and v have continuous second partial derivatives. The Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

imply

$$u_{xx} = v_{yx}, \qquad u_{yy} = -(v_x)_y = -v_{xy}.$$

Since mixed partials agree ($v_{yx} = v_{xy}$), we get

$$\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0.$$

Similarly, $\Delta v = 0$. □

REMARK 4.4. In real analysis, differentiability does not guarantee higher regularity. Even if a real-valued function f is differentiable at every point, its derivative f' need not be continuous, and the second derivative $f''(x_0)$ may fail to exist. For example, consider the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The first order derivative f' is not continuous at 0, and so $f''(0)$ does not exist.

In complex analysis, we will prove a remarkable **regularity** result: if f is holomorphic on an open set Ω (i.e., $f'(z)$ exists for every $z \in \Omega$), then $f''(z)$ also exists on Ω. In fact, all higher-order complex derivatives $f^{(n)}(z)$ exist on Ω for every $n \geq 1$.

We now ask the following question: Given a harmonic function u on Ω, can we find a function v such that $f = u + iv$ is holomorphic? If so, is v unique? Such a function v, if it exists, is called a **harmonic conjugate** of u.

THEOREM 4.5. *Let $D \subset \mathbb{C}$ be an open disk, and let $u \in C^2(D)$ be harmonic on D. Then there exists a harmonic conjugate v on D such that $f = u + iv$ is holomorphic on D. Moreover, v is unique up to an additive real constant.*

Before we prove this theorem, we illustrate the idea with an example.

EXAMPLE 4.6. Let $u(x, y) = x^2 - y^2$. We have seen u is harmonic on $\mathbb{C}$. If $f = u + iv$ is holomorphic, the Cauchy–Riemann equations require

$$v_x = -u_y = 2y, \qquad v_y = u_x = 2x.$$

Integrating $v_x = 2y$ with respect to x gives $v(x, y) = 2xy + C(y)$. Differentiating with respect to y yields $v_y = 2x + C'(y)$, and comparing with $v_y = 2x$ gives $C'(y) = 0$. Hence $C(y)$ is constant, and we may take

$$v(x, y) = 2xy.$$

Therefore $f(z) = u + iv = (x^2 - y^2) + i(2xy) = z^2$ is holomorphic, and v is a harmonic conjugate of u (unique up to an additive constant).

PROOF OF THEOREM 4.5. Assume $u \in C^2(D)$ is harmonic on an open disk $D \subset \mathbb{C}$. If $f = u + iv$ is holomorphic, then u and v must satisfy the Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x,$$

equivalently,

$$v_x = -u_y, \qquad v_y = u_x. \tag{4.1}$$

So we are led to solve (4.1) for v.

Define a vector field on D by

$$F(x, y) = (-u_y(x, y), u_x(x, y)).$$

Since $u \in C^2$, F is C^1. Moreover, its (scalar) curl is

$$\frac{\partial}{\partial x} F_2 - \frac{\partial}{\partial y} F_1 = \frac{\partial}{\partial x}(u_x) - \frac{\partial}{\partial y}(-u_y) = u_{xx} + u_{yy} = \Delta u = 0.$$

Thus F is curl-free on D. Because D is a disk (hence simply connected), a curl-free C^1 vector field on D admits a potential function: there exists a function v on D such that

$$\nabla v = F, \qquad \text{i.e.} \qquad v_x = -u_y, \;\; v_y = u_x.$$

Therefore (4.1) holds, so the Cauchy–Riemann equations are satisfied and $f = u + iv$ is holomorphic on D.

For uniqueness, if v and $\tilde{v}$ both satisfy (4.1), then

$$\nabla(v - \tilde{v}) = (0, 0),$$

so $v - \tilde{v}$ is constant on the connected set D. Hence v is unique up to an additive real constant. □

The disk D can be replaced by any simply-connected domain: Roughly, that is a domain without any hole.

EXAMPLE 4.7 (Failure on a non-simply connected domain). Let $\Omega = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and define

$$u(z) = \ln|z| = \frac{1}{2}\ln(x^2 + y^2).$$

A direct computation shows $\Delta u = 0$ on Ω, so u is harmonic on Ω.

If u had a harmonic conjugate v on Ω, then the function $f = u + iv$ would be holomorphic on Ω, and the Cauchy–Riemann equations would give

$$v_x = -u_y = -\frac{y}{x^2 + y^2}, \qquad v_y = u_x = \frac{x}{x^2 + y^2}.$$

Consider the unit circle $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Along γ we have $x = \cos t$, $y = \sin t$, so

$$\frac{dv}{dt} = v_x \frac{dx}{dt} + v_y \frac{dy}{dt} = \left(-\frac{y}{x^2 + y^2}\right)(-y) + \left(\frac{x}{x^2 + y^2}\right)x = \frac{x^2 + y^2}{x^2 + y^2} = 1.$$

Integrating from 0 to 2π yields

$$v(\gamma(2\pi)) - v(\gamma(0)) = \int_0^{2\pi} \frac{dv}{dt}\, dt = 2\pi.$$

But $\gamma(0) = \gamma(2\pi) = 1$, so the left-hand side must be 0, a contradiction. Therefore $u(z) = \ln|z|$ does *not* admit a harmonic conjugate on $\mathbb{C}^*$.

On the other hand, if we remove a ray to make the domain simply connected, e.g.

$$\Omega' = \mathbb{C} \setminus (-\infty, 0],$$

then u does admit a harmonic conjugate $v = \arg(z)$ on Ω', and $u + iv$ is holomorphic then.

REMARK 4.8. This example is directly related to the fact that the complex logarithm cannot be defined globally as a single-valued holomorphic function on $\mathbb{C}^*$. We will return to this topic later.

5. Power series

A **power series** (centered at 0) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n z^n, \qquad a_n \in \mathbb{C}.$$

It can be viewed as a generalization of polynomials (which correspond to the case $a_n = 0$ for all sufficiently large n).

In this section, we will show that a power series defines a holomorphic function inside its disk of convergence, and that its derivative can be computed term-by-term. In a later chapter, we will prove the converse: every holomorphic function admits a local power series expansion.

Before studying power series, we briefly recall some basic results about series in $\mathbb{C}$. We will skip most details here; Walter Rudin, *Principles of Mathematical Analysis*, is a good reference.

5.1. Basic results for series. In this section we recall some basic facts about infinite series in $\mathbb{C}$.

DEFINITION 5.1. Let $\sum_{n=0}^{\infty} c_n$ be a series with $c_n \in \mathbb{C}$, and define the partial sums

$$S_N = \sum_{n=0}^{N} c_n, \qquad N = 0, 1, 2, \dots .$$

We say that the series $\sum_{n=0}^{\infty} c_n$ **converges** (in $\mathbb{C}$) if the sequence $(S_N)_{N=0,1,2,\dots}$ converges in $\mathbb{C}$. In this case we define

$$\sum_{n=0}^{\infty} c_n := \lim_{N\to\infty} S_N.$$

DEFINITION 5.2. A series $\sum_{n=0}^{\infty} c_n$ is called **absolutely convergent** if the series

$$\sum_{n=0}^{\infty} |c_n|$$

converges (as a series of nonnegative real numbers).

PROPOSITION 5.3 (A necessary condition). *If $\sum_{n=0}^{\infty} c_n$ converges, then $c_n \to 0$ as $n \to \infty$.*

PROOF. If $S_N \to S$, then for $N \geq 1$,

$$c_N = S_N - S_{N-1} \longrightarrow S - S = 0.$$

□

PROPOSITION 5.4 (Cauchy criterion). *A series $\sum_{n=0}^{\infty} c_n$ converges in $\mathbb{C}$ if and only if its sequence of partial sums $(S_N)_{N=0,1,2,\dots}$ is a Cauchy sequence, i.e. for every $\varepsilon > 0$ there exists N such that for all $m > n \geq N$,*

$$\left| \sum_{k=n+1}^{m} c_k \right| = |S_m - S_n| < \varepsilon.$$

PROOF. This follows from the fact that $\mathbb{C}$ is complete: a sequence in $\mathbb{C}$ converges if and only if it is Cauchy. □

THEOREM 5.5 (Absolute convergence implies convergence). *If $\sum_{n=0}^{\infty} |c_n|$ converges, then $\sum_{n=0}^{\infty} c_n$ converges.*

PROOF. Let $S_N = \sum_{n=0}^{N} c_n$. For $m > n$, by the triangle inequality,

$$|S_m - S_n| = \left| \sum_{k=n+1}^{m} c_k \right| \leq \sum_{k=n+1}^{m} |c_k|.$$

Apply Cauchy criterion, $\sum_{n=0}^{\infty} c_n$ converges. □

PROPOSITION 5.6 (Comparison test). *Let $\{c_n\}$ be a sequence of complex numbers.*

(1) *If there exists a sequence of nonnegative real numbers* $\{a_n\}$ *such that*

$$|c_n| \le a_n \quad \text{for all } n,$$

and $\sum_{n=0}^{\infty} a_n$ *converges, then* $\sum_{n=0}^{\infty} c_n$ *absolutely converges.*

(2) *If* $c_n \ge 0$ *for all n and there exists a sequence of nonnegative real numbers* $\{a_n\}$ *such that*

$$c_n \ge a_n \quad \text{for all } n,$$

and $\sum_{n=0}^{\infty} a_n$ *diverges, then* $\sum_{n=0}^{\infty} c_n$ *diverges.*

PROOF. (1) Since $\sum a_n$ converges, for every $\varepsilon > 0$ there exists N such that for all $m > n \ge N$,

$$\sum_{k=n+1}^{m} a_k < \varepsilon.$$

Then by $|c_k| \le a_k$,

$$\left|\sum_{k=n+1}^{m} c_k\right| \le \sum_{k=n+1}^{m} |c_k| \le \sum_{k=n+1}^{m} a_k < \varepsilon.$$

By the Cauchy criterion, $\sum c_n$ absolutely converges.

(2) Let $C_N = \sum_{n=0}^{N} c_n$ and $A_N = \sum_{n=0}^{N} a_n$. If $c_n \ge a_n \ge 0$, then $C_N \ge A_N$ for all N. Since $\sum a_n$ diverges, (A_N) is unbounded, hence so is (C_N), so $\sum c_n$ diverges. □

5.2. An example: Geometric series. The series

$$\sum_{n=0}^{\infty} z^n.$$

is called the geometric series.

Let S_N denote the partial sum, i.e.,

$$S_N = \sum_{n=0}^{N} z^n = 1 + z + \cdots + z^N.$$

They can be calculated out explicitly as follows.

Multiply both sides by $1 - z$, we get

$$(1-z)S_N = (1-z)(1 + z + \cdots + z^N) = 1 - z^{N+1},$$

so if $z \ne 1$,

$$S_N = \frac{1 - z^{N+1}}{1 - z}.$$

If $|z| < 1$, then $|z^{N+1}| = |z|^{N+1} \to 0$ as $N \to \infty$, hence

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad (|z| < 1).$$

If $|z| \ge 1$, then $|z^n| = |z|^n$ which doesn't converge to zero. This shows the series diverges.

As a conclusion,

$$\sum_{n=0}^{\infty} z^n = \begin{cases} \dfrac{1}{1-z}, & |z| < 1, \\ \text{diverges}, & |z| \ge 1. \end{cases}$$

EXAMPLE 5.7. Write down a power series expansion for the rational function $f(z) = \frac{1}{4-z}$, and specify where the expansion is valid, i.e., where the series converges to the given function.

SOLUTION: Rewrite

$$\frac{1}{4-z} = \frac{1}{4} \cdot \frac{1}{1-\frac{z}{4}}.$$

Using the geometric series $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$ for $|w| < 1$, with $w = \frac{z}{4}$, we obtain

$$\frac{1}{4-z} = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{4^{n+1}}.$$

This expansion is valid for $\left|\frac{z}{4}\right| < 1$, i.e. for $|z| < 4$, and on this disk the series converges to $\frac{1}{4-z}$. □

5.3. Radius of convergence. Given a general power series centered at 0: $\sum_{n=0}^{\infty} a_n z^n$, define

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}},$$

with the conventions:

$$R = \infty \text{ if } \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0, \qquad R = 0 \text{ if } \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty.$$

THEOREM 5.8 (Radius of convergence). *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R as above. Then:*

(1) *If $|z| < R$, the series converges absolutely.*
(2) *If $|z| > R$, the series diverges.*

PROOF. (1) Fix z_0 with $|z_0| < R$. By definition of R,

$$|z_0| \cdot \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1.$$

Choose a number r such that

$$|z_0| \cdot \limsup_{n\to\infty} \sqrt[n]{|a_n|} < r < 1.$$

Further, there is some small $\varepsilon > 0$ so that

$$|z_0| \cdot (\limsup_{n\to\infty} \sqrt[n]{|a_n|} + \varepsilon) < r < 1.$$

By the definition of lim sup, there exists N such that for all $n \geq N$,

$$\sqrt[n]{|a_n|} \leq \limsup_{k\to\infty} \sqrt[k]{|a_k|} + \varepsilon.$$

It follows

$$|z_0| \sqrt[n]{|a_n|} < r.$$

Hence for all $n \geq N$,

$$|a_n||z_0|^n \leq r^n.$$

Since $\sum_{n=0}^{\infty} r^n$ converges for $0 \leq r < 1$, by comparison $\sum |a_n||z_0|^n$ converges; i.e. $\sum a_n z_0^n$ converges absolutely.

(2) If $|z_0| > R$, then

$$|z_0| \cdot \limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1.$$

Therefore there exists a subsequence $n_k \to \infty$ such that

$$|z_0| \sqrt[n_k]{|a_{n_k}|} > 1,$$

equivalently,

$$|a_{n_k} z_0^{n_k}| > 1.$$

In particular, $a_n z_0^n \nrightarrow 0$, so the series $\sum a_n z_0^n$ diverges. $\square$

What happens on the boundary $|z| = R$? In general there is no theorem that decides convergence when $|z| = R$; the answer depends on the coefficients. Here are some examples.

EXAMPLE 5.9. The radius of convergence of each of the following power series is $R = 1$.

(1) On $|z| = 1$, we have seen that $\sum_{n=0}^{\infty} z^n$ diverges.
(2) On $|z| = 1$, $\sum_{n=1}^{\infty} nz^n$ diverges since $|nz^n| = n \nrightarrow 0$.
(3) On $|z| = 1$, $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges absolutely, since

$$\sum_{n=1}^{\infty} \left| \frac{z^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.
(4) The situation for $\sum_{n=1}^{\infty} \frac{z^n}{n}$ is more delicate. In fact, it converges for $|z| = 1$ and $z \neq 1$, but diverges at $z = 1$. We explain this below.

To explain the last example, we recall the following test.

PROPOSITION 5.10 (Dirichlet test). *Let (a_n) be a sequence of complex numbers and let*

$$A_N = \sum_{k=0}^{N} a_k$$

be its partial sums. Assume:

(1) *the partial sums $(A_N)_{N=0,1,2,\dots}$ are uniformly bounded, i.e. $|A_N| \le M$ for all N;*
(2) *(b_n) is a real sequence with $b_n > 0$ and $b_n \searrow 0$.*

Then the series $\sum_{n=0}^{\infty} a_n b_n$ converges.

PROOF. Fix $m \le n$. Using $a_k = A_k - A_{k-1}$ (with the convention $A_{-1} = 0$), we have

$$\begin{aligned}
\sum_{k=m}^{n} a_k b_k &= \sum_{k=m}^{n} (A_k - A_{k-1}) b_k \\
&= \sum_{k=m}^{n} A_k b_k - \sum_{k=m}^{n} A_{k-1} b_k \\
&= \sum_{k=m}^{n} A_k b_k - \sum_{k=m-1}^{n-1} A_k b_{k+1} \\
&= A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}).
\end{aligned}$$

Taking absolute values and using $|A_k| \le M$,

$$\begin{aligned}
\left| \sum_{k=m}^{n} a_k b_k \right| &\le |A_n| b_n + |A_{m-1}| b_m + \sum_{k=m}^{n-1} |A_k| (b_k - b_{k+1}) \\
&\le M b_n + M b_m + M \sum_{k=m}^{n-1} (b_k - b_{k+1}).
\end{aligned}$$

Since (b_n) is decreasing, $b_k - b_{k+1} \geq 0$, and the telescoping sum gives

$$\sum_{k=m}^{n-1}(b_k - b_{k+1}) = b_m - b_n.$$

Therefore,

$$\left|\sum_{k=m}^{n} a_k b_k\right| \leq M b_n + M b_m + M(b_m - b_n) \leq 2M b_m.$$

Because $b_m \to 0$ as $m \to \infty$, the tail sums $\sum_{k=m}^{n} a_k b_k$ can be made arbitrarily small uniformly in $n \geq m$. By the Cauchy criterion, $\sum_{n=0}^{\infty} a_n b_n$ converges. □

Now return to the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ on $|z| = 1$. If $|z| = 1$ and $z \neq 1$, set $a_n = z^n$ and $b_n = \frac{1}{n}$. Then

$$A_N = \sum_{k=1}^{N} z^k = \frac{z(1 - z^N)}{1 - z}, \qquad \text{so} \qquad |A_N| \leq \frac{2}{|1 - z|},$$

hence (A_N) is uniformly bounded. Since $b_n \searrow 0$, Dirichlet's test implies that

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

converges for $|z| = 1$ and $z \neq 1$. At $z = 1$, it becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

5.4. Power series are holomorphic on the disk of convergence. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence $R > 0$, so f is a complex valued function defined on the disk $D(0; R) = \{z \in \mathbb{C} : |z| < R\}$. Define a new power series

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

The series defining g is obtained by differentiating the series for f term-by-term. Notice that g has the same radius of convergence as f.

THEOREM 5.11. *The function f is holomorphic on $D(0; R)$, and*

$$f'(z) = g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \qquad |z| < R.$$

PROOF. We prove the case $0 < R < \infty$; the case $R = \infty$ is similar (and easier).

Fix $z_0 \in D(0; R)$. Choose a number r such that

$$|z_0| < r < R.$$

For $N \geq 0$, denote the partial sums and tails by

$$S_N(z) = \sum_{n=0}^{N} a_n z^n, \qquad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n,$$

so that $f = S_N + E_N$. Similarly, define

$$\widetilde{S}_N(z) = \sum_{n=1}^{N} n a_n z^{n-1}, \qquad \widetilde{E}_N(z) = \sum_{n=N+1}^{\infty} n a_n z^{n-1},$$

so that $g = \widetilde{S}_N + \widetilde{E}_N$.

Let $h \to 0$ be such that $z_0 + h \in D(0; r)$ (hence $|z_0 + h| \le r$ for $|h|$ small). Consider

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0).$$

Using $f = S_N + E_N$ and $g = \widetilde{S}_N + \widetilde{E}_N$, we decompose it as

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \underbrace{\left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - \widetilde{S}_N(z_0)\right)}_{(1)} + \underbrace{\frac{E_N(z_0+h) - E_N(z_0)}{h}}_{(2)} + \underbrace{\left(-\widetilde{E}_N(z_0)\right)}_{(3)}.$$

For each fixed N, the function S_N is a polynomial, hence differentiable everywhere, with

$$S_N'(z) = \widetilde{S}_N(z).$$

Therefore, for each fixed N,

$$(1) \longrightarrow 0 \qquad \text{as } h \to 0.$$

Next, since $|z_0| < r < R$ and g has radius of convergence R, we have

$$\sum_{n=1}^{\infty} n|a_n|\, r^{n-1} < \infty.$$

In particular,

$$(3) = \left|\sum_{n=N+1}^{\infty} n a_n z_0^{n-1}\right| \le \sum_{n=N+1}^{\infty} n|a_n|\,|z_0|^{n-1} \le \sum_{n=N+1}^{\infty} n|a_n|\, r^{n-1} \xrightarrow[N\to\infty]{} 0.$$

It remains to estimate (2). Write

$$(2) = \left|\sum_{n=N+1}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h}\right| \le \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right|.$$

Using the factorization

$$(z_0+h)^n - z_0^n = h\Big((z_0+h)^{n-1} + (z_0+h)^{n-2} z_0 + \cdots + z_0^{n-1}\Big),$$

we obtain

$$\left|\frac{(z_0+h)^n - z_0^n}{h}\right| \le |z_0+h|^{n-1} + |z_0+h|^{n-2}|z_0| + \cdots + |z_0|^{n-1} \le n r^{n-1},$$

for $|h|$ sufficiently small (so that $|z_0 + h| \le r$). Hence

$$(2) \le \sum_{n=N+1}^{\infty} n|a_n|\, r^{n-1} \xrightarrow[N\to\infty]{} 0.$$

Now let $\varepsilon > 0$. Choose N large so that both $(2) < \varepsilon/3$ and $(3) < \varepsilon/3$ (uniformly for all sufficiently small h as above). With this N fixed, choose $\delta > 0$ so that $|h| < \delta$ implies $(1) < \varepsilon/3$. Then for $|h| < \delta$,

$$\left|\frac{f(z_0+h) - f(z_0)}{h} - g(z_0)\right| < \varepsilon.$$

Therefore the limit exists and equals $g(z_0)$, i.e. $f'(z_0) = g(z_0)$. Since z_0 was arbitrary, f is holomorphic on $D(0; R)$ and $f' = g$ there. □

COROLLARY 5.12 (Term-by-term differentiation to all orders). *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence $R > 0$*. Then* f *is holomorphic on* $D(0; R)$*, and for every integer* $k \geq 1$*, the k-th derivative* $f^{(k)}$ *exists on* $D(0; R)$ *and can be computed term-by-term:*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} a_n\, n(n-1)\cdots(n-k+1)\, z^{n-k}, \qquad |z| < R.$$

Moreover, each series defining $f^{(k)}$ *has the same radius of convergence* R*.*

PROOF. Apply Theorem 5.11 to f to obtain $f' = \sum_{n\geq 1} n a_n z^{n-1}$ on $D(0; R)$, and note that the derivative series has the same radius of convergence as f. Repeating the argument inductively gives the formula for $f^{(k)}$ and shows the radius of convergence remains R. □

EXAMPLE 5.13 (The exponential function). Define

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The ratio test shows this series converges absolutely for every $z \in \mathbb{C}$:

$$\frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|}{n+1} \longrightarrow 0,$$

so its radius of convergence is $R = \infty$. By term-by-term differentiation,

$$\frac{d}{dz} e^z = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = e^z.$$

EXAMPLE 5.14 (Sine and cosine). Define

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Both series converge absolutely for all $z \in \mathbb{C}$ (again by the ratio test), hence $R = \infty$. Term-by-term differentiation gives the familiar identities

$$(\cos z)' = -\sin z, \qquad (\sin z)' = \cos z.$$

REMARK 5.15 (Euler's identity). Using the power series definitions, one can prove that for every $z \in \mathbb{C}$,

$$e^{iz} = \cos z + i \sin z.$$

Notice, unlike the real-variable case, the functions $\cos z$ and $\sin z$ are *not* bounded on $\mathbb{C}$. Indeed, using Euler's identity $e^{iz} = \cos z + i \sin z$ (and $e^{-iz} = \cos z - i \sin z$), we have

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right).$$

Take $z = iy$ with $y \in \mathbb{R}$. Then $iz = -y$ and $-iz = y$, so

$$\cos(iy) = \frac{1}{2}\left(e^{-y} + e^{y}\right), \qquad \sin(iy) = \frac{1}{2i}\left(e^{-y} - e^{y}\right),$$

and both expressions are unbounded as $|y| \to \infty$.

REMARK 5.16. There are analogous functions, the hyperbolic sine and cosine:

$$\sinh z, \qquad \cosh z.$$

Their power-series definitions and basic identities are completely parallel to those of $\sin z$ and $\cos z$. You will see more details in this week's homework assignment.

EXAMPLE 5.17 (A basic geometric-series expansion). For $|z| < 1$, we have

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^n z^n.$$

This expansion is valid precisely on the disk $|z| < 1$ (it fails on $|z| \geq 1$ since the singularity is at $z = -1$).

EXAMPLE 5.18 (Differentiating the geometric series). Differentiating the previous series term-by-term (valid for $|z| < 1$), we obtain

$$\left(\frac{1}{1+z}\right)' = \sum_{n=1}^{\infty} (-1)^n n z^{n-1}.$$

Since $\left(\frac{1}{1+z}\right)' = -\frac{1}{(1+z)^2}$, we conclude that for $|z| < 1$,

$$\frac{1}{(1+z)^2} = -\sum_{n=1}^{\infty} (-1)^n n z^{n-1} = \sum_{m=0}^{\infty} (-1)^m (m+1) z^m = 1 - 2z + 3z^2 - 4z^3 + \cdots.$$

CHAPTER 3

Integration in the Complex Plane

1. Definition of Complex Integration

1.1. C^1 regular curves. By a **parametrized curve**, we mean a map

$$z : [a, b] \to \mathbb{C}, \qquad t \mapsto z(t).$$

Writing $z(t) = x(t) + iy(t)$, this is equivalent to a pair of real-valued maps

$$x, y : [a, b] \to \mathbb{R}, \qquad x = \Re z, \;\; y = \Im z.$$

DEFINITION 1.1. A parametrized curve $z : [a, b] \to \mathbb{C}$ is called **C^1 regular** if

$$z \in C^1([a, b]; \mathbb{C}) \quad \text{and} \quad z'(t) \neq 0 \;\; \forall t \in [a, b].$$

Equivalently, $x, y \in C^1([a, b]; \mathbb{R})$ and $(x'(t), y'(t)) \neq (0, 0)$ for all t.

DEFINITION 1.2. A parametrized curve $z : [a, b] \to \mathbb{C}$ is called

- **piecewise C^1 regular** if there exists a partition

$$a = a_0 < a_1 < \cdots < a_n = b$$

  such that each restriction $z|_{[a_i, a_{i+1}]}$ is C^1 regular;
- **simple** if z is injective on (a, b);
- a **loop** if $z(a) = z(b)$.

DEFINITION 1.3 (Reparametrization). Let

$$z_k : [a_k, b_k] \to \mathbb{C}, \qquad k = 1, 2,$$

be C^1 regular parametrized curves.

- They are called **orientation-preservingly equivalent** if there exists a C^1 bijection

$$\phi : [a_1, b_1] \to [a_2, b_2]$$

  such that ϕ is strictly increasing (equivalently, $\phi'(t) > 0$ on (a_1, b_1)) and

$$z_1 = z_2 \circ \phi.$$

- They are called **geometrically equivalent** if the same holds with ϕ either strictly increasing or strictly decreasing.

DEFINITION 1.4. An **oriented C^1 regular curve** is an orientation-preserving equivalence class of C^1 regular parametrized curves.

A (non-oriented) **C^1 regular curve** is a geometric equivalence class.

EXAMPLE 1.5. The circle centered at z_0 with radius R can be parametrized by

$$z_1 : [0, 2\pi] \to \mathbb{C}, \qquad z_1(t) = z_0 + Re^{it},$$

also by

$$z_2 : [0, \pi] \to \mathbb{C}, \qquad z_2(t) = z_0 + Re^{2it},$$

and by

$$z_3 : [0, 2\pi] \to \mathbb{C}, \qquad z_3(t) = z_0 + Re^{-it}.$$

The parametrizations z_1 and z_2 are orientation-preservingly equivalent, while z_1 and z_3 are not. (They represent opposite orientations.)

For a C^1 regular curve γ, define its **length** by

$$\mathrm{L}ength(\gamma) := \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt,$$

where $z : [a, b] \to \mathbb{C}$ is any C^1 regular parametrization of γ. We leave it to the reader to check this is independent of parametrization and of orientation.

1.2. Complex integral along a curve. Now let $\gamma \subset \mathbb{C}$ be an oriented C^1 regular curve, and let $f : \gamma \to \mathbb{C}$. We say f is **continuous on** γ if for one (equivalently, every) orientation-compatible C^1 regular parametrization

$$z : [a, b] \to \gamma,$$

the composition

$$f \circ z : [a, b] \to \mathbb{C}$$

is continuous.

DEFINITION 1.6. The **complex integral** of f over γ is

$$\int_\gamma f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt,$$

where $z : [a, b] \to \gamma$ is any orientation-compatible C^1 regular parametrization.

LEMMA 1.7. *The value $\int_\gamma f(z)\, dz$ is independent of the choice of orientation-compatible C^1 regular parametrization.*

PROOF. Assume $z : [a, b] \to \gamma$ and $\tilde{z} := z \circ \phi : [c, d] \to \gamma$ are two orientation-compatible parametrizations, where $\phi : [c, d] \to [a, b]$ is C^1, bijective, and strictly increasing. Then

$$\begin{aligned}
\int_c^d f(\tilde{z}(s))\tilde{z}'(s)\, ds &= \int_c^d f(z(\phi(s)))\, z'(\phi(s))\, \phi'(s)\, ds \\
&= \int_{\phi(c)}^{\phi(d)} f(z(t))\, z'(t)\, dt \qquad (t = \phi(s)) \\
&= \int_a^b f(z(t))\, z'(t)\, dt.
\end{aligned}$$

So the definition is well-defined. $\square$

EXAMPLE 1.8. Let C_R^+ be the counterclockwise oriented circle centered at 0 with radius R.

(1) For $f(z) = z$, using $z(t) = Re^{it}$, $t \in [0, 2\pi]$:

$$\int_{C_R^+} z\,dz = \int_0^{2\pi} z(t)z'(t)\,dt = \int_0^{2\pi} Re^{it} \cdot (iRe^{it})\,dt$$
$$= iR^2 \int_0^{2\pi} e^{2it}\,dt = 0.$$

(2) For $f(z) = 1/z$, again with $z(t) = Re^{it}$:

$$\int_{C_R^+} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{Re^{it}} \cdot iRe^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

In fact,

$$\frac{1}{2\pi i}\int_{C_R^+} \frac{1}{z}\,dz = 1,$$

which equals the winding number of C_R^+ about 0.

1.3. Basic properties of complex integrals. Complex integration of continuous functions along oriented C^1 regular curves has the following basic properties.

PROPOSITION 1.9. *Assume γ is an oriented C^1 regular curve, and f, g are continuous on γ. Then:*

(1) *For any $\alpha, \beta \in \mathbb{C}$,*

$$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha\int_\gamma f(z)\,dz + \beta\int_\gamma g(z)\,dz.$$

(2)

$$\int_{\gamma^-} f(z)\,dz = -\int_\gamma f(z)\,dz,$$

where γ^- denotes γ with reversed orientation.

(3)

$$\left|\int_\gamma f(z)\,dz\right| \le \sup_{z\in\gamma} |f(z)| \cdot Length(\gamma).$$

PROOF. (1) Let $z : [a, b] \to \gamma$ be an orientation-compatible parametrization. Then

$$\int_\gamma (\alpha f + \beta g)\,dz = \int_a^b (\alpha f(z(t)) + \beta g(z(t)))z'(t)\,dt$$
$$= \alpha\int_a^b f(z(t))z'(t)\,dt + \beta\int_a^b g(z(t))z'(t)\,dt$$
$$= \alpha\int_\gamma f(z)\,dz + \beta\int_\gamma g(z)\,dz.$$

(2) If $z : [a,b] \to \gamma$, define $z^-(t) := z(a+b-t)$, a parametrization of γ^-. Then

$$\begin{aligned}
\int_{\gamma^-} f(z)\,dz &= \int_a^b f(z^-(t))(z^-)'(t)\,dt \\
&= \int_a^b f(z(a+b-t))\,z'(a+b-t)\,(-1)\,dt \\
&= \int_b^a f(z(s))z'(s)\,ds \qquad (s = a+b-t) \\
&= -\int_a^b f(z(s))z'(s)\,ds \\
&= -\int_\gamma f(z)\,dz.
\end{aligned}$$

(3) Using the triangle inequality and the definition of length:

$$\begin{aligned}
\left|\int_\gamma f(z)\,dz\right| &= \left|\int_a^b f(z(t))z'(t)\,dt\right| \\
&\le \int_a^b |f(z(t))||z'(t)|\,dt \\
&\le \sup_{z\in\gamma} |f(z)| \int_a^b |z'(t)|\,dt \\
&= \sup_{z\in\gamma} |f(z)| \cdot Length(\gamma).
\end{aligned}$$

Here, the first inequality is obtained by applying the triangle inequality to the Riemann sums and then taking limits.

□

If γ is oriented piecewise C^1 regular, define

$$\int_\gamma f(z)\,dz$$

as the sum of integrals over smooth pieces. This is independent of the chosen piecewise decomposition and of the chosen compatible parametrizations on each piece.

PROPOSITION 1.10. *Assume γ is an oriented piecewise C^1 regular curve, and f, g are continuous on γ. Then all three statements in Proposition 1.9 remain valid.*

2. Primitives

In real-variable calculus, definite integrals can be computed using antiderivatives via the Fundamental Theorem of Calculus. In complex analysis, there is an analogous statement.

DEFINITION 2.1. Let $f : \Omega \to \mathbb{C}$ be a complex-valued function on an open set $\Omega \subset \mathbb{C}$. A **primitive** of f on Ω is a holomorphic function F on Ω such that

$$F'(z) = f(z) \qquad (z \in \Omega).$$

EXAMPLE 2.2. (1) The function $f(z) = z$ has a primitive $F(z) = \frac{1}{2}z^2$ on $\mathbb{C}$. More generally, if $c \in \mathbb{C}$, then $F(z) + c$ is also a primitive of f.

(2) A polynomial
$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \quad (a_n \neq 0)$$
has a primitive
$$F(z) = a_0 z + \frac{a_1}{2} z^2 + \cdots + \frac{a_n}{n+1} z^{n+1}.$$
For any constant $c \in \mathbb{C}$, $F(z) + c$ is also a primitive.

(3) A power series
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
with radius of convergence $R > 0$ has a primitive on $D(0; R)$, namely
$$F(z) = c + \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}, \qquad c \in \mathbb{C}.$$

(4) In real calculus, $\frac{d}{dx} \ln x = \frac{1}{x}$ on $(0, \infty)$. In complex analysis, this remains true only *locally*: one cannot define a single-valued holomorphic logarithm on all of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Equivalently, $\frac{1}{z}$ has no primitive on $\mathbb{C}^*$.

PROPOSITION 2.3. *Assume f is continuous on an open set $\Omega \subset \mathbb{C}$, and f has a primitive F on Ω. Let γ be an oriented piecewise C^1 regular curve in Ω, starting at z_0 and ending at z_1. Then*
$$\int_\gamma f(z)\, dz = F(z_1) - F(z_0).$$
In particular, if γ is a loop, then $\int_\gamma f(z)\, dz = 0$.

PROOF. The key ingredient is the Fundamental Theorem of Calculus.

First assume γ is C^1 regular, and let
$$z : [a, b] \to \Omega$$
be a C^1 regular parametrization of γ, with $z(a) = z_0$, $z(b) = z_1$. Then
$$\begin{aligned} \int_\gamma f(z)\, dz &= \int_a^b f(z(t))\, z'(t)\, dt \\ &= \int_a^b F'(z(t))\, z'(t)\, dt. \end{aligned}$$
By the chain rule,
$$\frac{d}{dt} F(z(t)) = F'(z(t))\, z'(t).$$
Hence
$$\begin{aligned} \int_a^b F'(z(t))\, z'(t)\, dt &= \int_a^b \frac{d}{dt} F(z(t))\, dt \\ &= F(z(b)) - F(z(a)) \\ &= F(z_1) - F(z_0), \end{aligned}$$
by the Fundamental Theorem of Calculus.

Now assume γ is piecewise C^1 regular. Take a partition
$$a = t_0 < t_1 < \cdots < t_n = b$$

such that each restriction $z|_{[t_i,t_{i+1}]}$ is C^1 regular. Applying the C^1 case on each subinterval,

$$\int_\gamma f(z)\,dz = \sum_{i=0}^{n-1}\big(F(z(t_{i+1})) - F(z(t_i))\big) = F(z(b)) - F(z(a)) = F(z_1) - F(z_0).$$

This completes the proof. □

EXAMPLE 2.4. Now we can apply Proposition 2.3 to see the function $f(z) = \frac{1}{z}$ has no primitive on $\Omega = \mathbb{C}^*$: Take $\gamma = C_1^+$, counterclockwise unit circle, then

$$\int_{C_1^+} \frac{1}{z} dz = 2\pi i \neq 0.$$

Now let's prove the result that all primitives of a given function f on a domain Ω can be obtained from one primitive by shifting constants.

COROLLARY 2.5. *Assume f is a complex valued function on a domain (i.e., a connected open set) Ω. If F is a primitive of f, then every primitive of f is*

$$F + c, \quad \text{with some constant } c \in \mathbb{C}.$$

In particular, all primitives of the zero function are constant functions.

PROOF. If F_1, F_2 are both primitives of f, we consider the function

$$F = F_1 - F_2.$$

Notice $F' = f - f = 0$, It is enough to show primitives of the zero function are constant functions.

To see this, take an arbitrary point $z_0 \in \Omega$ and consider the set

$$A := \{z \in \Omega | F(z) = F(z_0)\}.$$

Obviously A is not empty since $z_0 \in A$.

The set A is closed in Ω because A is the preimage of the closed set $\{F(z_0)\}$ in $\mathbb{C}$ for the continuous function $F : \Omega \to \mathbb{C}$.

Now let's use Proposition 2.3 to show A is also open in Ω, by showing every $w_0 \in A$ is an interior point of A.

First because Ω is open, there is $r > 0$ so that the disk

$$D(w_0; r) \subset \Omega.$$

Then for any point w in the disk, consider the complex integral of F' over the oriented line segment $\vec{w_0 w}$ (check rigorously this line segment stays inside $D(w_0; r)$):

$$\int_{\vec{w_0 w}} F'(z)dz = F(w) - F(w_0).$$

Since $F' = 0$, it follows $F(w) = F(w_0)$, which proves $D(w_0; r) \subset A$.

Now since Ω is assumed to be connected, a nonempty open and closed subset in Ω must be Ω itself. Thus $A = \Omega$. Equivalently, for any $z \in \Omega$, $F(z) = F(z_0)$, i.e., a constant. □

REMARK 2.6. It is a general topological result (see Theorem 6.15) that for open sets in $\mathbb{C}$, connectedness and path-connectedness are exactly the same.

Given any point $z \in \Omega$, if there is an oriented piecewise C^1 regular curve in Ω joining z_0 and z, then

$$F(z) - F(z_0) = \int_\gamma F'(w)dw = 0,$$

and this will also finish the proof.

However, path-connectedness only gives continuous paths. To apply complex integration, we need piecewise C^1 regular curves; obtaining them from continuous paths is standard but technical (via compactness), so we omit it and use the much simpler argument via connectedness here.

3. Cauchy's theorem

The central result in complex function theory is Cauchy's theorem. Roughly speaking, it says that on a domain with no holes (this is the intuitive meaning of simply connectedness), the integral of a holomorphic function along any closed piecewise C^1 curve is zero.

In this section we introduce this theorem, and first explain why such a statement is natural to expect.

3.1. A revisit of Green's theorem. Let $\Omega \subset \mathbb{C}$ be open, and let $D \subset \mathbb{C}$ be a bounded domain with no holes (i.e. simply connected), such that $\overline{D} \subset \Omega$. Assume ∂D is piecewise C^1.

The boundary ∂D carries a canonical orientation induced by D: namely, the positive orientation for which D always lies on the left as one traverses ∂D.

Let f be holomorphic on Ω. Write

$$f = u + iv, \qquad u, v : \Omega \to \mathbb{R}.$$

For the moment, assume that u and v are C^1 on Ω, which will allow us to apply Green's theorem.

We now compute $\int_{\partial D} f(z)\,dz$ in terms of u and v. Let $z(t) = x(t) + iy(t)$. Then

$$\begin{aligned} f(z(t))z'(t) &= (u(z(t)) + iv(z(t)))(x'(t) + iy'(t)) \\ &= \big(u(z(t))x'(t) - v(z(t))y'(t)\big) + i\big(u(z(t))y'(t) + v(z(t))x'(t)\big), \end{aligned}$$

on each C^1 regular piece.

Hence,

$$\begin{aligned} \int_{\partial D} f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\ &= \int_a^b \big(u(z(t))x'(t) - v(z(t))y'(t)\big)\,dt + i\int_a^b \big(u(z(t))y'(t) + v(z(t))x'(t)\big)\,dt \\ &= \int_{\partial D} (u\,dx - v\,dy) + i\int_{\partial D} (v\,dx + u\,dy). \end{aligned}$$

Applying Green's theorem to the two real line integrals, we get

$$\begin{aligned} \int_{\partial D} (u\,dx - v\,dy) &= -\iint_D \big(v_x + u_y\big)\ dx\,dy, \\ \int_{\partial D} (v\,dx + u\,dy) &= \iint_D \big(u_x - v_y\big)\ dx\,dy. \end{aligned}$$

Therefore

$$\int_{\partial D} f(z)\,dz = -\iint_D (v_x + u_y)\,dx\,dy + i\iint_D (u_x - v_y)\,dx\,dy.$$

Now the Cauchy–Riemann equations $u_x = v_y$, $u_y = -v_x$ imply both integrands vanish, so

$$\int_{\partial D} f(z)\,dz = 0.$$

In summary, under the additional assumption that $u, v \in C^1$, Green's theorem shows that the integral of a holomorphic function over ∂D is zero whenever f is holomorphic on a neighborhood of $\overline{D}$.

This already gives a very powerful computational consequence: many contour integrals of concrete holomorphic functions vanish immediately. However, as a foundational method it has a limitation: at this stage we have not yet proved that, for a general holomorphic f, the functions u, v are C^1.

This motivates Cauchy's integral theorem in full generality: it proves the same vanishing result for holomorphic functions without requiring an a priori C^1-assumption on u and v, and without relying on Green's theorem.

3.2. Cauchy's theorem in star shaped domain and Goursat's theorem. We recall the following definitions.

DEFINITION 3.1. A subset $D \subset \mathbb{C}$ is called **convex** if for any two points $z_0, z_1 \in D$, the line segment

$$\{(1-t)z_0 + tz_1 \, : \, 0 \le t \le 1\}$$

is contained in D.

A subset $D \subset \mathbb{C}$ is called **star-shaped** if there exists a point $z_* \in D$ such that for every $z \in D$, the line segment

$$\{(1-t)z_* + tz \, : \, 0 \le t \le 1\}$$

is contained in D.

Every disk is convex. Any convex set is star-shaped. The converse is false. For example, a five-point star region is star-shaped but not convex.

THEOREM 3.2. *[Cauchy's theorem for star-shaped domains] Let $D \subset \mathbb{C}$ be an open star-shaped domain, and let $f \in \mathcal{O}(D)$. Then f admits a primitive on D. Consequently, for every oriented piecewise C^1 regular loop γ in D,*

$$\int_\gamma f(z)\, dz = 0.$$

The key step in the proof is the following theorem of Goursat.

THEOREM 3.3 (Goursat's theorem). *Let $\Omega \subset \mathbb{C}$ be open, and let $T \subset \Omega$ be a closed triangle (i.e. the union of the triangle and its boundary ∂T). If $f \in \mathcal{O}(\Omega)$, then*

$$\int_{\partial T} f(z)\, dz = 0.$$

We postpone the proof of Goursat's theorem to the next subsection, and now explain how it implies Cauchy's theorem on a star-shaped domain.

PROOF OF CAUCHY'S THEOREM USING GOURSAT'S THEOREM. Since D is star-shaped, there is a point $z_0 \in D$ so that each $z \in D$ the line segment from z_0 to z, denoted by $\gamma_{z_0,z}$, lies in D.

Motivated by Proposition 2.3, define

$$F(z) := \int_{\gamma_{z_0,z}} f(w)\, dw, \qquad z \in D.$$

We show $F'(z) = f(z)$ for every $z \in D$.

Take $z \in D$, and let $h \in \mathbb{C}$ be small enough so that $z + h \in D$. Then

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left(\int_{\gamma_{z_0,z+h}} f(w)\,dw - \int_{\gamma_{z_0,z}} f(w)\,dw\right).$$

Let δ_h be the line segment from z to $z + h$. The closed path

$$\Gamma_h := \gamma^-_{z_0,z+h} \cup \gamma_{z_0,z} \cup \delta_h$$

is the boundary of a triangle contained in D. Hence, by Goursat's theorem,

$$\int_{\Gamma_h} f(w)\,dw = 0 \quad \Longrightarrow \quad \int_{\gamma_{z_0,z+h}} f(w)\,dw - \int_{\gamma_{z_0,z}} f(w)\,dw = \int_{\delta_h} f(w)\,dw.$$

Therefore

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h}\int_{\delta_h} f(w)\,dw.$$

Now subtract $f(z)$:

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} - f(z) &= \frac{1}{h}\int_{\delta_h} f(w)\,dw - \frac{1}{h}\int_{\delta_h} f(z)\,dw \\ &= \frac{1}{h}\int_{\delta_h} \big(f(w) - f(z)\big)\,dw.\end{aligned}$$

Using the estimate for contour integrals,

$$\begin{aligned}\left|\frac{1}{h}\int_{\delta_h} (f(w) - f(z))\,dw\right| &\le \frac{1}{|h|} \sup_{w\in\delta_h} |f(w) - f(z)| \cdot Length(\delta_h) \\ &= \sup_{w\in\delta_h} |f(w) - f(z)|.\end{aligned}$$

As $h \to 0$, the segment δ_h shrinks to $\{z\}$, and continuity of f at z gives

$$\sup_{w\in\delta_h} |f(w) - f(z)| \to 0.$$

Hence

$$\lim_{h\to 0}\left(\frac{F(z+h) - F(z)}{h} - f(z)\right) = 0,$$

so $F'(z) = f(z)$. Since $z \in D$ was arbitrary, F is a primitive of f on D.

By Proposition 2.3, for every oriented piecewise C^1 regular loop $\gamma \subset D$,

$$\int_\gamma f(z)\,dz = 0.$$

$\square$

REMARK 3.4. In the construction of the primitive $F(z) = \int_{\gamma_{z_0,z}} f(w)\,dw$ and the proof that $F'(z) = f(z)$, the only inputs are the continuity of f and the vanishing of $\int_{\partial T} f(z)\,dz$ for the triangular loops that appear in the argument. In particular, *once* the triangular-loop integrals are known to be zero, the holomorphicity of f is not used further in this step.

Of course, holomorphicity is essential is Cauchy's theorem, through Goursat's theorem, to guarantee that these triangular integrals vanish in the first place, as we will see next.

3.3. Proof of Goursat's theorem.

PROOF OF THEOREM 3.3. Let $T \subset \Omega$ be a closed triangle, and orient ∂T counterclockwise. Bisect each side of T. This decomposes T into four closed subtriangles

$$T_1, T_2, T_3, T_4,$$

each with the induced counterclockwise boundary orientation. Interior edges cancel in pairs, so

$$\int_{\partial T} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial T_j} f(z)\,dz.$$

Hence there exists $n_1 \in \{1,2,3,4\}$ such that

$$\left| \int_{\partial T^1_{n_1}} f(z)\,dz \right| = \max_{1\le j\le 4} \left| \int_{\partial T_j} f(z)\,dz \right|,$$

and therefore

$$\left| \int_{\partial T} f(z)\,dz \right| \le 4 \left| \int_{\partial T^1_{n_1}} f(z)\,dz \right|.$$

Repeat this construction on $T^1_{n_1}$: divide it into four subtriangles and pick $T^2_{n_2}$ with maximal boundary-integral modulus. Iterating, we obtain nested closed triangles

$$T^1_{n_1} \supset T^2_{n_2} \supset \cdots \supset T^k_{n_k} \supset \cdots$$

such that

$$\left| \int_{\partial T} f(z)\,dz \right| \le 4^k \left| \int_{\partial T^k_{n_k}} f(z)\,dz \right|, \qquad \operatorname{diam}\left(T^k_{n_k}\right) = 2^{-k}\operatorname{diam}(T),$$

where $\operatorname{diam}(T) := \max_{z_1,z_2\in T} |z_1 - z_2|$.

We now use the following general lemma to understand the intersection of the nested triangles.

LEMMA 3.5 (Nested compact sets). *Let*

$$K_1 \supset K_2 \supset \cdots \supset K_k \supset \cdots$$

be a nested sequence of nonempty compact subsets of $\mathbb{C}$, *then*

$$\bigcap_{k=1}^{\infty} K_k \neq \emptyset.$$

If further,

$$\operatorname{diam}(K_k) := \sup\{|z-w| : z, w \in K_k\} \longrightarrow 0 \quad (k \to \infty).$$

Then there exists a unique $z_0 \in \mathbb{C}$ *such that*

$$\bigcap_{k=1}^{\infty} K_k = \{z_0\}.$$

In fact, for the first conclusion

$$\bigcap_{k=1}^{\infty} K_k \neq \emptyset,$$

there is a purely topological proof using compactness (via the finite intersection property): if every finite intersection is nonempty, then the whole intersection is nonempty. We leave this standard proof to interested readers.

Here we present a direct metric proof for both results under the additional assumption $\mathrm{diam}(K_k) \to 0$, using sequences and completeness of $\mathbb{C}$.

PROOF. Since each $K_k \neq \emptyset$, choose $z_k \in K_k$ for each $k \geq 1$.

Step 1: (z_k) is Cauchy. Let $\varepsilon > 0$. Since $\mathrm{diam}(K_k) \to 0$, there exists N such that

$$\mathrm{diam}(K_N) < \varepsilon.$$

For any $m, n \geq N$, by nestedness we have $K_m \subset K_N$ and $K_n \subset K_N$, hence $z_m, z_n \in K_N$. Therefore

$$|z_m - z_n| \leq \mathrm{diam}(K_N) < \varepsilon.$$

So (z_k) is Cauchy in $\mathbb{C}$.

Because $\mathbb{C}$ is complete, there exists $z_0 \in \mathbb{C}$ such that

$$z_k \to z_0.$$

Step 2: $z_0 \in \bigcap_{k=1}^{\infty} K_k$. Fix $j \in \mathbb{N}$. For every $k \geq j$, nestedness gives $K_k \subset K_j$, so $z_k \in K_j$. Hence the tail $(z_k)_{k\geq j}$ lies in K_j. Since K_j is compact (thus closed) and $z_k \to z_0$, we conclude $z_0 \in K_j$.

As j is arbitrary, $z_0 \in K_j$ for all j, so

$$z_0 \in \bigcap_{k=1}^{\infty} K_k.$$

In particular, the intersection is nonempty.

Step 3: Uniqueness. Let $w \in \bigcap_{k=1}^{\infty} K_k$. Then for every k, both $z_0, w \in K_k$, hence

$$|z_0 - w| \leq \mathrm{diam}(K_k).$$

Let $k \to \infty$. Since $\mathrm{diam}(K_k) \to 0$, we get $|z_0 - w| = 0$, i.e. $w = z_0$.

Therefore

$$\bigcap_{k=1}^{\infty} K_k = \{z_0\}.$$

□

Apply this lemma to the nested triangles, we have

$$\bigcap_{k=1}^{\infty} T_{n_k}^{k} = \{z_0\}$$

for some $z_0 \in T \subset \Omega$.

Since f is complex differentiable at z_0, define

$$\eta(z) := \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0), & z \neq z_0, \\ 0, & z = z_0, \end{cases}$$

then $\eta(z) \to 0$ as $z \to z_0$, and

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\eta(z).$$

Thus

$$\int_{\partial T_{n_k}^k} f(z)\,dz = \int_{\partial T_{n_k}^k} f(z_0)dz + \int_{\partial T_{n_k}^k} f'(z_0)(z-z_0)\,dz + \int_{\partial T_{n_k}^k} (z-z_0)\eta(z)\,dz.$$

The first two integrals vanish, since their integrands admit primitives. It follows

$$\left|\int_{\partial T} f(z)\,dz\right| \le 4^k \left|\int_{\partial T_{n_k}^k} (z-z_0)\eta(z)\,dz\right|.$$

Using the third property in Proposition 1.9, we have

$$\begin{aligned}\left|\int_{\partial T} f(z)\,dz\right| &\le 4^k \sup_{z\in\partial T_{n_k}^k} |(z-z_0)\eta(z)|\, Length(\partial T_{n_k}^k)\\ &\le 4^k \sup_{z\in\partial T_{n_k}^k} |\eta(z)|\, \mathrm{diam}(T_{n_k}^k)\, Length(\partial T_{n_k}^k).\end{aligned}$$

Now

$$\mathrm{diam}(T_{n_k}^k) = 2^{-k}\mathrm{diam}(T), \qquad Length(\partial T_{n_k}^k) = 2^{-k} Length(\partial T),$$

hence

$$\left|\int_{\partial T} f(z)\,dz\right| \le \mathrm{diam}(T)\, Length(\partial T) \sup_{z\in\partial T_{n_k}^k} |\eta(z)|.$$

Because $\mathrm{diam}(T_{n_k}^k) \to 0$ and $\bigcap_k T_{n_k}^k = \{z_0\}$, the boundaries $\partial T_{n_k}^k$ lie arbitrarily close to z_0; therefore

$$\sup_{z\in\partial T_{n_k}^k} |\eta(z)| \longrightarrow 0.$$

Letting $k\to\infty$, we get

$$\int_{\partial T} f(z)\,dz = 0.$$

This proves Goursat's theorem. □

3.4. First applications of Cauchy's theorem. We now see an immediate consequence of Cauchy's theorem.

EXAMPLE 3.6. Compute

$$\int_{\partial D(i;3)} \left(e^z + z^2 + \sin z\right) dz.$$

SOLUTION. Let

$$f(z) = e^z + z^2 + \sin z.$$

Then f is entire, hence holomorphic on $\mathbb{C}$. The curve $\partial D(i;3)$ is an oriented piecewise C^1 regular loop in $\mathbb{C}$. Since $\mathbb{C}$ is star-shaped (in particular, a domain to which Cauchy's theorem applies), we get

$$\int_{\partial D(i;3)} f(z)\,dz = 0.$$

□

EXAMPLE 3.7. Compute

$$\int_{\partial D(i;3)} z^n\,dz, \qquad n\in\mathbb{Z}.$$

SOLUTION. We distinguish two cases.

(1) **Case** $n = 0, 1, 2, \ldots$.

Then z^n is entire on $\mathbb{C}$. Since $\partial D(i;3)$ is a piecewise C^1 loop in $\mathbb{C}$, Cauchy's theorem gives

$$\int_{\partial D(i;3)} z^n \, dz = 0.$$

(2) **Case** $n = -1, -2, \ldots$.

Fix $r > 0$ (for example $r = 1$) such that

$$\overline{D(0;r)} \subset D(i;3).$$

Let

$$A := \overline{D(i;3)} \setminus D(0;r).$$

Choose $\varepsilon > 0$ small and define four open half-planes

$$H_1 := \{z : \Re z > \varepsilon\}, \quad H_2 := \{z : \Im z > \varepsilon\}, \quad H_3 := \{z : \Re z < -\varepsilon\}, \quad H_4 := \{z : \Im z < -\varepsilon\}.$$

Each H_j is open and convex, and $0 \notin H_j$, hence z^n is holomorphic on each H_j.

Now cut A by the two lines $\Im z = \Re z$ and $\Im z = -\Re z$, obtaining four regions

$$A = A_1 \cup A_2 \cup A_3 \cup A_4, \qquad A_j \subset H_j \ (j = 1, 2, 3, 4),$$

where each A_j has piecewise C^1 boundary. Therefore, by Cauchy's theorem on each H_j,

$$\int_{\partial A_j} z^n \, dz = 0, \qquad j = 1, 2, 3, 4.$$

Summing over j, all interior cut-edges cancel in opposite orientations, so

$$\int_{\partial A} z^n \, dz = \sum_{j=1}^{4} \int_{\partial A_j} z^n \, dz = 0.$$

With the boundary orientation of A, this is

$$\int_{\partial D(i;3)} z^n \, dz - \int_{\partial D(0;r)} z^n \, dz = 0,$$

hence

$$\int_{\partial D(i;3)} z^n \, dz = \int_{\partial D(0;r)} z^n \, dz.$$

Now evaluate the inner circle as we have seen

- If $n = -1$, then

$$\int_{\partial D(0;r)} \frac{1}{z} \, dz = 2\pi i,$$

so

$$\int_{\partial D(i;3)} z^{-1} \, dz = 2\pi i.$$

- If $n \le -2$, since z^n has primitive,

$$\int_{\partial D(0;r)} z^n \, dz = 0,$$

and therefore

$$\int_{\partial D(i;3)} z^n \, dz = 0.$$

So in summary:

$$\int_{\partial D(i;3)} z^n\,dz = \begin{cases} 2\pi i, & n=-1, \\ 0, & n\neq -1. \end{cases}$$

$\square$

The Jordan curve theorem states that a simple loop in $\mathbb{C} \cong \mathbb{R}^2$ separates the plane into exactly two connected components: a bounded component (called the interior) and an unbounded component (called the exterior), and the curve is the common boundary of these two components. This theorem is surprisingly difficult to prove; we refer the reader to Munkres' Topology for a proof.

Now the above example can be generalized.

PROPOSITION 3.8 (Same integral on two nested loops). *Let γ_1, γ_2 be positively oriented piecewise C^1 regular simple loops in $\mathbb{C}$, and assume*

$$\overline{\mathrm{Int}(\gamma_1)} \subset \mathrm{Int}(\gamma_2).$$

Define

$$A := \mathrm{Int}(\gamma_2) \setminus \overline{\mathrm{Int}(\gamma_1)}.$$

If f is holomorphic on an open neighborhood of $\overline{A}$, then

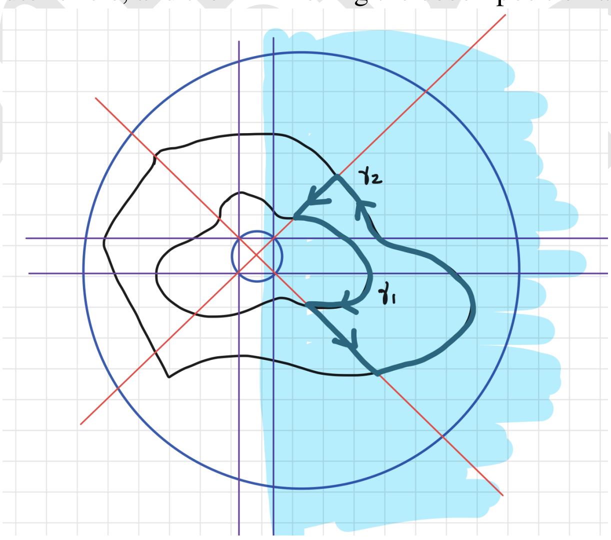
$$\int_{\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Equivalently,

$$\int_{\partial A} f(z)\,dz = 0,$$

where ∂A carries the boundary orientation induced from A.

We omit the details of the proof. The figure below illustrates a proof by choosing an inner circle and an outer circle, and then mimicking the decomposition argument in the previous example.



REMARK 3.9. The proposition above is a first geometric form of deformation invariance. Later we will extend it further: the assumption that the curves are simple can be removed. More generally, for a holomorphic function on a domain, the integral over a closed piecewise C^1 loop depends only on its homotopy class (with fixed domain). In particular, homotopic loops have the same integral.

EXAMPLE 3.10. Compute

$$\int_{\partial D(i;3)} \frac{1}{z^2 - 3z}\, dz.$$

SOLUTION. Set

$$f(z) = \frac{1}{z^2 - 3z} = \frac{1}{z(z-3)} = \frac{1}{3}\Big(\frac{1}{z-3} - \frac{1}{z}\Big).$$

The poles are at $z = 0$ and $z = 3$. Now

$$|0 - i| = 1 < 3, \qquad |3 - i| = \sqrt{10} > 3,$$

so $0 \in D(i;3)$ while $3 \notin \overline{D(i;3)}$.

Hence $\frac{1}{z-3}$ is holomorphic on a neighborhood of $\overline{D(i;3)}$, so by Cauchy's theorem,

$$\int_{\partial D(i;3)} \frac{1}{z-3}\, dz = 0.$$

Also, as in the previous example,

$$\int_{\partial D(i;3)} \frac{1}{z}\, dz = 2\pi i$$

(since the loop winds once counterclockwise around 0).

Therefore

$$\begin{aligned}\int_{\partial D(i;3)} \frac{1}{z^2 - 3z}\, dz &= \frac{1}{3}\int_{\partial D(i;3)} \frac{1}{z-3}\, dz - \frac{1}{3}\int_{\partial D(i;3)} \frac{1}{z}\, dz \\ &= 0 - \frac{1}{3}(2\pi i) = -\frac{2\pi i}{3}.\end{aligned}$$

□

EXAMPLE 3.11. Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

have radius of convergence $R > 0$, and let γ be a positively oriented piecewise C^1 regular simple loop with

$$\overline{\mathrm{Int}(\gamma)} \subset D(z_0; R), \qquad z_0 \in \mathrm{Int}(\gamma).$$

For $m \in \mathbb{Z}_{\geq 0}$, set

$$I_m := \int_\gamma \frac{f(z)}{(z - z_0)^m}\, dz.$$

If $m = 0$, then f is holomorphic on a neighborhood of $\overline{\mathrm{Int}(\gamma)}$, so by Cauchy's theorem

$$I_0 = 0.$$

Now let $m \geq 1$. Split

$$f(z) = \sum_{n=0}^{m-2} a_n (z - z_0)^n + a_{m-1}(z - z_0)^{m-1} + \sum_{n=m}^{\infty} a_n (z - z_0)^n,$$

hence

$$\frac{f(z)}{(z - z_0)^m} = \underbrace{\sum_{n=0}^{m-2} a_n (z - z_0)^{n-m}}_{H_1(z)} + \frac{a_{m-1}}{z - z_0} + \underbrace{\sum_{n=m}^{\infty} a_n (z - z_0)^{n-m}}_{H_2(z)}.$$

So

$$I_m = \int_\gamma H_1(z)\, dz + a_{m-1}\int_\gamma \frac{dz}{z - z_0} + \int_\gamma H_2(z)\, dz.$$

Here $\int_\gamma H_1(z)\,dz = 0$ (each term has a primitive on $\mathbb{C} \setminus \{z_0\}$), and $\int_\gamma H_2(z)\,dz = 0$ (holomorphic near $\overline{\mathrm{Int}(\gamma)}$, Cauchy theorem). Also

$$\int_\gamma \frac{dz}{z - z_0} = 2\pi i.$$

Therefore

$$I_m = 2\pi i\, a_{m-1} \qquad (m \geq 1).$$

Notice $a_{m-1} = \dfrac{f^{(m-1)}(z_0)}{(m-1)!}$, we obtain

$$I_0 = 0, \qquad I_m = 2\pi i\, \frac{f^{(m-1)}(z_0)}{(m-1)!} \quad (m \geq 1).$$

This example motivates Cauchy's integral formula which we will study in later chapters.

Homotopy of paths and multivalued functions

1. Homotopy in topology

Homotopy is a topological notion that will play a useful role in complex analysis, perhaps a bit unexpectedly.

DEFINITION 1.1 (Homotopy rel. A). Let X, Y be topological spaces and let $A \subset X$. We say that two continuous maps $\gamma_0, \gamma_1 : X \to Y$ are **homotopic relative to** A if there exists a continuous map

$$H : X \times [0,1] \to Y$$

such that

$$H(x,0) = \gamma_0(x), \qquad H(x,1) = \gamma_1(x) \quad (\forall x \in X),$$

and moreover

$$H(a,s) = \gamma_0(a) = \gamma_1(a) \qquad (\forall a \in A,\ \forall s \in [0,1]).$$

We write $\gamma_0 \simeq \gamma_1$ rel A.

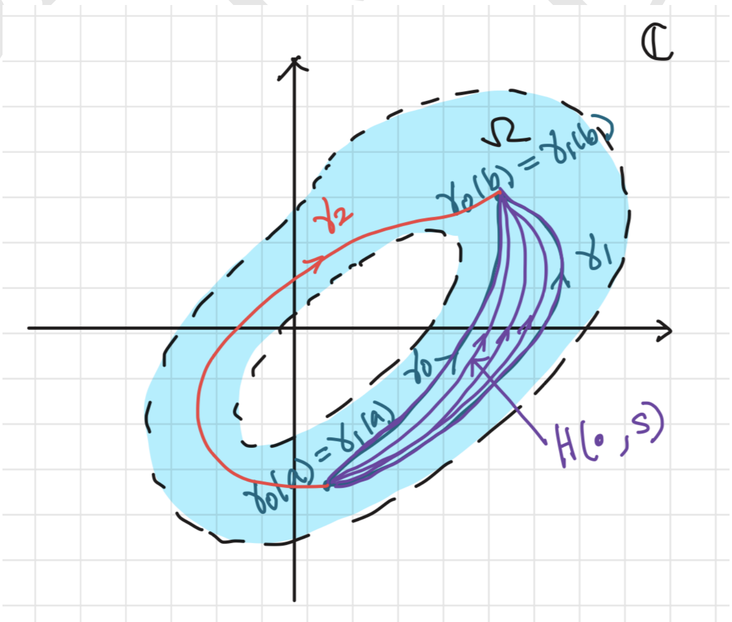
When $A = \emptyset$, we simply say γ_0 and γ_1 are **homotopic** and write $\gamma_0 \simeq \gamma_1$.

For our purposes in complex analysis, we will take $X = [a,b]$ with the subspace topology from $\mathbb{R}$, and $Y = \Omega \subset \mathbb{C}$ (typically Ω is open) with the subspace topology from $\mathbb{C}$. We will usually take

$$A = \{a, b\}.$$

Then $\gamma_0, \gamma_1 : [a,b] \to \Omega$ are two continuous paths in Ω with the same endpoints:

$$\gamma_0(a) = \gamma_1(a), \qquad \gamma_0(b) = \gamma_1(b).$$



EXAMPLE 1.2. Let $\Omega = \mathbb{C}$ and consider the two loops

$$\gamma_0(t) := 1, \qquad \gamma_1(t) := e^{2\pi i t}, \qquad 0 \le t \le 1.$$

Define

$$H(t,s) := (1-s) + s e^{2\pi i t}, \qquad (t,s) \in [0,1] \times [0,1].$$

Clearly H is a homotopy between γ_0 and γ_1, with

$$H(0,s) = H(1,s) = 1 \qquad (\forall s \in [0,1]).$$

Hence $\gamma_1 \simeq \gamma_0$ rel. $\{0,1\}$ in $\mathbb{C}$ via H.

Now let $\Omega = \mathbb{C}^*$ and consider the same loops viewed as maps into $\mathbb{C}^*$. Intuitively, any continuous homotopy deforming γ_0 to γ_1 would have to "pass through the puncture at the origin" which is impossible.

We will prove later, using a complex-analytic method, that γ_0 and γ_1 are *not* homotopic in $\mathbb{C}^*$ relative to the basepoint; this reflects the fact that $\mathbb{C}^*$ is not simply connected.

PROPOSITION 1.3 (Homotopy rel. endpoints is an equivalence relation). *Let $\Omega \subset \mathbb{C}$ be an open set and let $z_0, z_1 \in \Omega$. Consider*

$$\mathcal{P}(z_0, z_1; \Omega) := \{\gamma : [0,1] \to \Omega \textit{ continuous} \ : \ \gamma(0) = z_0, \ \gamma(1) = z_1\}.$$

Define a relation on $\mathcal{P}(z_0, z_1; \Omega)$ by declaring $\gamma_0 \sim \gamma_1$ if

$$\gamma_0 \simeq \gamma_1 \text{ rel } \{0,1\}.$$

Then $\sim$ is an equivalence relation on $\mathcal{P}(z_0, z_1; \Omega)$.

The proof is left to the reader.

A special case is $z_0 = z_1$ (so the paths are loops based at z_0). For this case, the set of homotopy classes (relative to $\{0,1\}$) is denoted by

$$\pi_1(\Omega; z_0).$$

REMARK 1.4 (The fundamental group). $\pi_1(\Omega; z_0)$ carries a natural group structure given by concatenation of loops (well-defined on homotopy classes rel. endpoints). This group is called the **fundamental group** of Ω based at z_0, and it is a central object in algebraic topology.

The following short exercises regarding the fundamental group are left to the interested reader:

(1) Describe the group operation on $\pi_1(\Omega; z_0)$ explicitly (concatenation of based loops), and check that it defines a group structure on homotopy classes (associativity, identity, inverses).
(2) Prove that if Ω is path-connected, then for any two basepoints $z_0, z_1 \in \Omega$, the fundamental groups $\pi_1(\Omega; z_0)$ and $\pi_1(\Omega; z_1)$ are isomorphic. One often omits the basepoint and writes simply $\pi_1(\Omega)$.
(3) Compute (or at least develop a clear intuition for) the fundamental group of the circle:

$$\pi_1(S^1) \cong \mathbb{Z}, \qquad \text{the additive group of integers}$$

where S^1 is the unit circle in $\mathbb{C}$. A more challenging task is to compute the fundamental group of the figure-eight (in topological term, the wedge sum $S^1 \vee S^1$) in $\mathbb{C}$.

DEFINITION 1.5 (Simply-connected domain). A *domain* $\Omega \subset \mathbb{C}$ (i.e. a connected open set) is called **simply-connected** if its fundamental group is trivial:

$$\pi_1(\Omega; z_0) \cong \{e\} \quad \text{for some (equivalently, any) } z_0 \in \Omega.$$

Equivalently, every based loop in Ω is homotopic (rel. endpoints) to the constant loop. In other words, any two paths in Ω with the same endpoints are homotopic relative to the endpoints.

PROPOSITION 1.6 (Star-shaped $\Rightarrow$ simply-connected). *Let $D \subset \mathbb{C}$ be star-shaped with star center $z_* \in D$. Then D is path-connected and simply-connected. As a corollary. any two path in a star-shaped domain with the same endpoints are homotopic relative to endpoints.*

PROOF. *Step 1: D is path-connected.* Given $z \in D$, the line segment

$$\gamma_z(t) := (1-t)z + tz_*, \qquad 0 \le t \le 1,$$

lies in D by star-shapedness, so every point can be joined to z_* by a path in D.

Step 2: Every loop is homotopic to a constant loop. Let $\gamma : [0,1] \to D$ be any loop based at z_*, i.e. $\gamma(0) = \gamma(1) = z_*$. Define

$$H(t,s) := (1-s)\gamma(t) + sz_*, \qquad (t,s) \in [0,1] \times [0,1].$$

For each fixed t, $H(t,s)$ runs along the line segment from $\gamma(t)$ to z_*, hence stays in D. Thus H is a homotopy in D with

$$H(t,0) = \gamma(t), \qquad H(t,1) = z_*,$$

and it fixes the basepoint:

$$H(0,s) = H(1,s) = z_* \qquad (\forall s \in [0,1]).$$

Therefore γ is homotopic (rel. endpoints) to the constant loop at z_*. Hence $\pi_1(D; z_*)$ is trivial and D is simply-connected. $\square$

In particular, a convex domain, for example an open disk, is path-connected and simply-connected.

2. Homotopy invariance of holomorphic integrals

Now let us come back to complex analysis. We begin with the following lemma, which shows that working with homotopy classes of *continuous* paths does not lose any information relevant to complex integration: every homotopy class contains a piecewise C^1 regular representative to which we can apply line integrals.

LEMMA 2.1 (Smoothing a path). *Let $\Omega \subset \mathbb{C}$ be open and $\gamma : [0,1] \to \Omega$ be continuous. Then γ is homotopic rel. $\{0,1\}$ to a polygonal path in Ω (hence to a piecewise C^1 regular path).*

PROOF. Let $K := \gamma([0,1])$, which is compact in Ω since $[0,1]$ is compact and γ is continuous.

Since Ω is open, for each $z \in K$ choose $r_z > 0$ such that $D(z; r_z) \subset \Omega$. Then $\{D(z; r_z/2)\}_{z\in K}$ is an open cover of K, hence admits a finite subcover: there exist $z_1, \dots, z_N \in K$ such that

$$K \subset \bigcup_{k=1}^{N} D\left(z_k; \frac{r_{z_k}}{2}\right).$$

Set

$$\varepsilon := \min_{1\le k\le N} \frac{r_{z_k}}{2} > 0.$$

Since γ is uniformly continuous on $[0,1]$, there exists $\delta > 0$ such that

$$|t-s| < \delta \implies |\gamma(t) - \gamma(s)| < \varepsilon \qquad (\forall t, s \in [0,1]).$$

Choose a partition $0 = t_0 < \cdots < t_m = 1$ with $t_{\ell+1} - t_\ell < \delta$. Fix ℓ. Choose k such that $\gamma(t_\ell) \in D(z_k; r_{z_k}/2)$, i.e.

$$|\gamma(t_\ell) - z_k| < \frac{r_{z_k}}{2}.$$

Then for any $s \in [t_\ell, t_{\ell+1}]$,

$$\begin{aligned} |\gamma(s) - z_k| &\le |\gamma(s) - \gamma(t_\ell)| + |\gamma(t_\ell) - z_k| \\ &< \varepsilon + \frac{r_{z_k}}{2} \le \frac{r_{z_k}}{2} + \frac{r_{z_k}}{2} = r_{z_k}, \end{aligned}$$

so $\gamma([t_\ell, t_{\ell+1}]) \subset D(z_k; r_{z_k})$.

Let σ be the polygonal path joining the points $\gamma(t_0), \ldots, \gamma(t_m)$ by straight line segments. On each interval $[t_\ell, t_{\ell+1}]$, the segment from $\gamma(t_\ell)$ to $\gamma(t_{\ell+1})$ lies in $D(z_k; r_{z_k})$ since the disk is convex. Define the linear homotopy on $[t_\ell, t_{\ell+1}]$ by

$$H_\ell(t, s) := (1 - s)\gamma(t) + s\,\sigma(t), \qquad s \in [0, 1].$$

Then $H_\ell(t, s) \in D(z_k; r_{z_k}) \subset \Omega$ for all $t \in [t_\ell, t_{\ell+1}]$, and it fixes the endpoints. Gluing these finitely many homotopies together gives a homotopy in Ω from γ to σ rel. $\{0, 1\}$. □

Now we are ready to present the main theorem of this section.

THEOREM 2.2 (Homotopy invariance of holomorphic integrals). *Let $\Omega \subset \mathbb{C}$ be open and let $f \in \mathcal{O}(\Omega)$. Let $\gamma_0, \gamma_1 : [0, 1] \to \Omega$ be oriented piecewise C^1 regular paths with the same endpoints*

$$\gamma_0(0) = \gamma_1(0) = z_0, \qquad \gamma_0(1) = \gamma_1(1) = z_1.$$

Assume γ_0 and γ_1 are homotopic in Ω relative to the endpoints $\{0, 1\}$. Then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

PROOF. Let $H : [0, 1] \times [0, 1] \to \Omega$ be a homotopy rel. endpoints from γ_0 to γ_1, and set

$$\gamma_s(t) := H(t, s) \qquad (t, s \in [0, 1]).$$

Note that $\gamma_s(0) = z_0$ and $\gamma_s(1) = z_1$ for all s.

Step 1: choose a uniform partition and a uniform disk cover. Since H is continuous on the compact square $[0, 1]^2$, it is uniformly continuous. Also, the image

$$K := H([0, 1]^2) \subset \Omega$$

is compact, hence has positive distance to $\mathbb{C} \setminus \Omega$; equivalently, there exists $\rho > 0$ such that

$$D(z; \rho) \subset \Omega \qquad (\forall z \in K).$$

By uniform continuity, choose $\delta > 0$ such that

$$|(t, s) - (t', s')| < \delta \quad \Longrightarrow \quad |H(t, s) - H(t', s')| < \rho/4.$$

Fix a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ with $t_{k+1} - t_k < \delta$ for all k.

Then for every $s \in [0, 1]$ and every $t \in [t_k, t_{k+1}]$,

$$|\gamma_s(t) - \gamma_s(t_k)| = |H(t, s) - H(t_k, s)| < \rho/4,$$

hence

$$\gamma_s([t_k, t_{k+1}]) \subset D(\gamma_s(t_k); \rho/4) \subset D(\gamma_s(t_k); \rho) \subset \Omega.$$

Step 2: define the integrals for all s via polygonal approximation. For each s, let σ_s be the polygonal path obtained by joining $\gamma_s(t_0), \gamma_s(t_1), \dots, \gamma_s(t_m)$ by straight line segments (on each interval $[t_k, t_{k+1}]$, σ_s is the affine parametrization of the segment $[\gamma_s(t_k), \gamma_s(t_{k+1})]$). This polygonal approximation is exactly the construction in Lemma 2.1, applied to the path $t \mapsto H(t, s)$ (with the same partition $0 = t_0 < \cdots < t_m = 1$).

Each segment of σ_s lies in $D(\gamma_s(t_k); \rho/4)$, hence σ_s is piecewise C^1 regular. Define

$$J(s) := \int_{\sigma_s} f(z)\,dz.$$

(So $J(0) = \int_{\gamma_0} f\,dz$ and $J(1) = \int_{\gamma_1} f\,dz$, because γ_0, γ_1 are already piecewise C^1 and homotopic rel. endpoints to σ_0, σ_1 by Lemma 2.1.)

Step 3: $J(s)$ is locally constant (open–closed argument). Fix $s_0 \in [0, 1]$. We claim that $J(s) = J(s_0)$ for all s sufficiently close to s_0. Indeed, if $|s - s_0| < \delta$, then for every k,

$$|\gamma_s(t_k) - \gamma_{s_0}(t_k)| = |H(t_k, s) - H(t_k, s_0)| < \rho/4.$$

Consider the closed piecewise C^1 loop $\Gamma_k(s)$ obtained by concatenating:

- the segment $[\gamma_s(t_k), \gamma_s(t_{k+1})]$ (traversed along σ_s);
- the segment $[\gamma_s(t_{k+1}), \gamma_{s_0}(t_{k+1})]$;
- the segment $[\gamma_{s_0}(t_{k+1}), \gamma_{s_0}(t_k)]$ (traversed backward along σ_{s_0});
- the segment $[\gamma_{s_0}(t_k), \gamma_s(t_k)]$.

All four edges lie in the disk $D(\gamma_{s_0}(t_k); \rho) \subset \Omega$: for example, $\gamma_s(t_k) \in D(\gamma_{s_0}(t_k); \rho/4)$, and the segment $[\gamma_s(t_k), \gamma_s(t_{k+1})]$ lies in $D(\gamma_s(t_k); \rho/4) \subset D(\gamma_{s_0}(t_k); \rho/2)$, etc. Hence $\Gamma_k(s)$ is contained in a convex disk where f is holomorphic. By Cauchy's theorem,

$$\int_{\Gamma_k(s)} f(z)\,dz = 0.$$

Expanding this identity gives

$$\int_{[\gamma_s(t_k), \gamma_s(t_{k+1})]} f\,dz - \int_{[\gamma_{s_0}(t_k), \gamma_{s_0}(t_{k+1})]} f\,dz = -\int_{[\gamma_s(t_{k+1}), \gamma_{s_0}(t_{k+1})]} f\,dz + \int_{[\gamma_s(t_k), \gamma_{s_0}(t_k)]} f\,dz.$$

Now sum over $k = 0, 1, \dots, m-1$. The right-hand side telescopes: all intermediate terms cancel, leaving only the endpoint segments $[\gamma_s(0), \gamma_{s_0}(0)]$ and $[\gamma_s(1), \gamma_{s_0}(1)]$, which are trivial because the homotopy is rel. endpoints. Therefore

$$\int_{\sigma_s} f\,dz = \int_{\sigma_{s_0}} f\,dz,$$

i.e. $J(s) = J(s_0)$ whenever $|s - s_0| < \delta$. Hence J is locally constant on $[0, 1]$.

Let

$$A := \{\, s \in [0, 1] : J(s) = J(0) \,\}.$$

Then A is nonempty and open (by local constancy), and also closed in $[0, 1]$. Since $[0, 1]$ is connected, we conclude $A = [0, 1]$, so $J(1) = J(0)$.

Step 4: return to γ_0, γ_1. By Lemma 2.1, each γ_s is homotopic rel. endpoints to σ_s, and this homotopy can be chosen inside the same disks used above; applying Cauchy's theorem to the resulting small loops shows

$$\int_{\gamma_s} f\,dz = \int_{\sigma_s} f\,dz \qquad (\forall s).$$

In particular,

$$\int_{\gamma_0} f\,dz = \int_{\sigma_0} f\,dz = J(0) = J(1) = \int_{\sigma_1} f\,dz = \int_{\gamma_1} f\,dz.$$

This proves the theorem. □

REMARK 2.3. A key advantage of homotopy invariance is that we do *not* require paths (or loops) to be simple, i.e., self-intersections of the curves are allowed.

EXAMPLE 2.4. We now give a complex-analytic proof that $\mathbb{C}^*$ is not simply-connected. Since $\frac{1}{z}$ is holomorphic on $\mathbb{C}^*$, if $\mathbb{C}^*$ were simply-connected then every based loop would be homotopic (rel. endpoints) to the constant loop at 1, hence would have integral 0. But for $\gamma_1(t) = e^{2\pi i t}$,

$$\int_{\gamma_1} \frac{dz}{z} = 2\pi i \neq 0.$$

Therefore $\mathbb{C}^*$ is not simply-connected.

Similarly, one can show that the based loops

$$\gamma_n(t) := e^{2\pi i n t} \qquad (0 \le t \le 1)$$

lie in distinct homotopy classes in $\mathbb{C}^*$. We leave the details to the reader to fill.

We end this section with the following remark.

REMARK 2.5 (Periods). Fix an open set $\Omega \subset \mathbb{C}$ and a basepoint $z_0 \in \Omega$. For each $f \in \mathcal{O}(\Omega)$, define

$$\Phi_f : \pi_1(\Omega; z_0) \longrightarrow (\mathbb{C}, +), \qquad \Phi_f([\gamma]) = \int_\gamma f(z)\,dz,$$

where $[\gamma]$ denotes the homotopy class (rel. endpoints) of a based loop at z_0. Homotopy-invariance implies Φ_f is well-defined.

Moreover, Φ_f is additive in loops:

$$\Phi_f([\gamma_1 \cdot \gamma_2]) = \Phi_f([\gamma_1]) + \Phi_f([\gamma_2]),$$

and additive in f:

$$\Phi_{\alpha f + \beta g} = \alpha\,\Phi_f + \beta\,\Phi_g \qquad (\alpha, \beta \in \mathbb{C}).$$

Thus we get a linear map

$$\Phi : \mathcal{O}(\Omega) \longrightarrow \operatorname{Hom}\big(\pi_1(\Omega; z_0), \mathbb{C}\big), \qquad f \longmapsto \Phi_f.$$

Its kernel consists precisely of those holomorphic functions admitting a primitive on Ω:

$$\ker(\Phi) = \{\, f \in \mathcal{O}(\Omega) : f \text{ has a primitive on } \Omega \,\}.$$

(Indeed, if $f = F'$ then $\int_\gamma f\,dz = 0$ for every loop γ; conversely, if all loop integrals vanish then $F(z) := \int_{\gamma_{z_0,z}} f(w)\,dw$ is well-defined and satisfies $F' = f$.)

We will return to this viewpoint when we study residues: residues measure the nontrivial periods coming from punctures.

3. The complex logarithm

For the real exponential function

$$\exp : \mathbb{R} \to \mathbb{R}^+,$$

the map is one-to-one, hence it has an inverse function, the real logarithm $\ln x$.

If we try to define a complex logarithm in the same way, we immediately meet a difficulty:

$$\exp : \mathbb{C} \to \mathbb{C}^*, \qquad \exp(z) = e^z$$

is not one-to-one on $\mathbb{C}$.

In this section we will see that the obstruction to defining a *single-valued* holomorphic logarithm on an open set $\Omega \subset \mathbb{C}^*$ is exactly the same as the obstruction to finding a primitive of $1/z$ on Ω. From a topological viewpoint, this is closely related to the fact that $\exp : \mathbb{C} \to \mathbb{C}^*$ is the universal covering map and that $\pi_1(\mathbb{C}^*)$ is nontrivial (via deck transformations). We will make these connections precise as we go.

3.1. Branches of complex logarithm. Recall the exponential map

$$\exp : \mathbb{C} \to \mathbb{C}^*, \qquad \exp(z) = e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

It is an entire function. Using Euler's identity,

$$e^{x+iy} = e^x(\cos y + i \sin y), \qquad x, y \in \mathbb{R}.$$

In particular,

$$|e^{x+iy}| = e^x > 0,$$

so $\exp(z) \neq 0$ for all $z \in \mathbb{C}$.

Given any $w \in \mathbb{C}^*$, write w in polar form

$$w = |w|e^{i\theta},$$

where θ is an *argument* of w (i.e. $e^{i\theta} = w/|w|$). To solve $e^z = w$ with $z = x + iy$, we compare moduli and arguments:

$$e^x = |w|, \qquad e^{iy} = e^{i\theta}.$$

Hence

$$x = \ln|w|, \qquad y = \theta + 2\pi k \quad (k \in \mathbb{Z}),$$

so every $w \in \mathbb{C}^*$ has infinitely many preimages under exp, indexed by $\mathbb{Z}$. In particular, $\exp : \mathbb{C} \to \mathbb{C}^*$ is surjective but not injective.

It is convenient to regard the logarithm as a *multivalued* function. We write $[\log(w)]$ for the full set of solutions of $e^z = w$, namely

$$[\log(w)] = \left\{ \ln|w| + i(\theta + 2\pi k) : \ k \in \mathbb{Z} \right\},$$

where θ is any fixed argument of w (different choices of θ give the same set).

Convention (principal argument). The unique argument in $(-\pi, \pi]$ is called the **principal argument** and is denoted by $\mathrm{Arg}(w)$. The corresponding single-valued logarithm

$$\mathrm{Log}(w) := \ln|w| + i\,\mathrm{Arg}(w)$$

is called the **principal logarithm**.

EXAMPLE 3.1. Compute the multivalued logarithms $[\log 1]$, $[\log i]$, $[\log(-1)]$, $[\log(-i)]$, and their principal logarithms.

SOLUTION. Recall

$$[\log w] = \{\ln |w| + i(\theta + 2\pi k) : k \in \mathbb{Z}\},$$

where θ is any argument of w, and the principal logarithm is

$$\mathrm{Log}(w) = \ln |w| + i\,\mathrm{Arg}(w), \qquad \mathrm{Arg}(w) \in (-\pi, \pi].$$

In all four cases below, $|w| = 1$, so $\ln |w| = 0$.

(1) $w = 1$. An argument is $\theta = 0$, so

$$[\log 1] = \{\, 2\pi i k : k \in \mathbb{Z}\,\}, \qquad \mathrm{Log}(1) = 0.$$

(2) $w = i$. An argument is $\theta = \frac{\pi}{2}$, so

$$[\log i] = \left\{\, i\left(\frac{\pi}{2} + 2\pi k\right) : k \in \mathbb{Z}\,\right\}, \qquad \mathrm{Log}(i) = \frac{\pi i}{2}.$$

(3) $w = -1$. An argument is $\theta = \pi$, so

$$[\log(-1)] = \{\, i(\pi + 2\pi k) : k \in \mathbb{Z}\,\}, \qquad \mathrm{Log}(-1) = \pi i.$$

(4) $w = -i$. An argument is $\theta = -\frac{\pi}{2}$ (equivalently $\frac{3\pi}{2}$), so

$$[\log(-i)] = \left\{\, i\left(-\frac{\pi}{2} + 2\pi k\right) : k \in \mathbb{Z}\,\right\}, \qquad \mathrm{Log}(-i) = -\frac{\pi i}{2}.$$

□

The multivalued logarithm $[\log w]$ is not a complex-valued function. To obtain a genuine single-valued logarithm, we introduce the following definition.

DEFINITION 3.2 (Branch of the logarithm). Let $\Omega \subset \mathbb{C}^*$ be open. A **branch of the logarithm** on Ω is a *holomorphic function*

$$\mathrm{Log}_\Omega : \Omega \to \mathbb{C}$$

such that

$$e^{\mathrm{Log}_\Omega(w)} = w \qquad (\forall w \in \Omega).$$

Equivalently, Log_Ω is a holomorphic inverse of exp on Ω.

REMARK 3.3. (1) If Log_Ω is a branch on Ω, then so is $\mathrm{Log}_\Omega + 2\pi i k$ for any $k \in \mathbb{Z}$. In particular, a branch is never unique.
(2) It is possible that Log_Ω does not exist.

The principal logarithm $\mathrm{Log}(w) = \ln |w| + i\mathrm{Arg}(w)$ is well-defined as a *function* on $\mathbb{C}^*$, but it cannot be continuous (hence not holomorphic) everywhere: crossing the *negative* real axis, the value of Arg jumps by 2π. To fix this, we remove a curve from $\mathbb{C}^*$ so that the argument can be chosen continuously (and then holomorphically).

DEFINITION 3.4 (Principal branch cut). Let

$$\Omega_0 := \mathbb{C} \setminus (-\infty, 0]$$

(the complex plane with the nonpositive real axis removed). On Ω_0 there is a unique continuous function

$$\mathrm{Arg} : \Omega_0 \to (-\pi, \pi]$$

such that $w = |w|e^{i\mathrm{Arg}(w)}$ for all $w \in \Omega_0$. The corresponding logarithm

$$\mathrm{Log} w := \ln|w| + i\mathrm{Arg}(w) \qquad (w \in \Omega_0)$$

is called the **principal branch** of the logarithm.

PROPOSITION 3.5. *The principal branch* $\mathrm{Log} : \Omega_0 = \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ *is holomorphic, and*

$$(\mathrm{Log} w)' = \frac{1}{w}, \qquad \mathrm{Log} 1 = 0.$$

PROOF. This can be seen from direct calculation: Write $w = x + iy$ and then

$$u(x, y) := \ln|w| = \frac{1}{2}\ln(x^2 + y^2), \qquad v(x, y) := \mathrm{Arg}(w).$$

On Ω_0 we have $u, v \in C^1$, and a direct computation gives

$$u_x = \frac{x}{x^2 + y^2}, \qquad u_y = \frac{y}{x^2 + y^2},$$

and (using $\theta = \mathrm{Arg}(w)$ so that $\tan\theta = y/x$ locally on Ω_0)

$$v_x = -\frac{y}{x^2 + y^2}, \qquad v_y = \frac{x}{x^2 + y^2}.$$

Hence $u_x = v_y$ and $u_y = -v_x$, so $\mathrm{Log} = u + iv$ is holomorphic on Ω_0. Moreover,

$$(\mathrm{Log} w)' = u_x + iv_x = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2} = \frac{1}{x + iy} = \frac{1}{w}.$$

Finally, $\mathrm{Log} 1 = \ln 1 + i\mathrm{Arg}(1) = 0$. □

In fact, there is the following more general theorem.

THEOREM 3.6. *Let* $\Omega \subset \mathbb{C}^*$ *be a domain. The following are equivalent:*

(i) *There exists a branch of the logarithm on* Ω*, i.e. a holomorphic function*

$$\mathrm{Log}_\Omega : \Omega \to \mathbb{C}$$

such that $e^{\mathrm{Log}_\Omega(z)} = z$ *for all* $z \in \Omega$*.*

(ii) *The function* $\frac{1}{z}$ *admits a primitive on* Ω*, i.e. there exists a holomorphic function* $F : \Omega \to \mathbb{C}$ *such that*

$$F'(z) = \frac{1}{z} \qquad (\forall z \in \Omega).$$

Moreover, whenever these conditions hold, we have:

(a) *Every branch* Log_Ω *satisfies*

$$(\mathrm{Log}_\Omega)'(z) = \frac{1}{z} \qquad (\forall z \in \Omega).$$

(b) *If* Log_1 *and* Log_2 *are two branches on* Ω*, then*

$$\mathrm{Log}_1 - \mathrm{Log}_2 \equiv 2\pi i\, k$$

for some $k \in \mathbb{Z}$*. In particular, fixing one value determines the branch uniquely: if* $\mathrm{Log}_1(z_0) = \mathrm{Log}_2(z_0)$ *for some* $z_0 \in \Omega$*, then* $\mathrm{Log}_1 \equiv \mathrm{Log}_2$ *on* Ω*.*

PROOF. (i)⇒(ii) and (a). If $e^{\mathrm{Log}_\Omega(z)} = z$ on Ω, then differentiating and using the chain rule gives

$$e^{\mathrm{Log}_\Omega(z)}(\mathrm{Log}_\Omega)'(z) = 1.$$

Since $e^{\mathrm{Log}_\Omega(z)} = z$, we obtain $(\mathrm{Log}_\Omega)'(z) = 1/z$, so Log_Ω is a primitive of $1/z$.

(ii)⇒(i). Assume $F'(z) = 1/z$ on Ω. Consider $G(z) := e^{F(z)}$. Then

$$\frac{G'(z)}{G(z)} = F'(z) = \frac{1}{z}.$$

Hence

$$\left(\frac{G(z)}{z}\right)' = \frac{G'(z)z - G(z)}{z^2} = 0,$$

so $G(z)/z$ is constant on the connected set Ω. Thus $G(z) = cz$ for some $c \in \mathbb{C}^*$. Choose any $a \in \mathbb{C}$ so that $e^a = c$, and define

$$\mathrm{Log}_\Omega(z) := F(z) - a.$$

Then $e^{\mathrm{Log}_\Omega(z)} = e^{F(z)}/c = G(z)/c = z$, so Log_Ω is a branch of the logarithm on Ω.

(b) (uniqueness up to $2\pi i\mathbb{Z}$). Let $\mathrm{Log}_1, \mathrm{Log}_2$ be two branches on Ω. Then by (a),

$$(\mathrm{Log}_1 - \mathrm{Log}_2)' = \frac{1}{z} - \frac{1}{z} = 0,$$

so $\mathrm{Log}_1 - \mathrm{Log}_2 \equiv c$ is constant on Ω. Moreover,

$$1 = \frac{e^{\mathrm{Log}_1(z)}}{e^{\mathrm{Log}_2(z)}} = e^{\mathrm{Log}_1(z) - \mathrm{Log}_2(z)} = e^c,$$

so $c \in 2\pi i\,\mathbb{Z}$. If $\mathrm{Log}_1(z_0) = \mathrm{Log}_2(z_0)$ for some z_0, then $c = 0$ and $\mathrm{Log}_1 \equiv \mathrm{Log}_2$. □

3.2. A topological view. Now we reconsider the exponential function

$$\exp : \mathbb{C} \to \mathbb{C}^*.$$

We have already seen that exp is holomorphic and surjective.

Fix $z_0 \in \mathbb{C}^*$ and choose $r > 0$ such that $D(z_0; r) \subset \mathbb{C}^*$. Then $\frac{1}{z}$ is holomorphic on $D(z_0; r)$, hence admits a primitive there. By Theorem 3.6, there exists a branch of the logarithm on $D(z_0; r)$. We choose the branch $\mathrm{Log}_{D(z_0;r)} : D(z_0; r) \to \mathbb{C}$ normalized by

$$\mathrm{Log}_{D(z_0;r)}(z_0) = \ln|z_0| + i\mathrm{Arg}(z_0).$$

Let

$$U := \mathrm{Log}_{D(z_0;r)}\big(D(z_0; r)\big) \subset \mathbb{C},$$

which is an open set in $\mathbb{C}$. Since any two branches on the disk differ by an additive constant $2\pi ik$, it follows that

$$\exp^{-1}\big(D(z_0; r)\big) = \bigsqcup_{k\in\mathbb{Z}} \big(U + 2\pi ik\big),$$

a disjoint (why?) union of open sets, and for each $k \in \mathbb{Z}$ the restriction

$$\exp : U + 2\pi ik \longrightarrow D(z_0; r)$$

is a homeomorphism (indeed a biholomorphism).

REMARK 3.7 (Covering maps, deck transformations). A continuous surjective map $p : Y \to X$ is called a **covering map** if every $x \in X$ has an open neighborhood D such that $p^{-1}(D)$ is a disjoint union of open sets on which p restricts to homeomorphisms onto D. Such a D is said to be **evenly covered**. The discussion above shows that $\exp : \mathbb{C} \to \mathbb{C}^*$ is a covering map.

Since $\mathbb{C}$ is simply-connected, $\exp : \mathbb{C} \to \mathbb{C}^*$ is in fact the **universal covering** of $\mathbb{C}^*$: The term universal means any connected covering of $\mathbb{C}^*$ factors through this covering. We will return to this viewpoint later when we consider branches of nth roots.

EXAMPLE 3.8 (The fundamental group of $\mathbb{C}^*$ is $\mathbb{Z}$). We sketch a complex-analytic proof that

$$\pi_1(\mathbb{C}^*; 1) \cong \mathbb{Z}.$$

Let $\gamma : [0,1] \to \mathbb{C}^*$ be a based loop at 1. Cover the compact set $\gamma([0,1])$ by finitely many small disks $D_j \subset \mathbb{C}^*$, so that on each D_j the function $1/z$ is holomorphic and hence admits a branch of the logarithm Log_j. Using that two branches on an overlap differ by $2\pi i k$, we may adjust constants so that the branches match at the junction points. In this way we obtain a continuous lift

$$\widetilde{\gamma} : [0,1] \to \mathbb{C}, \qquad e^{\widetilde{\gamma}(t)} = \gamma(t), \qquad \widetilde{\gamma}(0) = 0.$$

Since $\gamma(1) = 1$, we have $e^{\widetilde{\gamma}(1)} = 1$, hence

$$\widetilde{\gamma}(1) = 2\pi i\, n \quad \text{for a unique } n \in \mathbb{Z}.$$

In $\mathbb{C}$ (which is simply-connected), the path $\widetilde{\gamma}$ is homotopic rel. endpoints to the straight path $t \mapsto 2\pi i n t$. Composing this homotopy with exp shows that γ is homotopic in $\mathbb{C}^*$ (rel. basepoint) to the standard loop $\gamma_n(t) = e^{2\pi i n t}$.

Finally, if $\gamma_n \simeq \gamma_m$ in $\mathbb{C}^*$ rel. basepoint, then by homotopy invariance of holomorphic integrals,

$$2\pi i n = \int_{\gamma_n} \frac{dz}{z} = \int_{\gamma_m} \frac{dz}{z} = 2\pi i m,$$

so $n = m$. Therefore the based homotopy classes are indexed by $\mathbb{Z}$, and $\pi_1(\mathbb{C}^*; 1) \cong \mathbb{Z}$.

3.3. nth roots and their branches. Next we discuss nth roots, in parallel with the logarithm. Fix an integer $n \geq 2$. Consider the map

$$p_n : \mathbb{C}^* \to \mathbb{C}^*, \qquad p_n(z) = z^n.$$

For each $w \in \mathbb{C}^*$, the equation $z^n = w$ has exactly n distinct solutions in $\mathbb{C}^*$. It is therefore natural to regard the nth root as a *multivalued* function and write

$$[w^{1/n}] := \{\, z \in \mathbb{C}^* : \ z^n = w \,\}.$$

The map p_n is an n-sheeted covering of $\mathbb{C}^*$, which is not universal, and it is related to the universal covering $\exp : \mathbb{C} \to \mathbb{C}^*$ by the factorization

$$\exp = p_n \circ \pi_n, \qquad \pi_n : \mathbb{C} \to \mathbb{C}^*, \;\; \pi_n(z) = e^{z/n},$$

since $(e^{z/n})^n = e^z$.

$$\begin{tikzcd}
\mathbb{C} \arrow[dr, "\pi_n"] \arrow[dd, "\exp"'] & \\
& \mathbb{C}^* \arrow[dl, "p_n"] \\
\mathbb{C}^* &
\end{tikzcd}$$

Let $\Omega \subset \mathbb{C}^*$ be open. A **branch of the nth root** on Ω is a holomorphic function

$$\mathrm{Root}_\Omega : \Omega \to \mathbb{C}^*$$

such that

$$\mathrm{Root}_\Omega(w)^n = w \qquad (\forall w \in \Omega).$$

The following exercise is left to the reader.

EXERCISE 3.9. Let $\Omega \subset \mathbb{C}^*$ be a domain and fix an integer $n \geq 2$. Assume there exists a branch of the nth root on Ω. Prove that there are exactly n distinct branches of the nth root on Ω, and determine all of them.

In particular, if a branch Log_Ω of the logarithm exists on Ω, then

$$\mathrm{Root}_\Omega(w) := \exp\Big(\frac{1}{n}\mathrm{Log}_\Omega(w)\Big)$$

defines a branch of the nth root on Ω. We will see in a moment that all branches of roots over a domain in $\mathbb{C}^*$ can be constructed using a branch of logarithm.

EXAMPLE 3.10. Let

$$\Omega_0 := \mathbb{C} \setminus (-\infty, 0]$$

and let $\mathrm{Log} : \Omega_0 \to \mathbb{C}$ be the principal branch of the logarithm.

Define a branch of the square root as

$$\mathrm{Root}^+_{\Omega_0}(z) := \exp\Big(\frac{1}{2}\,\mathrm{Log} z\Big), \qquad z \in \Omega_0.$$

Then $\mathrm{Root}^+_{\Omega_0}$ is holomorphic on Ω_0 and satisfies

$$\big(\mathrm{Root}^+_{\Omega_0}(z)\big)^2 = \exp(\mathrm{Log} z) = z, \qquad \mathrm{Root}^+_{\Omega_0}(1) = 1.$$

A second branch is obtained by shifting the logarithm by $2\pi i$:

$$\mathrm{Root}^-_{\Omega_0}(z) := \exp\Big(\frac{1}{2}\,(\mathrm{Log} z + 2\pi i)\Big) = -\exp\Big(\frac{1}{2}\,\mathrm{Log} z\Big) = -\mathrm{Root}^+_{\Omega_0}(z).$$

Again,

$$\big(\mathrm{Root}^-_{\Omega_0}(z)\big)^2 = z, \qquad \text{but} \quad \mathrm{Root}^-_{\Omega_0}(1) = -1.$$

It is customary to denote

$$\sqrt{z} := \mathrm{Root}^+_{\Omega_0}(z),$$

and then $\mathrm{Root}^-_{\Omega_0}(z) = -\sqrt{z}$.

Notice, although Ω_0 admits infinitely many logarithm branches (they differ by $2\pi i k$, $k \in \mathbb{Z}$), shifting by $2\pi i k$ changes $\exp\big(\frac{1}{2}\mathrm{Log} z\big)$ only by the factor $e^{\pi i k} = (-1)^k$. Hence there are exactly two square-root branches on Ω_0, namely $\pm\sqrt{z}$.

Equivalently, we can describe these branches in polar form. Write $z = re^{i\theta}$ with $r = |z| > 0$ and $\theta = \operatorname{Arg}(z) \in (-\pi, \pi]$. Then

$$\sqrt{z} = \sqrt{r}\, e^{i\theta/2}.$$

In particular,

$$\sqrt{1} = 1, \qquad \sqrt{-1} = e^{i\pi/2} = i.$$

An exercise for the reader is to generalize this example to nth roots and show that there are in fact exactly n branches on Ω_0.

4. Logarithm and nth roots of a holomorphic function

We now generalize the previous discussion to holomorphic functions.

Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in \mathcal{O}(\Omega)$ be nowhere zero. Equivalently, f is a holomorphic map

$$f : \Omega \longrightarrow \mathbb{C}^*.$$

We ask whether f admits a holomorphic logarithm on Ω, namely whether there exists a holomorphic function $L : \Omega \to \mathbb{C}$ such that

$$\exp \circ L = f, \qquad \text{equivalently} \qquad e^{L(z)} = f(z) \qquad (\forall z \in \Omega).$$

In the special case where f is the inclusion map $\Omega \hookrightarrow \mathbb{C}^*$, such an L is precisely a branch of the complex logarithm on Ω.

Similarly, for a fixed integer $n \geq 2$, we ask whether f admits a holomorphic nth root on Ω, namely whether there exists a holomorphic function $g : \Omega \to \mathbb{C}^*$ such that

$$p_n \circ g = f, \qquad \text{equivalently} \qquad g(z)^n = f(z) \qquad (\forall z \in \Omega),$$

where $p_n : \mathbb{C}^* \to \mathbb{C}^*$ is given by $p_n(w) = w^n$. Again, when f is the inclusion map $\Omega \hookrightarrow \mathbb{C}^*$, such a g is precisely a branch of the nth root on Ω.

In fact, these questions naturally fit into the covering-map viewpoint: they ask whether the holomorphic map $f : \Omega \to \mathbb{C}^*$ can be *lifted* through the coverings

$$\exp : \mathbb{C} \to \mathbb{C}^*, \qquad p_n : \mathbb{C}^* \to \mathbb{C}^*.$$

In other words, we ask whether the following diagrams can be completed by a holomorphic lift:

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ \Omega \arrow[ur, dashed, "L"] \arrow[r, "f"'] & \mathbb{C}^* \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C}^* \arrow[d, "p_n"] \\ \Omega \arrow[ur, dashed, "g"] \arrow[r, "f"'] & \mathbb{C}^* \end{tikzcd}$$

We have already answered the first question in the special case where f is the inclusion map $\Omega \hookrightarrow \mathbb{C}^*$. By Theorem 3.6, a branch of the logarithm on Ω exists if and only if $\frac{1}{z}$ admits a primitive on Ω. Equivalently, this holds if and only if

$$\int_\gamma \frac{1}{z}\, dz = 0$$

for every oriented piecewise C^1 regular loop γ in Ω (see Proposition 2.3, the path-independence argument in the proof of Cauchy's theorem, and Remark 2.5).

We now generalize this criterion to an arbitrary nowhere-vanishing holomorphic function $f \in \mathcal{O}(\Omega)$, assuming for the moment that f' is continuous. (In fact, this extra assumption will be redundant: in later chapters we will prove that the derivative of a holomorphic function is again holomorphic, hence automatically continuous.)

THEOREM 4.1 (Criterion for a holomorphic logarithm). *Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in \mathcal{O}(\Omega)$ be nowhere zero and f' is continuous in Ω (this continuity assumption can be removed). Then the following are equivalent:*

(i) *There exists $L \in \mathcal{O}(\Omega)$ such that $e^L = f$ on Ω.*
(ii) *The function $\dfrac{f'}{f}$ has a primitive on Ω.*
(iii) *For every oriented piecewise C^1 regular loop γ in Ω,*

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0.$$

Moreover, when these conditions hold, any such L satisfies

$$L'(z) = \frac{f'(z)}{f(z)} \qquad (\forall z \in \Omega),$$

and L is unique up to addition of a constant $2\pi i k$ ($k \in \mathbb{Z}$) if we view $e^L = f$. (Equivalently, fixing one value $L(z_0)$ determines L uniquely.)

PROOF. (i)⇒(ii). If $e^L = f$, differentiate to get $e^L L' = f'$, hence $L' = f'/f$.

(ii)⇔(iii). This is exactly the primitive criterion: a continuous function has a primitive on Ω if and only if its integral over every loop is 0. (See the proof of Cauchy's theorem and also Remark 3.4.)

(ii)⇒(i). If $F' = \dfrac{f'}{f}$ on Ω, set $G := e^F/f$. Then

$$\frac{G'}{G} = F' - \frac{f'}{f} = 0,$$

so G is constant on the connected set Ω, say $G \equiv c \in \mathbb{C}^*$. Thus $e^F = cf$. Choose any $a \in \mathbb{C}$ so that $e^a = c$, and define $L := F - a$. Then $e^L = f$.

Finally, if L_1, L_2 both satisfy $e^{L_j} = f$, then $e^{L_1 - L_2} = 1$, hence $L_1 - L_2 \in 2\pi i\,\mathbb{Z}$. Since $L_1 - L_2$ is holomorphic, it must be constant, so $L_1 - L_2 \equiv 2\pi i k$. □

Next, we state the parallel theorem for nth root.

THEOREM 4.2 (Criterion for a holomorphic nth root). *Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in \mathcal{O}(\Omega)$ be nowhere zero. Further assume f' is continuous in Ω (this extra assumption can be removed), and fix $n \geq 2$. Then the following are equivalent:*

(i) *There exists $g \in \mathcal{O}(\Omega)$ such that $g^n = f$ on Ω.*
(ii) *For every oriented piecewise C^1 regular loop γ in Ω,*

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz \in 2\pi i\, n\mathbb{Z}.$$

PROOF. (i)⇒(ii). Assume $g^n = f$ on Ω with $g \in \mathcal{O}(\Omega)$. Then g nowhere vanishes since f nowhere vanishes. Differentiating gives

$$ng^{n-1}g' = f'.$$

Since $g^{n-1} = f/g$, we obtain

$$n\,\frac{g'}{g} = \frac{f'}{f}.$$

Let $\gamma : [0,1] \to \Omega$ be an oriented piecewise C^1 regular loop, then $g \circ \gamma : [0,1] \to \mathbb{C}^*$ is an oriented piecewise C^1 regular loop in $\mathbb{C}^*$.

Choose a partition $0 = t_0 < \cdots < t_N = 1$ and disks $D_k \subset \mathbb{C}^*$ such that $g\circ\gamma([t_{k-1}, t_k]) \subset D_k$. On each disk D_k we can choose a holomorphic logarithm, say Log_{D_k} such that the matching conditions

$$\mathrm{Log}_{D_k}(g(\gamma(t_k))) = \mathrm{Log}_{D_{k+1}}(g(\gamma(t_k))) \qquad (k = 1, \ldots, N-1)$$

hold.

Notice,

$$(\mathrm{Log}_{D_k} g)' = \frac{g'}{g},$$

so by this way, we have constructed a local primitive of $\frac{g'}{g}$ in an open neighborhood of $\gamma([t_{k-1}, t_k])$.

It follows

$$\int_{\gamma|_{[t_{k-1},t_k]}} \frac{g'(z)}{g(z)}\,dz = (\mathrm{Log}_{D_k} g)\big(\gamma(t_k)\big) - (\mathrm{Log}_{D_k} g)\big(\gamma(t_{k-1})\big).$$

Summing over k gives a telescoping sum by the matching conditions, hence

$$\int_\gamma \frac{g'(z)}{g(z)}\,dz = \mathrm{Log}_{D_N}\big(g\circ\gamma(1)\big) - \mathrm{Log}_{D_1}\big(g\circ\gamma(0)\big).$$

Since $\gamma(0) = \gamma(1)$ and then $g\circ\gamma(0) = g\circ\gamma(1)$, two branches of logarithms at the same point differ by an element of $2\pi i\,\mathbb{Z}$, it follows that

$$\int_\gamma \frac{g'(z)}{g(z)}\,dz \in 2\pi i\,\mathbb{Z}.$$

Therefore,

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = n\int_\gamma \frac{g'(z)}{g(z)}\,dz \in 2\pi i\,n\mathbb{Z},$$

as desired.

(ii)⇒(i). Fix a basepoint $z_0 \in \Omega$ and choose $c \in \mathbb{C}^*$ such that $c^n = f(z_0)$. Set

$$h(z) := \frac{1}{n}\frac{f'(z)}{f(z)}.$$

Then h is continuous on Ω, so line integrals $\int_\alpha h(z)\,dz$ along oriented piecewise C^1 curves are well-defined. Moreover, (ii) says that for every oriented piecewise C^1 regular loop γ in Ω,

$$\int_\gamma h(z)\,dz \in 2\pi i\,\mathbb{Z}.$$

For each $z \in \Omega$, choose an oriented piecewise C^1 path $\alpha_{z_0,z}$ from z_0 to z and define

$$g(z) := c\exp\left(\int_{\alpha_{z_0,z}} h(w)\,dw\right).$$

We claim that g is independent of the choice of the path. Indeed, if α and β are two paths from z_0 to z, then $\alpha * \overline{\beta}$ is a loop, so

$$\int_\alpha h - \int_\beta h = \int_{\alpha*\overline{\beta}} h \in 2\pi i\,\mathbb{Z}.$$

Exponentiating gives $\exp(\int_\alpha h) = \exp(\int_\beta h)$, hence $g(z)$ is well-defined.

Now we prove g is holomorphic in Ω. Fix $z \in \Omega$ and choose a disk $D := D(z; r) \subset \Omega$. Let η be any oriented piecewise C^1 regular loop in D. Since D is convex, we can shrink η continuously to a constant loop inside D by the family

$$\eta_s(t) := z_* + s\big(\eta(t) - z_*\big), \qquad 0 \le s \le 1,$$

where $z_* \in D$ is fixed. For each s, η_s is again a loop in Ω, so by (ii),

$$I(s) := \int_{\eta_s} h(z)\, dz \in 2\pi i\, \mathbb{Z}.$$

On the other hand, since h is continuous and $(t, s) \mapsto \eta_s(t)$ is continuous, the integral $I(s)$ depends continuously on s. As $2\pi i\, \mathbb{Z}$ is a discrete subset of $\mathbb{C}$, $I(s)$ must be constant. Since $I(0) = 0$, we obtain $I(1) = 0$, i.e.

$$\int_{\eta} h(z)\, dz = 0.$$

Therefore $\int_\eta h\, dz = 0$ for every loop $\eta \subset D$, and by the primitive criterion h admits a holomorphic primitive H on D with $H' = h$.

Now fix once and for all a piecewise C^1 path α from z_0 to z. For $\zeta \in D$, let $\beta_{z,\zeta}$ be the straight line segment from z to ζ (which lies in D since D is convex), and define the path

$$\alpha_\zeta := \alpha * \beta_{z,\zeta}$$

(concatenation: first follow α from z_0 to z, then follow $\beta_{z,\zeta}$ from z to ζ). By path-independence, we may compute $g(\zeta)$ using this particular path α_ζ.

Since $H' = h$ on D, the function H is a primitive of h on D, hence

$$\int_{\beta_{z,\zeta}} h(w)\, dw = H(\zeta) - H(z).$$

Therefore

$$\int_{\alpha_\zeta} h = \int_\alpha h + \int_{\beta_{z,\zeta}} h = \int_\alpha h + H(\zeta) - H(z),$$

and so

$$g(\zeta) = c \exp\left(\int_{\alpha_\zeta} h \right) = \left(c\, e^{\int_\alpha h - H(z)} \right) e^{H(\zeta)} =: C\, e^{H(\zeta)},$$

where $C := \left(c\, e^{\int_\alpha h - H(z)} \right)$ is a constant. Since H is holomorphic on D, this shows that g is holomorphic on D. As z is arbitrary, $g \in \mathcal{O}(\Omega)$.

Finally, we verify that $g^n = f$ on Ω. By definition,

$$g(z)^n = c^n \exp\left(\int_{\alpha_{z_0,z}} \frac{f'(w)}{f(w)}\, dw \right).$$

Cover the path $f \circ \alpha_{z_0,z}$ by finitely many small disks $D_1, \cdots, D_N$ on which a branch of logarithm exists, and choose these branches to match on overlaps. As the telescoping argument we used in previous proofs, all intermediate terms cancel, leaving

$$\int_{\alpha_{z_0,z}} \frac{f'(w)}{f(w)}\, dw = \mathrm{Log}_{D_N} f(z) - \mathrm{Log}_{D_1} f(z_0),$$

so

$$g(z)^n = c^n \frac{f(z)}{f(z_0)} = f(z),$$

since $c^n = f(z_0)$.

We are done.

□

The logarithm condition requires all periods—that is, all loop integrals of $\frac{f'}{f}$—to vanish, whereas the nth-root condition only requires these periods to vanish modulo $2\pi i n$. Thus it may happen that f admits a holomorphic branch of an nth root on Ω even though f admits no holomorphic logarithm on Ω. Here is an example.

EXAMPLE 4.3 (A square root may exist even when a logarithm does not). Let

$$\Omega := \mathbb{C} \setminus [0,1], \qquad f(z) := z(z-1).$$

Then f has no zeros on Ω, so $f : \Omega \to \mathbb{C}^*$ is holomorphic. Moreover,

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \frac{1}{z-1} \qquad (z \in \Omega).$$

Let γ be a counterclockwise oriented piecewise C^1 regular simple loop in Ω whose interior contains the segment $[0,1]$. Then γ winds once around both 0 and 1, hence

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma \left(\frac{1}{z} + \frac{1}{z-1}\right) dz = 2\pi i + 2\pi i = 4\pi i.$$

This implies two things:

(1) By Theorem 4.1, f admits no holomorphic logarithm on Ω since condition (iii) fails for this loop.
(2) By Theorem 4.2 with $n = 2$, f admits a holomorphic square root on Ω (since $4\pi i \in 2\pi i \cdot 2\mathbb{Z}$ and $\pi_1(\Omega; 2) \cong \mathbb{Z}$).

Concretely, one can write down the two branches as follows. For $z \in \Omega$, write

$$z = r_0 e^{i\theta_0}, \qquad z - 1 = r_1 e^{i\theta_1},$$

where $r_0 = |z| > 0$, $r_1 = |z-1| > 0$, and $\theta_0, \theta_1 \in (-\pi, \pi]$.

Define

$$g_+(z) := \sqrt{r_0 r_1}\, e^{\frac{i}{2}(\theta_0+\theta_1)}.$$

It is holomorphic in Ω, even though θ_0, θ_1 are NOT continuous in z!

The other branch is

$$g_-(z) := -g_+(z) \in \mathcal{O}(\Omega),$$

and they both satisfy $g(z)^2 = f(z)$.

The next example shows that this phenomenon does *not* occur for the inclusion map $f(z) = z$. In other words, for a domain $\Omega \subset \mathbb{C}^*$, if Ω admits a branch of the nth root, then it also admits a branch of the logarithm.

EXAMPLE 4.4 (A branch of $z^{1/n}$ forbids winding around 0). Let $\Omega \subset \mathbb{C}^*$ be a domain and fix $n \geq 2$. Assume there exists a holomorphic branch of the nth root on Ω, i.e. a holomorphic function

$$\mathrm{Root}_\Omega : \Omega \to \mathbb{C}^*, \qquad \mathrm{Root}_\Omega(z)^n = z.$$

Apply Theorem 4.2 to the inclusion map $f(z) = z$. Since $f'/f = 1/z$, we obtain

$$\int_\gamma \frac{dz}{z} \in 2\pi i\, n\mathbb{Z} \qquad \text{for every loop } \gamma \text{ in } \Omega.$$

Now, if Ω contains a loop γ which is not homotopic in $\mathbb{C}^*$ to a constant loop, then one can find a piecewise C^1 simple loop $\hat{\gamma} \subset \Omega$, built from subarcs of γ, which encloses 0. (See the remark below for the proof of this fact using complex analysis.) Hence

$$\int_{\hat{\gamma}} \frac{dz}{z} = \pm 2\pi i,$$

which is *not* in $2\pi i\, n\mathbb{Z}$ for $n \geq 2$. Therefore, if a branch of the nth root exists on Ω, then every loop in Ω must be homotopic in $\mathbb{C}^*$ to the constant loop, and consequently

$$\int_{\gamma} \frac{dz}{z} = 0 \qquad \text{for every loop } \gamma \subset \Omega.$$

Therefore Ω admits a branch of the logarithm Log_{Ω}. This branch then determines all branches of the nth root. (See Exercise: 3.9.)

REMARK 4.5. Suppose $\Omega \subset \mathbb{C}^*$ contains a loop γ which is not homotopic in $\mathbb{C}^*$ to a constant loop. Then

$$\int_{\gamma} \frac{1}{z}\, dz \neq 0.$$

Since Ω is open, we may perturb γ slightly inside Ω and assume (without changing the value of the integral) that γ is a piecewise C^1 regular loop with only finitely many *transverse* self-intersections.

Let P be the (finite) set of self-intersection points of γ. Transversality implies that each $p \in P$ has exactly two distinct preimages in $S^1 = [0,1]/\{0,1\}$; denote them by t_p^-, t_p^+. Let

$$T := \bigcup_{p \in P} \{t_p^-, t_p^+\} \subset S^1$$

be the corresponding finite set of marked parameter values.

We now extract simple loops from γ. Choose $t_0 \in S^1 \setminus T$ and traverse S^1 in the positive direction. Let b be the first time at which the image meets a previously visited point, i.e. there exists $a < b$ with $\gamma(a) = \gamma(b)$, and b is minimal with this property. By minimality and transversality, the subpath $\gamma|_{[a,b]}$ has no self-intersections except at its endpoints, hence it forms a piecewise C^1 *simple* loop; denote it by η_1, oriented as γ. Remove the parameter interval (a, b) and repeat the same construction on the remaining parts of the parameter circle. Since T is finite, this procedure terminates after finitely many steps and yields simple loops

$$\eta_1, \ldots, \eta_m,$$

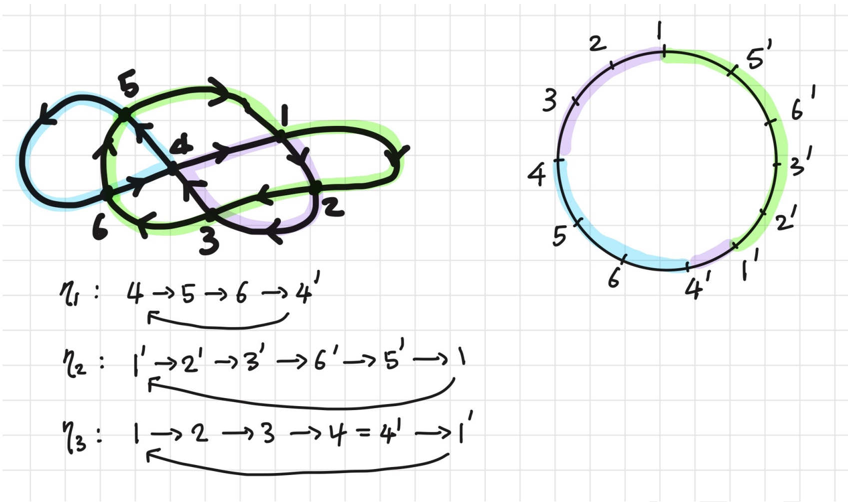
each obtained by concatenating subarcs of γ and each inheriting the orientation of γ. Moreover, the chosen intervals cover S^1 in cyclic order, so after a reparametrization we may view γ as the concatenation $\gamma = \eta_1 * \cdots * \eta_m$. Therefore, by additivity of contour integrals under concatenation,

$$\int_{\gamma} \frac{1}{z}\, dz = \sum_{j=1}^{m} \int_{\eta_j} \frac{1}{z}\, dz.$$

Since the left-hand side is nonzero, we must have $\int_{\eta_j} \frac{1}{z}\, dz \neq 0$ for some j. Let $\hat{\gamma} := \eta_j$. Then $\hat{\gamma} \subset \Omega$ is a piecewise C^1 simple loop built from subarcs of γ and

$$\int_{\hat{\gamma}} \frac{1}{z}\, dz \neq 0.$$

In particular, $\hat{\gamma}$ encloses 0. This proves the fact we used above.


η₁ : 4 → 5 → 6 → 4'
η₂ : 1' → 2' → 3' → 6' → 5' → 1
η₃ : 1 → 2 → 3 → 4 = 4' → 1'

CHAPTER 5

Cauchy's integral formula and its applications

1. Cauchy's integral formula and Cauchy's inequality

We first see a motivating example.

EXAMPLE 1.1. Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is a power series centered at z_0 with radius of convergence $R > 0$. Let $D = D(z_0; r)$ be a disk with $0 < r < R$ and let ∂D carry the positive (counterclockwise) orientation. We compute the following integrals over ∂D.

(1) Since $f \in \mathcal{O}(D(z_0; R))$, Cauchy's theorem gives

$$\int_{\partial D} f(z)\, dz = 0.$$

(2) Write

$$\frac{f(z)}{z - z_0} = \frac{a_0}{z - z_0} + a_1 + a_2(z - z_0) + \cdots.$$

The tail $a_1 + a_2(z - z_0) + \cdots$ is holomorphic on a neighborhood of $\overline{D}$, hence admits a primitive on D, so its integral over ∂D is 0. Therefore

$$\int_{\partial D} \frac{f(z)}{z - z_0}\, dz = \int_{\partial D} \frac{a_0}{z - z_0}\, dz = a_0 \cdot 2\pi i = 2\pi i\, f(z_0).$$

(3) Similarly,

$$\frac{f(z)}{(z - z_0)^2} = \frac{a_0}{(z - z_0)^2} + \frac{a_1}{z - z_0} + a_2 + a_3(z - z_0) + \cdots,$$

and all terms except $\frac{a_1}{z - z_0}$ have primitives on D. Hence

$$\int_{\partial D} \frac{f(z)}{(z - z_0)^2}\, dz = 2\pi i\, a_1 = \frac{2\pi i}{1!} f'(z_0).$$

(4) Likewise,

$$\int_{\partial D} \frac{f(z)}{(z - z_0)^3}\, dz = 2\pi i\, a_2 = \frac{2\pi i}{2!} f''(z_0).$$

In general, for $m \geq 0$,

$$\int_{\partial D} \frac{f(z)}{(z - z_0)^{m+1}}\, dz = \frac{2\pi i}{m!} f^{(m)}(z_0).$$

We will show that this formula holds for all holomorphic functions $f \in \mathcal{O}(\Omega)$.

THEOREM 1.2 (Cauchy's integral formula). *Let $\Omega \subset \mathbb{C}$ be open and let $f \in \mathcal{O}(\Omega)$. Let D be a disk with $\overline{D} \subset \Omega$ and $z_0 \in D$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0}\, dz.$$

PROOF. We write

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-z_0}\,dz &= \frac{1}{2\pi i}\int_{\partial D}\frac{f(z_0)+(f(z)-f(z_0))}{z-z_0}\,dz\\ &= \frac{1}{2\pi i}\int_{\partial D}\frac{f(z_0)}{z-z_0}\,dz+\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)-f(z_0)}{z-z_0}\,dz\\ &= f(z_0)+\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)-f(z_0)}{z-z_0}\,dz,\end{aligned}$$

since $\int_{\partial D}\frac{dz}{z-z_0}=2\pi i$.

It remains to show

$$\int_{\partial D}\frac{f(z)-f(z_0)}{z-z_0}\,dz=0.$$

Choose $\varepsilon>0$ so small that $\overline{D(z_0;\varepsilon)}\subset D$. Set

$$g(z):=\frac{f(z)-f(z_0)}{z-z_0},\qquad z\in D\setminus\{z_0\}.$$

Then g is holomorphic on $\Omega\setminus\{z_0\}$.

By the homotopy invariance of holomorphic integral (or by Cauchy's theorem on a region with two boundary components), we have

$$\int_{\partial D}g(z)\,dz=\int_{\partial D(z_0;\varepsilon)}g(z)\,dz.$$

In particular, the RHS is independent of $\varepsilon>0$.

Now use complex differentiability of f at z_0: write

$$g(z)=f'(z_0)+\phi(z),\qquad \lim_{z\to z_0}\phi(z)=0.$$

It follows

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)-f(z_0)}{z-z_0}\,dz &= \frac{1}{2\pi i}\int_{\partial D(z_0;\varepsilon)}g(z)\,dz\\ &= \frac{1}{2\pi i}\int_{\partial D(z_0;\varepsilon)}\big(f'(z_0)+\phi(z)\big)\,dz\\ &= \frac{1}{2\pi i}\int_{\partial D(z_0;\varepsilon)}\phi(z)\,dz,\end{aligned}$$

since $\int_{\partial D(z_0;\varepsilon)}f'(z_0)\,dz=0$.

Now we estimate

$$\left|\frac{1}{2\pi i}\int_{\partial D(z_0;\varepsilon)}\phi(z)\,dz\right|\le\frac{1}{2\pi}\,\mathrm{L}ength(\partial D(z_0;\varepsilon))\sup_{|z-z_0|=\varepsilon}|\phi(z)|=\varepsilon\sup_{|z-z_0|=\varepsilon}|\phi(z)|.$$

This tends to 0 as $\varepsilon\to 0^+$, hence

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)-f(z_0)}{z-z_0}\,dz=0,$$

and the desired identity follows:

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-z_0}\,dz=f(z_0).$$

□

EXAMPLE 1.3. Evaluate

$$\int_{\gamma}\frac{e^z+z^2+3}{z-2}\,dz.$$

(1) If $\gamma = \partial D(0;3)$ (counterclockwise), then $2 \in D(0;3)$, so by Cauchy's integral formula,

$$\int_\gamma \frac{e^z + z^2 + 3}{z-2}\, dz = 2\pi i\big(e^2 + 2^2 + 3\big) = 2\pi i\,(e^2 + 7).$$

(2) If $\gamma = \partial D(0;1)$ (counterclockwise), then $2 \notin D(0;1)$, and the integrand is holomorphic on a neighborhood of $\overline{D(0;1)}$. Hence by Cauchy's theorem,

$$\int_\gamma \frac{e^z + z^2 + 3}{z-2}\, dz = 0.$$

EXAMPLE 1.4. We can redo the integral

$$\int_\gamma \frac{1}{z(z-1)}\, dz, \qquad \gamma = \partial D\Big(1; \frac{1}{2}\Big) \text{ (counterclockwise)},$$

using Cauchy's integral formula.

Since $D(1;\frac{1}{2})$ contains 1 but not 0, the function $f(z) = \frac{1}{z}$ is holomorphic on a neighborhood of $\overline{D(1;\frac{1}{2})}$. Write

$$\frac{1}{z(z-1)} = \frac{f(z)}{z-1}.$$

Then by Cauchy's integral formula (with $z_0 = 1$),

$$\int_\gamma \frac{1}{z(z-1)}\, dz = \int_\gamma \frac{f(z)}{z-1}\, dz = 2\pi i\, f(1) = 2\pi i.$$

Next, we improve Cauchy's integral formula to arbitrary order derivatives.

THEOREM 1.5 (Cauchy's integral formula for derivatives). *Let $\Omega \subset \mathbb{C}$ be open and let $f \in \mathcal{O}(\Omega)$. Then f has derivatives of all orders on Ω. Moreover, if D is a disk with $\overline{D} \subset \Omega$ and $z_0 \in D$, then for every $n \in \mathbb{Z}^{\geq 0}$,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

PROOF. For $n = 0$ this is exactly Cauchy's integral formula:

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0}\, dz.$$

Assume the result holds for some $n = k \geq 0$, and we prove it for the case $n = k+1$.

For any $z_0 \in D$, we can take a disk D that contains z_0 and the closure $\overline{D} \subset \Omega$.

Now for h sufficiently small so that $z_0 + h \in D$, by the induction hypothesis we have $f^{(k)}$ exists for every point in Ω, and

$$f^{(k)}(z_0 + h) = \frac{k!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z - (z_0+h))^{k+1}}\, dz, \qquad f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-z_0)^{k+1}}\, dz.$$

Hence

$$\begin{aligned}
\frac{f^{(k)}(z_0+h) - f^{(k)}(z_0)}{h} &= \frac{k!}{2\pi i} \int_{\partial D} f(z)\, \frac{1}{h} \left(\frac{1}{(z-(z_0+h))^{k+1}} - \frac{1}{(z-z_0)^{k+1}} \right) dz \\
&= \frac{k!}{2\pi i} \int_{\partial D} f(z)\, \frac{(z-z_0)^{k+1} - (z-(z_0+h))^{k+1}}{h\,(z-(z_0+h))^{k+1}(z-z_0)^{k+1}}\, dz.
\end{aligned}$$

Using the factorization

$$A^{k+1} - B^{k+1} = (A-B) \sum_{j=0}^{k} A^{k-j} B^j, \qquad A = z - z_0, \;\; B = z - (z_0 + h),$$

we get

$$\frac{(z-z_0)^{k+1}-(z-(z_0+h))^{k+1}}{h}=\sum_{j=0}^{k}(z-z_0)^{k-j}(z-(z_0+h))^j.$$

Therefore

$$\frac{f^{(k)}(z_0+h)-f^{(k)}(z_0)}{h}=\frac{k!}{2\pi i}\int_{\partial D} f(z)\,\frac{\sum_{j=0}^{k}(z-z_0)^{k-j}(z-(z_0+h))^j}{(z-z_0-h)^{k+1}(z-z_0)^{k+1}}\,dz.$$

Now let $h\to 0$. For $z\in\partial D$, the denominator never vanishes, so the integrand converges uniformly on ∂D to

$$f(z)\cdot\frac{(k+1)}{(z-z_0)^{k+2}}.$$

Hence we may pass the limit through the integral, and obtain

$$f^{(k+1)}(z_0)=\lim_{h\to 0}\frac{f^{(k)}(z_0+h)-f^{(k)}(z_0)}{h}=\frac{k!}{2\pi i}\int_{\partial D} f(z)\,\frac{k+1}{(z-z_0)^{k+2}}\,dz=\frac{(k+1)!}{2\pi i}\int_{\partial D}\frac{f(z)}{(z-z_0)^{k+2}}\,dz.$$

This completes the induction. □

EXAMPLE 1.6. Use Cauchy's integral formula to compute

$$\int_{\partial D_3(0)}\frac{e^z+z^2+3}{(z-2)^2}\,dz.$$

SOLUTION. Let $f(z):=e^z+z^2+3$, which is entire. Since $2\in D_3(0)$, Cauchy's integral formula for derivatives (with $n=1$) gives

$$\frac{1}{2\pi i}\int_{\partial D_3(0)}\frac{f(z)}{(z-2)^2}\,dz=f'(2).$$

Compute $f'(z)=e^z+2z$, hence $f'(2)=e^2+4$. Therefore

$$\int_{\partial D_3(0)}\frac{e^z+z^2+3}{(z-2)^2}\,dz=2\pi i\,(e^2+4).$$

□

A corollary of Cauchy's integral formula is the following Cauchy inequality.

THEOREM 1.7 (Cauchy inequality). *Let $\Omega\subset\mathbb{C}$ be open and let $f\in\mathcal{O}(\Omega)$. Fix $z_0\in\Omega$ and $R>0$ such that $\overline{D(z_0;R)}\subset\Omega$. Then for every $n\in\mathbb{Z}^{\geq 0}$,*

$$|f^{(n)}(z_0)|\leq\frac{n!}{R^n}\,\|f\|_{\partial D(z_0;R)},\qquad \|f\|_{\partial D(z_0;R)}:=\sup_{|z-z_0|=R}|f(z)|.$$

PROOF. By Cauchy's integral formula for derivatives,

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_{\partial D(z_0;R)}\frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

Taking norms and using the estimate $\left|\int_\gamma g(z)\,dz\right|\leq\sup_\gamma|g|\,Length(\gamma)$ (from Lemma 1.9), we obtain

$$\begin{aligned}|f^{(n)}(z_0)|&\leq\frac{n!}{2\pi}\sup_{|z-z_0|=R}\left|\frac{f(z)}{(z-z_0)^{n+1}}\right|\cdot Length(\partial D(z_0;R))\\&=\frac{n!}{2\pi}\sup_{|z-z_0|=R}\frac{|f(z)|}{R^{n+1}}\cdot 2\pi R=\frac{n!}{R^n}\,\|f\|_{\partial D(z_0;R)}.\end{aligned}$$

□

An important application of Cauchy's inequality is the following Liouville's theorem.

THEOREM 1.8 (Liouville's theorem). *If f is entire and bounded, then f is constant.*

PROOF. Assume f is entire and $|f(z)| \le M$ for all $z \in \mathbb{C}$. For any point $z_0 \in \mathbb{C}$ and any $R > 0$, apply Theorem 1.7 with $n = 1$ to the disk $D(z_0; R)$:

$$|f'(z_0)| \le \frac{1!}{R} \|f\|_{\partial D(z_0;R)} \le \frac{M}{R}.$$

Letting $R \to \infty$ yields $f'(z_0) = 0$. Since z_0 is arbitrary, we have $f' \equiv 0$ on $\mathbb{C}$. By Corollary 2.5, f is constant. □

As an example, we use Liouville's theorem to prove the fundamental theorem of algebra.

EXAMPLE 1.9 (Fundamental theorem of algebra). Every nonconstant complex polynomial has a zero in $\mathbb{C}$.

PROOF. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \qquad (a_n \neq 0,\ n \ge 1).$$

Assume for contradiction that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then the function

$$f(z) := \frac{1}{P(z)}$$

is entire.

We claim that f is bounded. Since $|P(z)| \to \infty$ as $|z| \to \infty$, there exists $R > 0$ such that

$$|P(z)| \ge 1 \qquad (|z| \ge R),$$

hence $|f(z)| \le 1$ for $|z| \ge R$. On the closed disk $\overline{D(0;R)}$, the continuous function $|f|$ attains a maximum:

$$M := \max_{|z| \le R} |f(z)| < \infty.$$

Therefore $|f(z)| \le \max\{M, 1\}$ for all $z \in \mathbb{C}$, so f is bounded.

By Liouville's theorem, f must be constant, hence P is constant, a contradiction. Therefore P has at least one zero in $\mathbb{C}$. □

REMARK 1.10 (Counting zeros with multiplicity). Once we know that a nonconstant polynomial has at least one complex root, we can strengthen the conclusion of the Fundamental Theorem of Algebra and show that a polynomial p of degree n has exactly n zeros in $\mathbb{C}$ *counted with multiplicity*, by a purely algebraic factorization argument.

Let $p(z)$ be a nonconstant polynomial of degree n. By the Fundamental Theorem of Algebra, there exists $z_1 \in \mathbb{C}$ such that $p(z_1) = 0$. Dividing $p(z)$ by $z - z_1$, the remainder must be the constant $p(z_1) = 0$, hence $(z - z_1)$ divides $p(z)$ and we can write

$$p(z) = (z - z_1) q_1(z),$$

where $q_1(z)$ is a polynomial of degree $n - 1$. If q_1 is nonconstant, we can apply the theorem again to find a root z_2 of q_1, and factor

$$q_1(z) = (z - z_2) q_2(z).$$

Iterating this process, we obtain

$$p(z) = c\,(z - z_1)(z - z_2) \cdots (z - z_n),$$

for some constant $c \in \mathbb{C}^*$, where the z_j's are complex numbers (not necessarily distinct).

2. Holomorphic functions are analytic

A complex-valued function f defined on an open set $\Omega \subset \mathbb{C}$ is called **(complex) analytic** if for every $z_0 \in \Omega$, f admits a power series expansion about z_0 in some neighborhood of z_0.

We have seen that every analytic function is holomorphic. In this section, we will use Cauchy's integral formula to prove the converse: every holomorphic function is analytic.

THEOREM 2.1. *Let $\Omega \subset \mathbb{C}$ be open and let $f \in \mathcal{O}(\Omega)$. Fix $z_0 \in \Omega$ and a radius $R > 0$ such that $\overline{D(z_0; R)} \subset \Omega$. Then for every $z \in D(z_0; R)$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the series converges on $D(z_0; R)$ and

$$a_n = \frac{1}{2\pi i} \int_{\partial D(z_0;R)} \frac{f(w)}{(w - z_0)^{n+1}}\, dw = \frac{f^{(n)}(z_0)}{n!}.$$

PROOF. Fix $z \in D(z_0; R)$. By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(z_0;R)} \frac{f(w)}{w - z}\, dw.$$

Write

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}}.$$

For $w \in \partial D(z_0; R)$ we have $|w - z_0| = R$, while $|z - z_0| < R$, hence

$$\left| \frac{z - z_0}{w - z_0} \right| < 1.$$

Therefore the geometric series gives

$$\frac{1}{1 - \frac{z - z_0}{w - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n,$$

and so

$$\frac{1}{w - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}.$$

Since f is continuous on the compact circle $\partial D(z_0; R)$, there is $M > 0$ with $|f(w)| \leq M$ on $\partial D(z_0; R)$. Moreover, for $w \in \partial D(z_0; R)$,

$$\left| f(w) \frac{(z - z_0)^n}{(w - z_0)^{n+1}} \right| \leq M\, \frac{|z - z_0|^n}{R^{n+1}}.$$

The majorant series $\sum_{n=0}^{\infty} M \frac{|z - z_0|^n}{R^{n+1}}$ converges, so we may integrate term-by-term:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D(z_0;R)} f(w) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}\, dw \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial D(z_0;R)} \frac{f(w)}{(w - z_0)^{n+1}}\, dw \right) (z - z_0)^n \\ &=: \sum_{n=0}^{\infty} a_n (z - z_0)^n, \end{aligned}$$

which proves the power series expansion on $D(z_0; R)$.

Finally, by Cauchy's formula for derivatives,

$$a_n = \frac{1}{2\pi i}\int_{\partial D(z_0;R)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw = \frac{f^{(n)}(z_0)}{n!}.$$

□

We remark that in the proof of Theorem 2.1, the only assumption on the disk is that

$$\overline{D(z_0;R)} \subset \Omega.$$

The specific value of R is irrelevant. In particular, if f is entire, then for every $z_0 \in \mathbb{C}$ and every $R > 0$ the restriction of f to $D(z_0;R)$ has a power series expansion about z_0. Hence every entire function admits a global power series expansion about any center:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (z \in \mathbb{C}), \qquad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

REMARK 2.2 (Analyticity in the real and complex settings). The terminology *analytic* is not restricted to complex-valued functions. A real-valued function u on an open interval $I \subset \mathbb{R}$ is called **(real) analytic** if for every $x_0 \in I$ there exists $r > 0$ and real coefficients c_n such that

$$u(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n \qquad (|x-x_0| < r).$$

For complex-valued functions we have proved the fundamental equivalence

$$\text{holomorphic} \iff \text{(complex) analytic}.$$

For real-valued functions, analyticity is *stronger* than smoothness:

$$\text{(real) analytic} \Rightarrow C^\infty,$$

but the converse is false.

Here is a typical example you should have seen in real analysis class. Define

$$u(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $u \in C^\infty(\mathbb{R})$ and all derivatives satisfy $u^{(n)}(0) = 0$, so the Taylor series at 0 is identically 0, yet $u(x) > 0$ for $x \neq 0$. Hence u is not real analytic at 0.

3. The identity theorem

We now prove the identity theorem for holomorphic functions. It implies that a holomorphic function is either identically zero, or else its zeros are isolated. This is another key difference between holomorphic functions and smooth functions: it is easy to construct a nonzero smooth function that vanishes on a nonempty open subset. (See Remark 3.9.)

THEOREM 3.1 (Identity theorem). *Let $\Omega \subset \mathbb{C}$ be a domain and let $f \in \mathcal{O}(\Omega)$. If there exists a sequence of* distinct *points $z_n \in \Omega$ with $z_n \to z_0 \in \Omega$ and $f(z_n) = 0$ for all n, then $f \equiv 0$ on Ω.*

PROOF. Since f is continuous, $f(z_0) = \lim_{n\to\infty} f(z_n) = 0$. Choose $R > 0$ such that $\overline{D(z_0; R)} \subset \Omega$. By Theorem 2.1, f admits a power series expansion on $D(z_0; R)$:

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m.$$

If $f \not\equiv 0$ on $D(z_0; R)$, let $k \geq 0$ be the smallest index with $a_k \neq 0$. Then

$$f(z) = (z - z_0)^k \, \phi(z), \qquad \phi(z) := \sum_{m=0}^{\infty} a_{m+k} (z - z_0)^m,$$

and ϕ is holomorphic on $D(z_0; R)$ with $\phi(z_0) = a_k \neq 0$. By continuity, $\phi(z) \neq 0$ for all z in some smaller disk $D(z_0; r)$. But $z_n \to z_0$ implies $z_n \in D(z_0; r)$ for all large n, and then

$$0 = f(z_n) = (z_n - z_0)^k \phi(z_n)$$

forces $z_n = z_0$ for all large n, contradicting that the z_n are distinct. Hence $f \equiv 0$ on a neighborhood of z_0.

Now define

$$A := \{z \in \Omega : f \text{ vanishes on some neighborhood of } z\}.$$

We have $A \neq \emptyset$ and A is open by definition.

Now we prove A is also closed. For this, take a convergent sequence $z_m \in A$ that converges to $z \in \Omega$. Then for each m, $f(z_m) = 0$. Therefore (z_m) is a sequence of zeros converging to z just as the assumption we started with, and we have proved that f vanishes in an open neighborhood of z, and thus $z \in A$. This proves A is closed in Ω.

Since Ω is connected and A is nonempty, open, and closed, we conclude $A = \Omega$, so $f \equiv 0$ on Ω. $\square$

REMARK 3.2 (A boundary accumulation of zeros does not force $f \equiv 0$). The identity theorem requires that the zeros have an accumulation point *inside* the domain. If a sequence of distinct zeros $z_n \in \Omega$ only accumulates at a boundary point $z_0 \in \partial\Omega$, then a nonzero holomorphic function may still vanish at every z_n.

For example, let $\Omega = D(0; 1)$ and

$$f(z) := \sin\left(\frac{1}{1-z}\right) \in \mathcal{O}(\Omega).$$

If we set $z_n := 1 - \frac{1}{n\pi}$, then $z_n \in \Omega$, $z_n \to 1 \notin \Omega$, and

$$f(z_n) = \sin(n\pi) = 0 \qquad (\forall n),$$

but $f \not\equiv 0$ on Ω.

DEFINITION 3.3 (Extensions and analytic continuation). Let $U \subset V$ be subsets of $\mathbb{C}$.

(1) Suppose $f : U \to \mathbb{C}$ and $\tilde{f} : V \to \mathbb{C}$ satisfy $\tilde{f}|_U = f$, i.e.

$$\tilde{f}(z) = f(z) \qquad (\forall z \in U).$$

Then we say that $\tilde{f}$ is an **extension** of f from U to V.

(2) Suppose moreover that U and V are open, and that $f \in \mathcal{O}(U)$ and $\tilde{f} \in \mathcal{O}(V)$. Then we say that $\tilde{f}$ is a **holomorphic extension/continuation** (or **analytic extension/continuation**) of f from U to V.

From the identity theorem we obtain the following uniqueness statement for analytic continuation.

COROLLARY 3.4 (Uniqueness of analytic continuation). *Let $\Omega \subset \mathbb{C}$ be a domain and let $U \subset \Omega$ be a nonempty open subset. If $f \in \mathcal{O}(U)$ admits analytic continuations $\tilde{f}_1, \tilde{f}_2 \in \mathcal{O}(\Omega)$, i.e.*

$$\tilde{f}_1|_U = f = \tilde{f}_2|_U,$$

then $\tilde{f}_1 \equiv \tilde{f}_2$ on Ω. In particular, f has at most one *holomorphic continuation from U to Ω.*

PROOF. Let $g := \tilde{f}_1 - \tilde{f}_2 \in \mathcal{O}(\Omega)$. Then $g \equiv 0$ on U. Choose any $z_0 \in U$ and $r > 0$ such that $D(z_0; r) \subset U$. Define

$$z_n := z_0 + \frac{r}{2n} \qquad (n \geq 1).$$

Then $z_n \in D(z_0; r) \subset U$, the points z_n are distinct, and $z_n \to z_0$. Moreover $g(z_n) = 0$ for all n. By the identity theorem, $g \equiv 0$ on Ω. Hence $\tilde{f}_1 \equiv \tilde{f}_2$ on Ω. □

REMARK 3.5 (Nonexistence of analytic continuation). A holomorphic function on an open set need not admit any analytic contiuation to a larger domain. For instance, $f(z) = \dfrac{1}{z}$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ cannot be extended holomorphically to $\mathbb{C}$.

PROOF IDEA (SEVERAL QUICK CONTRADICTIONS). Suppose $\tilde{f} \in \mathcal{O}(\mathbb{C})$ satisfies $\tilde{f}(z) = 1/z$ for all $z \neq 0$. Then $\tilde{f}$ cannot be continuous at 0 since $\frac{1}{z}$ is unbounded as $z \to 0$; contradiction. (You can also obtain contradictions by looking at $\tilde{f}'(0)$ from the definition, by integrating $\tilde{f}$ around a small circle centered at 0, by applying Liouville's theorem to $z\tilde{f}(z)$, or by using Cauchy's integral formula to force a value at 0 that is incompatible with $\tilde{f}(z) = 1/z$ on $\mathbb{C}^*$.) □

Given a holomorphic function f on a domain U, one can ask whether it admits an analytic contiuation to a larger domain $V \supset U$. A point where extension is impossible is called a non-removable singularity of the continuation. We will introduce the terms of singularity in the next chapter. Here 0 is a non-removable singularity of $\dfrac{1}{z}$, in fact, a simple pole singularity.

EXAMPLE 3.6. A power series may extend beyond its disk of convergence. For example, consider the geometric series

$$\sum_{n=0}^{\infty} z^n.$$

It has radius of convergence 1, hence defines a holomorphic function on $D(0; 1)$. On $|z| < 1$ we have the identity

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

Since $\frac{1}{1-z}$ is holomorphic on $\mathbb{C} \setminus \{1\}$, this identity provides an analytic continuation of the power series from $D(0; 1)$ to $\mathbb{C} \setminus \{1\}$. The obstruction to continuing further is the pole at $z = 1$.

EXAMPLE 3.7 (More analytic continuations from power series). (1) On $|z| < 1$ we have

$$-\mathrm{Log}(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n},$$

where Log is the principal logarithm on $\mathbb{C} \setminus (-\infty, 1]$.

Thus the power series defines a holomorphic function on $D(0; 1)$ which admits an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

(2) Consider the power series

$$F(z) := \sum_{n=0}^{\infty} z^{n!}.$$

The convergence radius is 1, hence defines a holomorphic function on $D(0;1)$. In fact, the unit circle $|z| = 1$ is a natural boundary for F so that there is no holomorphic extension beyond the unit disk. Here is some explanation.

REASON: Indeed, if ζ is a root of unity, say $\zeta^N = 1$, then for every $n \geq N$ we have

$$\zeta^{n!} = 1.$$

Write

$$F(r\zeta) = \sum_{n=0}^{N-1} (r\zeta)^{n!} + \sum_{n=N}^{\infty} r^{n!} \xrightarrow[r\to 1^-]{} \infty.$$

Since roots of unity are dense on the unit circle, every neighborhood of every point of $|z| = 1$ contains such a ζ.

If F admitted a holomorphic continuation across some boundary point, then that continuation would be locally bounded near that point, contradicting the fact that $F(r\zeta) \to \infty$ along a sequence $r\zeta$ approaching it from inside the disk.

Hence F cannot be analytically continued across any point of $|z| = 1$, so the unit circle is a natural boundary. □

REMARK 3.8. Analytic continuation is actually a very important terminology throughout mathematics. For example, the famous *Riemann Hypothesis* concerns the analytic continuation of the *Riemann zeta function*. Originally, the zeta function is defined by the convergent series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \qquad \Re(z) > 1.$$

A fundamental result in complex analysis is that $\zeta(z)$ admits an analytic continuation to $\mathbb{C} \setminus \{1\}$. The Riemann Hypothesis asserts that all nontrivial zeros of this analytically continued function (that is, all zeros other than the negative even integers $-2, -4, -6, \dots$) lie on the line

$$\Re(z) = \frac{1}{2}.$$

REMARK 3.9 (Smooth extensions are highly non-unique). If we replace "holomorphic" by "smooth" (real C^∞), then extensions are typically abundant and far from unique. For example, define on $\mathbb{R}^2$ the smooth "bump" function

$$g(x,y) := \begin{cases} \exp\left(-\dfrac{1}{1-x^2-y^2}\right), & x^2+y^2 < 1, \\ 0, & x^2+y^2 \geq 1. \end{cases}$$

Then g is C^∞ on $\mathbb{R}^2$ and satisfies $g \equiv 0$ on the complement of the unit disk, but g is not identically zero. In particular, the zero function on $\{x^2 + y^2 \geq 1\}$ admits many different smooth extensions to $\mathbb{R}^2$.

4. Morera's theorem and its applications

4.1. Morera's theorem. Morera's theorem can be viewed as a converse to Goursat's theorem (Theorem 3.3): it gives an integral characterization of holomorphicity, which turns out to be very useful.

THEOREM 4.1 (Morera). *Let $D \subset \mathbb{C}$ be an open set and let $f : D \to \mathbb{C}$ be continuous. Assume that for every triangle T with $\overline{T} \subset D$,*

$$\int_{\partial T} f(z)\,dz = 0.$$

Then f is holomorphic on D.

PROOF. We prove that f is holomorphic by constructing a primitive F and showing $F' = f$.

As in the proof of Cauchy's theorem (via Goursat's theorem), the assumption

$$\int_{\partial T} f(z)\,dz = 0 \quad \text{for every triangle } T \text{ with } \overline{T} \subset D$$

implies that the line integral of f is path independent in the disk D. Hence, fixing $z_0 \in D$, we may define

$$F(z) := \int_{\gamma_{z_0,z}} f(w)\,dw, \qquad z \in D,$$

where $\gamma_{z_0,z}$ is any oriented piecewise C^1 path in D from z_0 to z. This is well-defined.

Moreover, exactly the same computation as before shows that F is complex differentiable and

$$F'(z) = f(z) \qquad (\forall z \in D).$$

In particular note here we only need to assume f is continuous, instead of holomorphic.

Therefore F is holomorphic on D. Since $f = F'$, it follows from Theorem 1.5 that f is holomorphic on D as well. □

4.2. Glue holomorphic functions. Now we are going to use Morera's theorem to discuss the *gluing* of holomorphic functions. We begin with a special case, usually called the **Schwarz reflection principle**.

A set $\Omega \subset \mathbb{C}$ is called **symmetric about the real axis** if

$$z \in \Omega \quad \Longleftrightarrow \quad \bar{z} \in \Omega.$$

For such a set, we write

$$\Omega^+ := \{z \in \Omega : \Im z > 0\}, \qquad \Omega^- := \{z \in \Omega : \Im z < 0\}, \qquad I := \{z \in \Omega : \Im z = 0\}.$$

Clearly,

$$\Omega = \Omega^+ \cup I \cup \Omega^-.$$

If Ω is open in $\mathbb{C}$, then Ω^+ and Ω^- are open (in $\mathbb{C}$).

The Schwarz reflection principle addresses the following question: given a holomorphic function f on Ω^+, when can we extend it holomorphically to all of Ω? By the uniqueness of holomorphic continuation (Corollary **??**), if such an extension exists, it is unique.

THEOREM 4.2 (Schwarz Reflection Principle). *Assume f is holomorphic over Ω^+ and can be continuously extended to I, which means there exists a continuous function*

$$\tilde{f} : \Omega^+ \cup I \to \mathbb{C}$$

so that $\tilde{f}|_{\Omega^+} = f$. Then if $\tilde{f}|_I$ is a real function, f has a (unique) analytic contiuation to Ω.

THEOREM 4.3 (Schwarz reflection principle). *Assume f is holomorphic on Ω^+ and extends continuously to $\Omega^+ \cup I$, i.e. there exists*

$$\tilde{f} : \Omega^+ \cup I \to \mathbb{C}$$

continuous such that $\tilde{f}|_{\Omega^+} = f$. If $\tilde{f}|_I$ is real-valued, then f admits a unique holomorphic continuation to Ω.

PROOF. Define $F : \Omega \to \mathbb{C}$ by

$$F(z) := \begin{cases} f(z), & z \in \Omega^+, \\ \tilde{f}(z), & z \in I, \\ \overline{f(\bar{z})}, & z \in \Omega^-. \end{cases}$$

We first show that F is continuous on Ω. On Ω^+ and Ω^- this is clear. Let $x \in I$. If $z_n \to x$ with $z_n \in \Omega^+ \cup I$, then

$$F(z_n) = \tilde{f}(z_n) \to \tilde{f}(x) = F(x)$$

by continuity of $\tilde{f}$. If $z_n \to x$ with $z_n \in \Omega^-$, then $\bar{z}_n \to x$ with $\bar{z}_n \in \Omega^+$, so

$$F(z_n) = \overline{f(\bar{z}_n)} \to \overline{\tilde{f}(x)} = \tilde{f}(x) = F(x),$$

since $\tilde{f}(x) \in \mathbb{R}$. Thus F is continuous on Ω.

Next we show that F is holomorphic on $\Omega^+ \cup \Omega^-$. On Ω^+, this is obvious since $F = f$.

If $z \in \Omega^-$, then for h small enough,

$$\frac{F(z+h) - F(z)}{h} = \frac{\overline{f(\bar{z} + \bar{h})} - \overline{f(\bar{z})}}{h} = \overline{\frac{f(\bar{z} + \bar{h}) - f(\bar{z})}{\bar{h}}}.$$

Since f is holomorphic at $\bar{z} \in \Omega^+$, taking $h \to 0$ gives

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \overline{f'(\bar{z})}.$$

Hence F is holomorphic on Ω^-.

It remains to prove holomorphicity at points of I. Fix $x \in I$, and choose a disk D centered at x with $\overline{D} \subset \Omega$. We apply Morera's theorem on D.

Let T be any triangle with $\overline{T} \subset D$. For $\varepsilon > 0$ small, set

$$T_\varepsilon^+ := \{z \in T : \Im z > \varepsilon\}, \qquad T_\varepsilon^- := \{z \in T : \Im z < -\varepsilon\}, \qquad T_\varepsilon := \{z \in T : -\varepsilon \le \Im z \le \varepsilon\}.$$

Then

$$\int_{\partial T} F(z)\, dz = \int_{\partial T_\varepsilon^+} F(z)\, dz + \int_{\partial T_\varepsilon^-} F(z)\, dz + \int_{\partial T_\varepsilon} F(z)\, dz.$$

Since F is holomorphic on Ω^+ and Ω^-, the first two integrals are 0. Thus

$$\int_{\partial T} F(z)\, dz = \int_{\partial T_\varepsilon} F(z)\, dz.$$

Now F is continuous on the compact set $\overline{T}$, hence bounded there. The boundary ∂T_ε consists of two short vertical segments of total length $O(\varepsilon)$, whose contribution therefore tends to 0, together with two horizontal segments L_ε^+ and L_ε^-, one just above and one just below the real axis. By continuity of F, the integrals over $L_\varepsilon^\pm$ converge to the integrals over the same segment on I, but with opposite orientations; since F is continuous, these two limiting integrals cancel. Therefore

$$\lim_{\varepsilon\to 0}\int_{\partial T_\varepsilon} F(z)\,dz = 0.$$

Hence $\int_{\partial T} F(z)\,dz = 0$ for every triangle $T \subset D$. By Morera's theorem, F is holomorphic on D, and therefore at x.

Thus F is holomorphic on all of Ω. By construction $F|_{\Omega^+} = f$, so F is a holomorphic continuation of f. Its uniqueness follows from Corollary 3.4. □

The proof of the Schwarz reflection principle extends essentially verbatim to the following more general gluing result.

THEOREM 4.4 (Symmetry principle). *Let $\Omega \subset \mathbb{C}$ be an open set, and suppose that a piecewise C^1 regular curve I divides Ω into two open subsets Ω^+ and Ω^-. Assume that*

$$f^\pm \in \mathcal{O}(\Omega^\pm)$$

and that each $f^\pm$ extends continuously to I (we continue to denote these extensions by $f^\pm$). If moreover

$$f^+(z) = f^-(z) \qquad (z \in I),$$

then the function $F : \Omega \to \mathbb{C}$ defined by

$$F(z) := \begin{cases} f^+(z), & z \in \Omega^+ \cup I, \\ f^-(z), & z \in \Omega^-, \end{cases}$$

is holomorphic on Ω.

REMARK 4.5. The nontrivial point in these Schwarz-type symmetry principles is that for holomorphic functions, mere continuous gluing is enough to guarantee holomorphic gluing. This is another important difference from smooth real-valued functions.

For example, consider

$$f^+(x,y) = y \qquad \text{on } \Omega^+ := \{(x,y) \in \mathbb{R}^2 : y > 0\},$$

and

$$f^-(x,y) = y \qquad \text{on } \Omega^- := \{(x,y) \in \mathbb{R}^2 : y < 0\}.$$

They agree continuously on the boundary line $y = 0$, and each is smooth on its own side. Their continuous gluing is the function

$$F(x,y) = |y|,$$

which is continuous on $\mathbb{R}^2$ and smooth on Ω^+ and Ω^-, but is not even differentiable on the x-axis. Thus, unlike the holomorphic case, continuous gluing is not enough to guarantee smooth gluing.

4.3. Sequences of holomorphic functions. As another application of Morera's theorem, we obtain a fundamental stability result for limits of holomorphic functions.

Before stating it, let us recall two famous examples from real analysis due to Weierstrass.

REMARK 4.6. (1) The Weierstrass approximation theorem states that for every continuous function

$$f : [a, b] \to \mathbb{R}$$

is the uniform limit of a sequence of polynomials.

(2) On the other hand, Weierstrass also constructed examples of continuous nowhere differentiable functions, for instance

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

for suitable constants $0 < a < 1$ and odd integer b, satisfying an additional condition such as

$$ab > 1 + \frac{3\pi}{2}.$$

(3) These examples show that in real analysis, even uniform convergence on a compact set does not in general preserve differentiability.

The situation is much better in complex analysis: holomorphicity is preserved under (locally) uniform limits.

THEOREM 4.7. *Let $\Omega \subset \mathbb{C}$ be open, and let $(f_n)_{n\in\mathbb{Z}^+}$ be a sequence of holomorphic functions. Assume that $f_n \to f$ uniformly on every compact subset $K \subset \Omega$. Then f is holomorphic on Ω.*

Moreover, for each $m \geq 1$, the derivatives $f_n^{(m)}$ converge uniformly to $f^{(m)}$ on every compact subset of Ω.

PROOF. We prove the two statements separately.
Step 1: f is holomorphic. Fix $z_0 \in \Omega$. Choose $\varepsilon > 0$ such that $\overline{D(z_0; \varepsilon)} \subset \Omega$. To apply Morera's theorem on $D(z_0; \varepsilon)$, let T be any triangle with $\overline{T} \subset D(z_0; \varepsilon)$.

Since ∂T is compact and $f_n \to f$ uniformly on compact subsets of Ω, we have uniform convergence on ∂T. Hence

$$\int_{\partial T} f(z)\, dz = \int_{\partial T} \lim_{n\to\infty} f_n(z)\, dz = \lim_{n\to\infty} \int_{\partial T} f_n(z)\, dz = 0,$$

because each f_n is holomorphic, so $\int_{\partial T} f_n(z)\, dz = 0$. By Morera's theorem, f is holomorphic on $D(z_0; \varepsilon)$, hence at z_0. Since $z_0 \in \Omega$ was arbitrary, $f \in \mathcal{O}(\Omega)$.
Step 2: $f_n^{(m)} \to f^{(m)}$ uniformly on compact subsets. We first prove the case $m = 1$, and then iterate.

Let $K \subset \Omega$ be compact. For each $z \in K$, choose $r_z > 0$ such that

$$\overline{D(z; 2r_z)} \subset \Omega.$$

Then $\{D(z; r_z)\}_{z\in K}$ is an open cover of K. By compactness, there exist $z_1, \ldots, z_N \in K$ such that

$$K \subset \bigcup_{j=1}^{N} D(z_j; r_j), \qquad r_j := r_{z_j}.$$

Set

$$r := \min_{1\leq j\leq N} r_j > 0, \qquad \widetilde{K} := \bigcup_{j=1}^{N} \overline{D(z_j; 2r_j)}.$$

Then $\widetilde{K}$ is compact and contained in Ω. Moreover, for every $z \in K$,

$$\overline{D(z;r)} \subset \widetilde{K}.$$

Indeed, if $z \in D(z_j; r_j)$ and $w \in D(z; r)$, then

$$|w - z_j| \leq |w - z| + |z - z_j| < r + r_j \leq 2r_j,$$

so $w \in D(z_j; 2r_j) \subset \widetilde{K}$.

Now $f_n \to f$ uniformly on $\widetilde{K}$. Hence for every $\epsilon > 0$, there exists N_0 such that for all $n \geq N_0$,

$$\sup_{w \in \widetilde{K}} |f_n(w) - f(w)| < \epsilon.$$

Fix $z \in K$. Since $f_n - f$ is holomorphic on $D(z; r)$, Cauchy's estimate gives

$$|f_n'(z) - f'(z)| \leq \frac{1}{r} \sup_{|w-z|=r} |f_n(w) - f(w)| \leq \frac{1}{r} \sup_{w \in \widetilde{K}} |f_n(w) - f(w)| \leq \frac{\epsilon}{r}.$$

Since this bound is uniform in $z \in K$, we conclude that $f_n' \to f'$ uniformly on K.

Applying the same argument to the sequence $f_n' \to f'$, then to $f_n'' \to f''$, and so on, proves inductively that for every $m \geq 1$,

$$f_n^{(m)} \to f^{(m)}$$

uniformly on every compact subset of Ω. □

REMARK 4.8. We make two remarks about the assumptions in the theorem above.

(1) The assumption that $f_n \to f$ uniformly on every compact subset of Ω is often called **compact convergence** (or **local uniform convergence**). This is much less restrictive than uniform convergence on the whole domain, and is the natural notion in complex analysis.

For example, if we work on $\Omega = \mathbb{C}$ and require that a sequence of entire functions $\{f_n\}$ converges uniformly to an entire function f on all of $\mathbb{C}$, then for each n,

$$f_n - f$$

is a bounded entire function. By Liouville's theorem, $f_n - f$ must be constant. Hence

$$f_n = f + c_n \qquad \text{for some constants } c_n \in \mathbb{C},\ c_n \to 0.$$

So uniform convergence on all of $\mathbb{C}$ is far too restrictive and rules out most interesting examples.

A standard example is

$$E_n(z) := \sum_{k=0}^{n} \frac{z^k}{k!},$$

which converges compactly to e^z, but not uniformly on $\mathbb{C}$.

(2) Another basic example is the sequence

$$f_n(z) = z^n, \qquad n = 1, 2, \ldots,$$

on the unit disk $D(0; 1)$. This sequence converges pointwise to 0 on $D(0; 1)$, but it does *not* converge uniformly on $D(0; 1)$.

On the other hand, if $K \subset D(0; 1)$ is compact, then the continuous function $z \mapsto |z|$ attains its maximum on K, say

$$r := \max_{z \in K} |z|.$$

Since $K \subset D(0;1)$, we have $0 \le r < 1$. Therefore, for every $z \in K$,

$$|f_n(z)| = |z|^n \le r^n.$$

Because $r^n \to 0$, it follows that $f_n \to 0$ uniformly on K. Hence $z^n \to 0$ compactly on $D(0;1)$, although not uniformly on the whole disk.

4.4. Holomorphic functions defined in terms of integrals. We have just proved that locally uniform limits of holomorphic functions are again holomorphic. The following theorem may be viewed as an integral version of that result, since the integral can be approximated by Riemann sums of holomorphic functions. (See Remark 4.10 below.)

THEOREM 4.9. *Let $\Omega \subset \mathbb{C}$ be open, and let*

$$g : \Omega \times [0,1] \to \mathbb{C}$$

be continuous. Assume moreover that for each $s \in [0,1]$, the function

$$g_s := g(\cdot, s) : \Omega \to \mathbb{C}$$

is holomorphic. Define

$$f(z) := \int_0^1 g(z,s)\,ds.$$

Then f is holomorphic on Ω.

PROOF. We apply Morera's theorem.

Fix a triangle T with $\overline{T} \subset \Omega$. Since g is continuous on the compact set $\partial T \times [0,1]$, we may use Fubini's theorem to write

$$\int_{\partial T} f(z)\,dz = \int_{\partial T}\int_0^1 g(z,s)\,ds\,dz = \int_0^1 \int_{\partial T} g(z,s)\,dz\,ds.$$

For each fixed s, the function g_s is holomorphic on Ω, so by Goursat's theorem,

$$\int_{\partial T} g(z,s)\,dz = 0.$$

Hence

$$\int_{\partial T} f(z)\,dz = \int_0^1 0\,ds = 0.$$

Since f is continuous on Ω (as an integral of a continuous function), Morera's theorem implies that f is holomorphic on Ω. □

REMARK 4.10. There is another proof using Theorem 4.7 and we leave full details to the reader. For each $n \ge 1$, define the Riemann sum

$$f_n(z) := \frac{1}{n}\sum_{k=1}^{n} g\left(z, \frac{k}{n}\right).$$

Each f_n is holomorphic on Ω, and by uniform continuity of g on $K \times [0,1]$ for every compact $K \subset \Omega$, the sequence (f_n) converges uniformly to f on K. Hence f is holomorphic by Theorem 4.7.

CHAPTER 6

Local properties of singularities

As we have seen from the previous chapters, holomorphic functions are remarkably rigid, and much of complex analysis is concerned with how far a holomorphic function can be extended and what prevents such an extension. In this context, singularities arise naturally.

A singularity is not merely a “bad point” where a formula breaks down. Very often, singularities capture essential information about the function. For example, analytic continuation is governed by the nature of singularities: some are removable, some are poles, and some lead to much more complicated local behavior.

Singularities also play an important role in applications, especially in contour integration as we will see, where the contribution comes from the singularities inside the contour.

In this chapter, we will focus mainly on the local properties of isolated singularities. We will also introduce the Riemann sphere, and briefly discuss more general Riemann surfaces and holomorphic maps. This point of view is very helpful for understanding poles and meromorphic functions.

1. Isolated singularities

In general, if $S \subset \Omega$, we say that a point $z_0 \in S$ is an **isolated point** of S if there exists an open disk D centered at z_0 such that

$$D \cap S = \{z_0\}.$$

Now suppose $\Omega \subset \mathbb{C}$ is an open set, $S \subset \Omega$ is a closed subset, and

$$f : \Omega \setminus S \to \mathbb{C}$$

is holomorphic. Then we may think of S as a set of possible singularities of f.

If $z_0 \in S$ is an isolated point of S, equivalently, if there exists $r > 0$ such that

$$D^*(z_0; r) \subset \Omega \setminus S,$$

then z_0 is called an **isolated singularity** of f.

EXAMPLE 1.1. Consider the function

$$f(z) = \frac{1}{\sin(1/z)}.$$

This function is holomorphic on

$$\mathbb{C} \setminus \left(\{0\} \cup \left\{ \frac{1}{n\pi} : n \in \mathbb{Z} \setminus \{0\} \right\} \right),$$

since $\sin w = 0$ exactly when $w = n\pi$, $n \in \mathbb{Z}$.

The singular set is

$$S := \{0\} \cup \left\{ \frac{1}{n\pi} : n \in \mathbb{Z} \setminus \{0\} \right\}.$$

This set accumulates at 0, since

$$\frac{1}{n\pi} \to 0 \qquad (n \to \infty).$$

Therefore 0 is *not* an isolated singularity of f. However, every other point of S is an isolated singularity.

In this chapter, we will mainly study isolated singularities.

1.1. Local behavior of isolated zeros. Before classifying isolated singularities, let us first recall some basic facts about isolated zeros of holomorphic functions. The relation between zeros and singularities will become especially transparent later from the point of view of the Riemann sphere.

Suppose f is a holomorphic function on an open set $\Omega \subset \mathbb{C}$. A point $z_0 \in \Omega$ is called an **isolated zero** of f if

$$f(z_0) = 0.$$

By the Identity Theorem 3.1, if $f \not\equiv 0$ in a domain Ω, then every zero of f in Ω is isolated. Locally, near an isolated zero z_0, the function f admits a particularly simple form.

PROPOSITION 1.2. *Let $\Omega \subset \mathbb{C}$ be open and connected, and let $f \in \mathcal{O}(\Omega)$ with $f \not\equiv 0$. If $z_0 \in \Omega$ is a zero of f, then there exist an open disk $D \subset \Omega$ centered at z_0, a nowhere vanishing holomorphic function $g \in \mathcal{O}(D)$, and a positive integer n such that*

$$f(z) = (z - z_0)^n g(z) \qquad (z \in D).$$

PROOF. By the Identity Theorem 3.1, z_0 is an isolated zero of f, so after shrinking to a small disk $D \subset \Omega$ centered at z_0, we may assume that z_0 is the only zero of f in D.

Since f is holomorphic, it has a power series expansion about z_0:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad (z \in D).$$

Because $f(z_0) = 0$, we have $a_0 = 0$. Since $f \not\equiv 0$ on D, there exists a smallest integer $n \geq 1$ such that

$$a_n \neq 0.$$

Thus we may factor

$$f(z) = (z - z_0)^n g(z), \qquad g(z) := \sum_{k=0}^{\infty} a_{n+k} (z - z_0)^k.$$

The function g is holomorphic on D, and nowhere vanishing on D. In particular, $g(z_0) = a_n \neq 0$. $\square$

The integer n is uniquely determined by f. Indeed, if

$$f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z),$$

where g and h are holomorphic and nowhere vanishing near z_0, then $n = m$. For if $n < m$, then for $z \neq z_0$,

$$g(z) = (z - z_0)^{m-n} h(z).$$

Letting $z = z_0$, there is $g(z_0) = 0$, a contradiction.

This uniquely determined integer n is called the **order** (or **multiplicity**) of the zero z_0. A zero of order 1 is called a **simple zero**.

EXAMPLE 1.3. (1) The point $z_0 = 0$ is a simple zero of

$$f(z) = z, \qquad \sin z, \qquad e^z - 1.$$

This is easy to see from the power series expansions at 0:

$$\begin{aligned} \sin z &= z - \frac{z^3}{3!} + \cdots = z(1 - \frac{z^2}{3!} + \cdots), \\ e^z - 1 &= z + \frac{z^2}{2!} + \cdots = z(1 + \frac{z}{2!} + \cdots). \end{aligned}$$

(2) The point $z_0 = 1$ is a zero of order 2 of

$$f(z) = (z-1)^2(z-i).$$

More general, let $p(z)$ be a nonconstant polynomial. By the Fundamental Theorem of Algebra, p can be written as

$$p(z) = a\prod_{j=1}^{m}(z-\alpha_j)^{n_j},$$

where $a \neq 0$, the $\alpha_j \in \mathbb{C}$ are distinct, and $n_j \in \mathbb{Z}^+$. Then for each j, the point α_j is a zero of order n_j of p in $\mathbb{C}$. Thus, for polynomials on $\mathbb{C}$, the order of a zero agrees exactly with the usual multiplicity of the corresponding root.

If one is only considering p as a holomorphic function on some open set $\Omega \subset \mathbb{C}$, then of course one only discusses those zeros α_j that lie in Ω.

1.2. Types of isolated singularities. We start with three examples of isolated singularities.

EXAMPLE 1.4. Let

$$f(z) = \frac{z^2}{z}, \qquad z \in \mathbb{C}^*.$$

Then $z_0 = 0$ is an isolated singularity of f. In fact, this singularity is very mild: since $f(z) = z$ for $z \neq 0$, we can define $f(0) = 0$, and then f extends to the holomorphic function

$$f(z) = z$$

on all of $\mathbb{C}$. Such a singularity is called a **removable singularity**.

DEFINITION 1.5. Let f be holomorphic on $\Omega^* := \Omega \setminus \{z_0\}$, where $z_0 \in \Omega$. We say that z_0 is a **removable singularity** of f if f admits a holomorphic extension to Ω.

Clearly if such a holomorphic extension exists, then $\lim_{z\to z_0} f(z)$ exists and is exactly the value of the extension function takes at z_0.

EXAMPLE 1.6. Let

$$f(z) = \frac{1}{z}, \qquad z \in \mathbb{C}^*.$$

Then $z_0 = 0$ is again an isolated singularity. This time, however, f cannot be extended holomorphically across 0. This is the basic example of a **pole**.

DEFINITION 1.7. Let f be holomorphic on $\Omega^* := \Omega \setminus \{z_0\}$, where $z_0 \in \Omega$. We say that z_0 is a **pole** of f if f is nowhere vanishing near z_0 on some punctured neighborhood, and $1/f$ extends holomorphically across z_0 with a zero at z_0. The order of this zero is called the **order** of the pole. A pole with order 1 is also called a **simple** role.

Notice that if z_0 is a pole, then

$$\lim_{z\to z_0} |f(z)| = +\infty.$$

EXAMPLE 1.8. Let

$$f(z) = e^{1/z}, \qquad z \in \mathbb{C}^*.$$

Then $z_0 = 0$ is also an isolated singularity, but again f cannot be extended holomorphically across 0. This singularity is much more complicated than a pole; it is called an **essential singularity**.

DEFINITION 1.9. Let f be holomorphic on $\Omega^* := \Omega \setminus \{z_0\}$, where $z_0 \in \Omega$. If z_0 is neither a removable singularity nor a pole, then z_0 is called an **essential singularity** of f.

This above example $f(z) = e^{1/z}$ has $z_0 = 0$ as an essential singularity. We look at the limit behavior of f at 0.

Write z in polar form:

$$z = re^{i\theta}.$$

Then

$$\frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r},$$

so

$$f(z) = e^{1/z} = \exp\Big(\frac{\cos\theta}{r}\Big)\exp\Big(-i\frac{\sin\theta}{r}\Big).$$

Hence

$$|f(z)| = \exp\Big(\frac{\cos\theta}{r}\Big).$$

Now the behavior depends strongly on the direction in which $z \to 0$:

(1) If $\theta = 0$, i.e. $z = r > 0$, then

$$f(z) = e^{1/r} \to \infty.$$

(2) If $\theta = \pi$, i.e. $z = -r < 0$, then

$$f(z) = e^{-1/r} \to 0.$$

(3) If $\theta = \frac{\pi}{2}$, i.e. $z = ir$, then

$$f(z) = e^{-i/r},$$

so $|f(z)| = 1$, but the argument oscillates more and more rapidly as $r \to 0$. In particular, the limit does not exist.

Thus $f(z) = e^{1/z}$ has no single limiting behavior as $z \to 0$: along different paths, it can tend to ∞, tend to 0, or oscillate. This shows that the singularity at 0 is much more complicated than a removable singularity or a pole.

In fact, near an essential singularity, the behavior is much wilder: in every punctured neighborhood of 0, the function $e^{1/z}$ comes arbitrarily close to every nonzero complex value. This is the Casorati–Weierstrass theorem which we will discuss later.

The following theorem of Riemann gives us a simple way to classify isolated singularities by the behavior of $f(z)$ as $z \to z_0$.

THEOREM 1.10 (Riemann's theorem on removable singularities). *Let $\Omega \subset \mathbb{C}$ be open, let $z_0 \in \Omega$, and let f be holomorphic on $\Omega^* = \mathcal{O}(\Omega \setminus \{z_0\})$.*

Assume that f is bounded near z_0. Then f extends holomorphically across z_0. In another word, z_0 is a removable singularity if and only if f is bounded near z_0.

PROOF. It suffices to work locally near z_0. Choose $R > 0$ such that

$$\overline{D(z_0; R)} \subset \Omega$$

and f is bounded on $D(z_0; R) \setminus \{z_0\}$, say

$$|f(z)| \leq M \qquad (0 < |z - z_0| < R).$$

Define, for $z \in D(z_0; R)$,

$$g(z) := \frac{1}{2\pi i} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

By the Cauchy integral construction, g is holomorphic on $D(z_0; R)$.

We claim that

$$g(z) = f(z) \qquad (0 < |z - z_0| < R).$$

Fix such a point z. Choose $r > 0$ and $\varepsilon > 0$ so small that the closed disks

$$\overline{D(z; r)}, \qquad \overline{D(z_0; \varepsilon)}$$

are disjoint and both lie in $D(z_0; R)$.

Apply Cauchy's theorem to the region

$$D(z_0; R) \setminus \left(\overline{D(z; r)} \cup \overline{D(z_0; \varepsilon)}\right),$$

on which the function

$$\zeta \longmapsto \frac{f(\zeta)}{\zeta - z}$$

is holomorphic. Its boundary consists of the outer circle $\partial D(z_0; R)$, the small circle $\partial D(z; r)$ with opposite orientation, and the small circle $\partial D(z_0; \varepsilon)$ with opposite orientation. Hence

$$\int_{|\zeta - z_0| = R} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{|\zeta - z| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{|\zeta - z_0| = \varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Dividing by $2\pi i$, we get

$$g(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int_{|\zeta - z_0| = \varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

The first term is equal to $f(z)$ by Cauchy's integral formula.

For the second term, if $\varepsilon > 0$ is sufficiently small, then on the circle $|\zeta - z_0| = \varepsilon$ we have

$$|\zeta - z| \geq \frac{1}{2}|z - z_0|.$$

Therefore

$$\left| \frac{1}{2\pi i} \int_{|\zeta - z_0| = \varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| \leq \frac{1}{2\pi} \cdot (2\pi\varepsilon) \cdot \frac{M}{\frac{1}{2}|z - z_0|} = \frac{2M}{|z - z_0|}\, \varepsilon.$$

Hence this term tends to 0 as $\varepsilon \to 0$. Therefore

$$g(z) = f(z) \qquad (0 < |z - z_0| < R).$$

Thus g is a holomorphic extension of f across z_0. □

As a corollary, we obtain the following characterization of poles.

COROLLARY 1.11. *Let f be holomorphic on $\Omega^* := \Omega \setminus \{z_0\}$. Then z_0 is a pole of f if and only if*

$$|f(z)| \to +\infty \qquad \text{as } z \to z_0.$$

PROOF. Assume first that z_0 is a pole of f. Then by definition, $1/f$ extends holomorphically across z_0, and

$$\left(\frac{1}{f}\right)(z_0) = 0.$$

Since $1/f$ is continuous at z_0, we have

$$\frac{1}{f(z)} \to 0 \qquad \text{as } z \to z_0.$$

Hence

$$|f(z)| = \frac{1}{|1/f(z)|} \to +\infty.$$

Conversely, assume

$$|f(z)| \to +\infty \qquad \text{as } z \to z_0.$$

Then for z sufficiently close to z_0, we have $f(z) \neq 0$, and

$$\frac{1}{f(z)} \to 0 \qquad \text{as } z \to z_0.$$

In particular, $1/f$ is bounded near z_0. By Riemann's theorem on removable singularities, $1/f$ extends holomorphically across z_0, and the extension takes value 0 at z_0. Therefore z_0 is a pole of f. $\square$

EXAMPLE 1.12. (1) We have seen that the point $z_0 = 0$ is an essential singularity of

$$f(z) = e^{1/z}.$$

More concretely,
- $\lim_{z\to 0,\, z\in\mathbb{R}^+} e^{1/z} = +\infty$;
- $\lim_{z\to 0,\, z\in\mathbb{R}^-} e^{1/z} = 0$;
- along the imaginary axis, $e^{1/z}$ oscillates rapidly, so

$$\lim_{z\to 0,\, z\in i\mathbb{R}} e^{1/z}$$

does not exist.

Thus the behavior near 0 is much more complicated than in the removable or pole case.

More generally, if $p(z)$ has a pole at z_0, then

$$e^{p(z)}$$

typically has an essential singularity at z_0. For example,

$$e^{1/z^2}, \qquad e^{1/(z-1)}$$

have essential singularities at 0 and 1, respectively.

(2) The point $z_0 = 0$ is also an essential singularity of

$$f(z) = \sin(1/z), \qquad \cos(1/z).$$

For example,

$$\sin(1/z) = 0 \quad \text{when} \quad z = \frac{1}{n\pi}, \qquad n \in \mathbb{Z} \setminus \{0\},$$

and

$$\sin(1/z) = 1 \quad \text{when} \quad z = \frac{1}{\frac{\pi}{2} + 2n\pi}, \qquad n \in \mathbb{Z}.$$

Both sequences tend to 0, so $\sin(1/z)$ has no limit as $z \to 0$. Hence the singularity is not removable.

On the other hand, $\sin(1/z)$ does not tend to ∞ as $z \to 0$, since along the first sequence above it is identically 0. Therefore the singularity is not a pole. It follows that 0 is an essential singularity of $\sin(1/z)$.

The same reasoning applies to $\cos(1/z)$.

1.3. Behavior near an essential singularity. We have seen that the function

$$f(z) = e^{1/z}$$

has no single limiting behavior as $z \to 0$: along different paths, it can tend to ∞, tend to 0, or oscillate.

In fact, the behavior is much wilder. Fix any $w \neq 0$. Solving

$$e^{1/z} = w$$

is equivalent to solving

$$\frac{1}{z} = \log w,$$

where $\log w$ is multivalued. Thus

$$\frac{1}{z} = \ln|w| + i(\mathrm{Arg} w + 2k\pi), \qquad k \in \mathbb{Z},$$

and hence

$$z = \frac{1}{\ln|w| + i(\mathrm{Arg} w + 2k\pi)}, \qquad k \in \mathbb{Z}.$$

These give infinitely many distinct solutions, and as $|k| \to \infty$, they satisfy

$$z \to 0.$$

So in every punctured neighborhood of 0, the function $e^{1/z}$ takes every nonzero complex value infinitely many times.

In fact, this is a special case of the following deep theorem.

THEOREM 1.13 (Great Picard Theorem). *Let z_0 be an isolated essential singularity of a holomorphic function f. Then in every punctured neighborhood of z_0, the function f takes every complex value, with at most one exception, infinitely many times.*

In the example above, 0 is the exception point.

The proof of the Great Picard Theorem requires substantially deeper techniques than those developed in this course, so we will not present it here. Instead, we will prove a much simpler but still very important result, the Casorati–Weierstrass theorem. Although it is weaker, it already shows that the behavior near an essential singularity is extremely wild.

THEOREM 1.14 (Casorati–Weierstrass Theorem). *Let z_0 be an isolated essential singularity of a holomorphic function f. Then for every $r > 0$, the image*

$$f\left(D^*(z_0; r)\right), \quad D^*(z_0; r) = \{0 < |z| < r\},$$

is dense in $\mathbb{C}$.

PROOF. We argue by contradiction. Suppose there exists $r > 0$ such that

$$f\left(D^*(z_0; r)\right)$$

is not dense in $\mathbb{C}$. Then there exists a point $w_0 \in \mathbb{C}$, an $\varepsilon > 0$, and a punctured neighborhood $D^*(z_0; r)$ such that

$$f(z) \notin D(w_0; \varepsilon) \qquad \text{for all } z \in D^*(z_0; r).$$

In other words,

$$|f(z) - w_0| \geq \varepsilon \qquad \text{for all } z \in D^*(z_0; r).$$

Consider the function

$$g(z) := \frac{1}{f(z) - w_0}.$$

Then g is holomorphic on $D^*(z_0; r)$, and moreover

$$|g(z)| \leq \frac{1}{\varepsilon} \qquad \text{for all } z \in D^*(z_0; r).$$

Thus g is bounded near z_0. By Riemann's theorem on removable singularities 1.10, g extends holomorphically across z_0.

Now there are two cases:

Case 1: $g(z_0) \neq 0$. Then g is nonzero in some neighborhood of z_0, so

$$f(z) - w_0 = \frac{1}{g(z)}$$

extends holomorphically across z_0. Hence f itself extends holomorphically across z_0, so z_0 is a removable singularity. This contradicts the assumption that z_0 is essential.

Case 2: $g(z_0) = 0$. Then g has a zero at z_0, so $1/g$ has a pole at z_0. But

$$\frac{1}{g(z)} = f(z) - w_0,$$

hence $f - w_0$ has a pole at z_0, and therefore f also has a pole at z_0. Again this contradicts the assumption that z_0 is essential.

In either case we obtain a contradiction. Therefore $f\left(D^*(z_0; r)\right)$ must be dense in $\mathbb{C}$ for every $r > 0$. $\square$

1.4. Local characterization near a pole. We now study poles more closely. Recall that an isolated singularity is a pole precisely when its reciprocal extends holomorphically and vanishes at that point.

From Proposition 1.2, we obtain the following local description of a holomorphic function near a pole.

PROPOSITION 1.15. *Let f be holomorphic on $\Omega^* := \Omega \setminus \{z_0\}$, and assume that z_0 is a pole of f. Then there exist an open disk $D \subset \Omega$ centered at z_0, a nowhere vanishing holomorphic function h on D, and a positive integer n such that*

$$f(z) = \frac{h(z)}{(z - z_0)^n} \qquad (z \in D^* := D \setminus \{z_0\}).$$

The integer n is the order of the pole at z_0.

PROOF. Since z_0 is a pole of f, the function $1/f$ extends holomorphically across z_0, and the extension has z_0 as a zero. By Theorem 1.2, after shrinking to a small disk $D \subset \Omega$ centered at z_0, we may write

$$\frac{1}{f(z)} = (z - z_0)^n g(z),$$

where g is holomorphic and nowhere vanishing on D, and $n \geq 1$ is the order of the zero of $1/f$ at z_0. Hence on D^*,

$$f(z) = \frac{1}{(z - z_0)^n g(z)} = \frac{h(z)}{(z - z_0)^n}, \qquad \text{where } h(z) := \frac{1}{g(z)}.$$

Since g is nowhere vanishing and holomorphic on D, so is h. $\square$

EXAMPLE 1.16. (1) The functions

$$f(z) = \frac{1}{z}, \qquad \frac{1}{\sin z}, \qquad \frac{1}{e^z - 1}$$

all have $z_0 = 0$ as a simple pole. Indeed, $\sin z$ and $e^z - 1$ both have simple zeros at 0.

(2) The function

$$f(z) = \frac{1}{(z-1)^2(z-i)^3}$$

has $z_0 = 1$ as a pole of order 2, and $z_0 = i$ as a pole of order 3.

More generally, consider

$$f(z) = \frac{g(z)}{(z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_k)^{n_k}},$$

where $z_1, \ldots, z_k$ are distinct points in $\mathbb{C}$, each $n_j \in \mathbb{Z}_{>0}$, and g is holomorphic and nonvanishing at $z_1, \ldots, z_k$. Then f has a pole of order n_j at z_j for each $j = 1, \ldots, k$.

Equivalently, if

$$f(z) = \frac{g(z)}{p(z)},$$

where p is a polynomial and g is holomorphic and nonvanishing at every root of p, then the poles of f occur exactly at the roots of p, and the order of each pole is the multiplicity of the corresponding root of p.

(3) When a function has the form

$$f(z) = \frac{g(z)}{h(z)},$$

possible singularities arise from zeros of h. If g also vanishes at such a point z_0, then one must compare the orders of vanishing of g and h at z_0 to determine the type of singularity.

For example,

$$f(z) = \frac{z^2}{e^z - 1}$$

has a removable singularity at $z_0 = 0$. Indeed,

$$e^z - 1 = z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right),$$

so near 0,

$$f(z) = \frac{z}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots},$$

and the denominator is nowhere zero near 0. The holomorphic extension takes value 0 at $z = 0$.

On the other hand,

$$g(z) = \frac{z}{(e^z - 1)^2}$$

has $z_0 = 0$ as a simple pole. Indeed, writing $e^z - 1 = z\,h(z)$, where h is holomorphic and nowhere vanishing near 0, we get

$$g(z) = \frac{z}{(z\,h(z))^2} = \frac{1}{z\,h(z)^2},$$

which has exactly one factor of z in the denominator.

As a corollary, we obtain the following local Laurent expansion near a pole.

THEOREM 1.17. *If f has a pole of order n at z_0, then on some small disk D centered at z_0, f can be written as*

$$f(z) = \frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{c_{-1}}{z-z_0} + g(z), \tag{1.1}$$

where g is holomorphic on D, the coefficients

$$c_{-n}, c_{-(n-1)}, \ldots, c_{-1} \in \mathbb{C},$$

and $c_{-n} \neq 0$.

PROOF. By Proposition 1.15, on some small disk D centered at z_0, we may write

$$f(z) = \frac{h(z)}{(z-z_0)^n},$$

where h is holomorphic on D and $h(z_0) \neq 0$.

Expand h into its power series at z_0:

$$h(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k.$$

Then

$$f(z) = \frac{h(z)}{(z-z_0)^n} = \frac{\sum_{k=0}^{\infty} a_k (z-z_0)^k}{(z-z_0)^n}.$$

Separating the first n terms, we get

$$f(z) = \frac{a_0}{(z-z_0)^n} + \frac{a_1}{(z-z_0)^{n-1}} + \cdots + \frac{a_{n-1}}{z-z_0} + \sum_{k=n}^{\infty} a_k (z-z_0)^{k-n}.$$

Now define

$$c_{-n} = a_0, \quad c_{-(n-1)} = a_1, \quad \ldots, \quad c_{-1} = a_{n-1},$$

and

$$g(z) := \sum_{k=n}^{\infty} a_k (z-z_0)^{k-n}.$$

Then g is holomorphic on D, and (1.1) follows. Since $h(z_0) = a_0 \neq 0$, we have $c_{-n} \neq 0$. □

The sum of the n negative-power terms

$$P_f(z) := \frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \frac{c_{-1}}{z-z_0}$$

is called the **principal part** of f at the pole z_0.

The coefficient c_{-1} is called the **residue** of f at z_0, and is denoted by

$$\mathrm{Res}_{z=z_0} f := c_{-1}.$$

Notice that among the terms in the principal part, only the term

$$\frac{c_{-1}}{z-z_0}$$

fails to have a holomorphic primitive on the punctured disk. Therefore, if γ is a counterclockwise loop surrounding z_0, and z_0 is the only singularity of f inside γ, then

$$\oint_\gamma f(z)\,dz = 2\pi i\,\mathrm{Res}_{z=z_0} f.$$

This is the simplest form of the residue formula. We will discuss the general case in the next section. To calculate residues, the following formula is useful.

PROPOSITION 1.18. *Let z_0 be a pole of order n of f. Then*

$$\mathrm{Res}_{z=z_0} f = \frac{1}{(n-1)!}\,\frac{d^{\,n-1}}{dz^{\,n-1}}\big((z-z_0)^n f(z)\big)\Big|_{z=z_0}.$$

In particular, if z_0 is a simple pole of f, then

$$\mathrm{Res}_{z=z_0} f = \lim_{z\to z_0}(z-z_0)f(z).$$

Let us look at some examples.

EXAMPLE 1.19. (1) Consider

$$f(z) = \frac{e^{-z}}{1+z^2}.$$

Find the singularities in $\mathbb{C}$, determine their types, and calculate the residues at the poles.

SOLUTION. The singularities occur where

$$1+z^2 = 0,$$

namely at $z = \pm i$. Since

$$1+z^2 = (z-i)(z+i),$$

and $e^{-z} \neq 0$ everywhere, both $z = i$ and $z = -i$ are simple poles.

Write

$$f(z) = \frac{e^{-z}}{(z-i)(z+i)}.$$

Then

$$\mathrm{Res}_{z=i} f = \lim_{z\to i}(z-i)f(z) = \frac{e^{-z}}{z+i}\Big|_{z=i} = \frac{e^{-i}}{2i}.$$

Similarly,

$$\mathrm{Res}_{z=-i} f = \lim_{z\to -i}(z+i)f(z) = \frac{e^{-z}}{z-i}\Big|_{z=-i} = -\frac{e^{i}}{2i}.$$

□

(2) Consider

$$f(z) = \frac{(z^2-1)^2}{z^2(z-2)(2z-1)}.$$

Find the singularities in $\mathbb{C}$, determine their types, and calculate the residues at the poles.

SOLUTION. The singularities are at

$$z = 0, \qquad z = 2, \qquad z = \frac{1}{2}.$$

The point $z = 0$ is a pole of order 2, while $z = 2$ and $z = \frac{1}{2}$ are simple poles.

For the simple poles,

$$\mathrm{Res}_{z=2} f = \lim_{z \to 2} (z-2) f(z) = \left. \frac{(z^2-1)^2}{z^2(2z-1)} \right|_{z=2} = \frac{9}{12} = \frac{3}{4}.$$

Also,

$$\mathrm{Res}_{z=\frac{1}{2}} f = \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2} \right) f(z) = \lim_{z \to \frac{1}{2}} \frac{(z^2-1)^2}{2z^2(z-2)} = -\frac{3}{4}.$$

For the pole at $z = 0$, since it is of order 2, we use the residue formula:

$$\mathrm{Res}_{z=0} f = \left. \frac{d}{dz} \left(\frac{(z^2-1)^2}{(z-2)(2z-1)} \right) \right|_{z=0}.$$

A direct computation gives

$$\mathrm{Res}_{z=0} f = \frac{5}{4}.$$

Another way is to use the identity

$$(z^2-1)^2 = z^4 - 2z^2 + 1,$$

so

$$f(z) = \frac{z^2}{(z-2)(2z-1)} - \frac{2}{(z-2)(2z-1)} + \frac{1}{z^2(z-2)(2z-1)}.$$

Near $z = 0$, the first two terms are holomorphic, so they do not contribute to the residue. Hence

$$\mathrm{Res}_{z=0} f = \mathrm{Res}_{z=0} \frac{1}{z^2(z-2)(2z-1)}.$$

Now

$$\frac{1}{z^2(z-2)(2z-1)} = \frac{1}{2z^2} \cdot \frac{1}{1-\frac{z}{2}} \cdot \frac{1}{1-2z}.$$

Thus

$$\frac{1}{z^2(z-2)(2z-1)} = \frac{1}{2z^2} \left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots \right) \left(1 + 2z + (2z)^2 + \cdots \right).$$

To obtain the residue, we need the coefficient of z^{-1}. Since there is a factor $1/(2z^2)$ outside, this comes from the coefficient of z^1 in

$$\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots \right) \left(1 + 2z + (2z)^2 + \cdots \right).$$

That coefficient is

$$2 + \frac{1}{2} = \frac{5}{2}.$$

Hence

$$\mathrm{Res}_{z=0} f = \frac{1}{2} \cdot \frac{5}{2} = \frac{5}{4}.$$

□

2. Riemann sphere and an introduction to Riemann surfaces

2.1. The extended complex plane and the Riemann sphere. Consider the sphere

$$\Sigma := \left\{ (X, Y, Z) \in \mathbb{R}^3 \,:\, X^2 + Y^2 + Z^2 = \frac{1}{4} \right\}.$$

We equip Σ with the subspace topology induced from the Euclidean topology of $\mathbb{R}^3$. Since Σ is a closed and bounded subset of $\mathbb{R}^3$, it is compact.

Let

$$N = \left(0, 0, \frac{1}{2}\right), \qquad S = \left(0, 0, -\frac{1}{2}\right)$$

be the north and south poles. Then

$$U_N := \Sigma \setminus \{N\}, \qquad U_S := \Sigma \setminus \{S\}$$

are open subsets of Σ, and

$$\Sigma = U_N \cup U_S.$$

We now define two coordinate charts by stereographic projection. Projecting from the north pole N onto the plane $Z = -\frac{1}{2}$, and identifying that plane with $\mathbb{C}$ by

$$(X, Y, -\tfrac{1}{2}) \longmapsto X + iY,$$

we obtain a homeomorphism

$$\pi_N : U_N \to \mathbb{C}.$$

Similarly, projecting from the south pole S onto the plane $Z = \frac{1}{2}$, and identifying that plane with $\mathbb{C}$ by

$$(X, Y, \tfrac{1}{2}) \longmapsto X - iY,$$

we obtain a homeomorphism

$$\pi_S : U_S \to \mathbb{C}.$$

(The use of $X - iY$ rather than $X + iY$ in the second chart is important: with this choice, the transition map becomes holomorphic.)

On the overlap $U_N \cap U_S = \Sigma \setminus \{N, S\}$, the transition map is

$$\pi_S \circ \pi_N^{-1} : \mathbb{C}^* \to \mathbb{C}^*, \qquad z \longmapsto \frac{1}{z},$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Since $z \mapsto 1/z$ is biholomorphic (i.e., the map and its inverse map are both holomorphic.) on $\mathbb{C}^*$, these two charts are compatible.

Thus Σ, equipped with the charts (U_N, π_N) and (U_S, π_S), carries a natural complex structure.

A third chart (U, ϕ) on Σ is called **compatible** with these two charts if all transition maps are biholomorphic wherever they are defined. For example, stereographic projection from any other point of the sphere onto a suitable plane also gives a compatible chart.

The **Riemann sphere structure** on Σ is defined to be the collection of all complex charts on Σ that are compatible with (U_N, π_N) and (U_S, π_S). In other words, it is the maximal complex atlas determined by these two stereographic charts. We denote the resulting Riemann surface by $\widehat{\mathbb{C}}$.

2.2. General Riemann surfaces. More generally, a **Riemann surface** (also called a **1-dimensional complex manifold**) is a Hausdorff and second countable topological space equipped with a maximal complex atlas, that is, a collection of charts (U, ϕ), where U is open and

$$\phi : U \to \phi(U) \subset \mathbb{C}$$

is a homeomorphism onto an open subset of $\mathbb{C}$, such that all transition maps are biholomorphic.

The Riemann sphere constructed above is a Riemann surface whose underlying topological space is the sphere.

We now look at another example.

EXAMPLE 2.1 (The complex torus). Fix $\omega \in \mathbb{C} \setminus \mathbb{R}$, and let

$$\Lambda_\omega := \mathbb{Z} + \mathbb{Z}\omega \subset \mathbb{C}.$$

Define an equivalence relation on $\mathbb{C}$ by

$$z_1 \sim z_2 \qquad \Longleftrightarrow \qquad z_1 - z_2 \in \Lambda_\omega.$$

Let

$$X_\omega := \mathbb{C}/\sim$$

be the quotient space, and denote by

$$\pi : \mathbb{C} \to X_\omega$$

the quotient map. Topologically, X_ω is a torus.

We now define a complex atlas on X_ω. For each point $[z] \in X_\omega$, choose a representative $z \in \mathbb{C}$, and then choose a sufficiently small open disk $D_z \subset \mathbb{C}$ centered at z such that

$$D_z \cap (D_z + \lambda) = \emptyset \qquad \text{for all } \lambda \in \Lambda_\omega \setminus \{0\}.$$

Then the restriction

$$\pi|_{D_z} : D_z \to \pi(D_z)$$

is a homeomorphism. Hence we obtain a chart

$$\phi_z : \pi(D_z) \to D_z \subset \mathbb{C}$$

defined by

$$\phi_z([w]) = w,$$

where w is the unique representative of the class $[w]$ lying in D_z.

A point of X_ω belongs to infinitely many such charts, since it has infinitely many representatives in $\mathbb{C}$. If two such charts overlap, then the corresponding transition map is given by translation by an element of Λ_ω, and hence is biholomorphic.

Therefore X_ω carries a natural Riemann surface structure. It is called a **complex torus**.

REMARK 2.2. Another natural way to encounter complex tori is through multi-valued functions such as

$$\sqrt{z(z-z_0)(z-z_1)}, \qquad (z_0 \neq 0,\ z_1 \neq 0,\ z_0 \neq z_1).$$

In order to make such a function single-valued, one is naturally led to pass from the complex plane to a two-sheeted branched covering. After compactifying by adding the point at infinity, one obtains a complex torus.

These two constructions of complex tori are related by the Weierstrass $\wp$-function, which we will not discuss in further detail here.

This way of introducing Riemann surfaces from multi-valued functions reflects one of Riemann's original motivations for introducing the concept of a Riemann surface.

2.3. Holomorphic maps between Riemann surfaces. The existence of complex charts allows us to speak about holomorphic maps between Riemann surfaces.

DEFINITION 2.3. Let X and Y be Riemann surfaces, and let

$$F : X \to Y$$

be a continuous map. We say that F is **holomorphic** if for every point $p \in X$, there exist charts

$$(U, \phi) \quad \text{around } p, \qquad (V, \psi) \quad \text{around } F(p),$$

with $F(U) \subset V$, such that the coordinate expression

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V) \subset \mathbb{C}$$

is a holomorphic function.

If F is bijective and both F and F^{-1} are holomorphic, then F is called a **biholomorphic map**, or a **biholomorphism**. If there exists a biholomorphic map between Riemann surfaces X and Y, then we say that X and Y are **biholomorphically equivalent**.

Notice that the above notion is independent of the choice of charts. This follows from the compatibility of the charts: changing charts replaces

$$\psi \circ F \circ \phi^{-1}$$

by compositions with biholomorphic transition maps, and such compositions preserve holomorphicity.

We will use $\mathcal{O}(X, Y)$ to denote the set of holomorphic maps from the Riemann surface X to the Riemann surface Y.

REMARK 2.4. If two Riemann surfaces are biholomorphically equivalent, then from the point of view of complex analysis they should be regarded as the same Riemann surface. The theory of Riemann surfaces studies such surfaces only up to biholomorphic equivalence.

A fundamental, but highly nontrivial, consequence of the uniformization theorem is that every compact Riemann surface whose underlying topological space is the sphere is biholomorphic to the Riemann sphere $\widehat{\mathbb{C}}$. In this sense, the sphere carries a unique complex structure up to biholomorphism.

The situation is different for the torus. We constructed a complex torus X_ω using ω. Different values of ω can define Riemann surfaces that are not biholomorphic. More precisely,

$$X_\omega \cong X_{\omega'}$$

if and only if

$$\omega' = \frac{a\omega + b}{c\omega + d}$$

for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z}).$$

Thus the biholomorphism classes of complex tori are parametrized by

$$\mathbb{H}/PSL_2(\mathbb{Z}).$$

For compact Riemann surfaces of genus $g > 1$, there are continuously many non-biholomorphic complex structures. In fact, the set of biholomorphism classes has a rich geometric structure and is, away from finitely many exceptional points, a complex manifold of dimension $3g - 3$.

These spaces of equivalence classes are called the **moduli spaces** of Riemann surfaces. Such spaces play an important role in many areas of modern mathematics.

2.4. Pole singularities revisited. The Riemann sphere provides a very natural geometric interpretation of poles. A pole is not merely a point where a function "blows up"; rather, it is a point where the function takes the value ∞.

Let $\Omega \subset \mathbb{C}$ be open, and let

$$f : \Omega \setminus \{z_0\} \to \mathbb{C}$$

be holomorphic. Suppose that z_0 is a pole of f. Since Ω is itself a Riemann surface, we may define a map

$$F : \Omega \to \widehat{\mathbb{C}}$$

by

$$F(z) = \begin{cases} f(z), & z \neq z_0, \\ \infty, & z = z_0. \end{cases}$$

Using the local coordinate $w = 1/\zeta$ near ∞ on $\widehat{\mathbb{C}}$, one sees that F is holomorphic at z_0: indeed, near z_0,

$$\frac{1}{f(z)}$$

extends holomorphically and vanishes at z_0. Thus a pole may be viewed as an ordinary holomorphic point once the target is enlarged from $\mathbb{C}$ to $\widehat{\mathbb{C}}$.

REMARK 2.5. Strictly speaking, $\widehat{\mathbb{C}}$ is the Riemann surface structure carried by the sphere Σ. Under the chart

$$\pi_N : \Sigma \setminus \{N\} \to \mathbb{C},$$

the complement of the north pole N is identified with the complex plane, and the point $\infty \in \widehat{\mathbb{C}}$ corresponds to $N \in \Sigma$. Thus the map F above may equally well be viewed as a holomorphic map

$$F : \Omega \to \Sigma$$

which agrees with f on $\Omega \setminus \{z_0\}$ under the identification $\Sigma \setminus \{N\} \cong \mathbb{C}$, and sends the pole z_0 to the north pole N.

DEFINITION 2.6. Let $\Omega \subset \mathbb{C}$ be open. A function

$$f : \Omega \setminus Z \to \mathbb{C}$$

is called **meromorphic** on Ω if

(1) $Z \subset \Omega$ has no limit points in Ω when $Z \neq \emptyset$;
(2) f is holomorphic on $\Omega \setminus Z$;
(3) every point of Z is a pole of f.

EXAMPLE 2.7. (1) The function

$$f(z) = z + \frac{1}{z}$$

is meromorphic on $\mathbb{C}$. It is holomorphic on $\mathbb{C} \setminus \{0\}$, and $z = 0$ is a simple pole.

(2) More generally, every rational function

$$f(z) = \frac{p(z)}{q(z)},$$

where p, q are polynomials with no common factor, is meromorphic on $\mathbb{C}$. Indeed, f is holomorphic away from the zeros of q, and each zero of q gives a pole.

(3) Not every meromorphic function on $\mathbb{C}$ is rational. For example,

$$f(z) = \frac{1}{\sin z}$$

is meromorphic on $\mathbb{C}$, but it is not rational. Its poles occur exactly at the zeros of $\sin z$, namely at

$$z = n\pi, \qquad n \in \mathbb{Z}.$$

We denote by $\mathcal{M}(\Omega)$ the set of meromorphic functions on Ω. By the same argument as in the case of a single pole, a function on Ω is meromorphic if and only if it may be regarded as a holomorphic map

$$\Omega \to \widehat{\mathbb{C}}$$

which is not identically equal to ∞. In other words, $\mathcal{M}(\Omega)$ may be identified with the set of holomorphic maps

$$\Omega \to \widehat{\mathbb{C}}$$

other than the constant map to ∞.

2.5. The singularity at ∞. The Riemann sphere allows us to treat the point ∞ just like an ordinary point. Under the identification of the extended complex plane with the Riemann sphere, the point ∞ corresponds to the north pole N. Thus neighborhoods of ∞ may be understood as neighborhoods of the north pole, and the stereographic projection from the south pole gives a chart containing N.

Under this chart, a convenient local coordinate near ∞ is

$$w = \frac{1}{z}.$$

In this coordinate, points near ∞ correspond to points near 0 in the w-plane. Equivalently, a punctured neighborhood of ∞ corresponds to a punctured neighborhood of 0.

To understand a function f defined near ∞, we consider the function

$$g(w) := f(1/w)$$

near 0. In this way, the behavior of f near ∞ is translated into the behavior of g near 0.

DEFINITION 2.8. Let f be defined on

$$\{z \in \mathbb{C} : |z| > R\}$$

for some $R > 0$. We say that f has an **isolated singularity at ∞** if the associated function

$$g(w) = f(1/w)$$

has 0 as an isolated singularity. Equivalently, f is holomorphic outside some large disk.

Once f has an isolated singularity at ∞, we classify it by passing to $g(w) = f(1/w)$.

DEFINITION 2.9. Suppose f has an isolated singularity at ∞, and let

$$g(w) := f(1/w).$$

We say that f has

(1) a **removable singularity at** ∞ if g has a removable singularity at 0;
(2) a **pole of order** n **at** ∞ if g has a pole of order n at 0;
(3) an **essential singularity at** ∞ if g has an essential singularity at 0.

In other words, the singularity type of f at ∞ is exactly the singularity type of $g(w) = f(1/w)$ at 0.

EXAMPLE 2.10. (1) Consider the function

$$f(z) = \frac{1}{\sin z}$$

has poles at

$$z = n\pi, \qquad n \in \mathbb{Z},$$

and these poles have arbitrarily large modulus. Equivalently,

$$g(w) = f(1/w) = \frac{1}{\sin(1/w)}$$

has singularities at

$$w = \frac{1}{n\pi}, \qquad n \in \mathbb{Z} \setminus \{0\},$$

which accumulate at 0. Thus ∞ is not an isolated singularity of $1/\sin z$.

(2) Let

$$f(z) = \frac{1}{z}.$$

Then

$$g(w) = f(1/w) = w.$$

Since g extends holomorphically across 0, the function f has a removable singularity at ∞.

(3) Let

$$f(z) = z^n, \qquad n \geq 1.$$

Then

$$g(w) = f(1/w) = \frac{1}{w^n}.$$

Thus g has a pole of order n at 0, so f has a pole of order n at ∞.

(4) Let

$$f(z) = e^z.$$

Then

$$g(w) = e^{1/w}.$$

Since $e^{1/w}$ has an essential singularity at 0, the function e^z has an essential singularity at ∞.

DEFINITION 2.11. A function f is called **meromorphic on the extended complex plane** if

(1) f is meromorphic on $\mathbb{C}$, and
(2) ∞ is either a removable singularity or a pole of f.

REMARK 2.12. Since the extended complex plane, which is identified with the Riemann sphere $\widehat{\mathbb{C}}$, is compact, a meromorphic function on $\widehat{\mathbb{C}}$ can have only finitely many poles.

We denote by $\mathcal{M}(\widehat{\mathbb{C}})$ the set of meromorphic functions on $\widehat{\mathbb{C}}$. By the same reasoning as before, every such function may equivalently be viewed as a holomorphic map

$$\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

which is not identically equal to ∞.

In fact, we have the following theorem.

THEOREM 2.13. *A function is meromorphic on $\widehat{\mathbb{C}}$ if and only if it is rational, i.e.,*

$$\mathcal{M}(\widehat{\mathbb{C}}) = \mathbb{C}(z).$$

PROOF. We first show that every rational function is meromorphic on $\widehat{\mathbb{C}}$.

Clearly it is the case when $f = 0$. Now let

$$f(z) = \frac{p(z)}{q(z)},$$

where p, q are nonzero polynomials. After canceling common factors, we may assume that p and q have no common factor. Then f is holomorphic on $\mathbb{C}$ away from the finitely many zeros of q, and each zero of q gives a pole of f. Thus f is meromorphic on $\mathbb{C}$.

It remains to consider the point ∞.

If $\deg p \le \deg q$, then $|f(z)|$ is bounded for $|z|$ large. By Riemann's removable singularity theorem 1.10, ∞ is a removable singularity.

If $\deg p > \deg q$, then $f(z) \to \infty$ as $z \to \infty$, so ∞ is a pole by the corollary of Riemann's removable singularity theorem 1.11.

Hence every rational function is meromorphic on $\widehat{\mathbb{C}}$.

Conversely, let $f \in \mathcal{M}(\widehat{\mathbb{C}})$. Since $\widehat{\mathbb{C}}$ is compact, the set of poles of f is finite. Let the poles in $\mathbb{C}$ be

$$z_1, \dots, z_k.$$

For each $j = 1, \dots, k$, let

$$P_j(z) = \sum_{m=1}^{n_j} \frac{c_{j,-m}}{(z - z_j)^m}$$

be the principal part of the Laurent expansion of f at z_j.

We now also analyze the point ∞. Since f is meromorphic on $\widehat{\mathbb{C}}$, the point ∞ is either removable or a pole. Thus

$$g(w) := f(1/w)$$

has at $w = 0$ either a removable singularity or a pole. Hence near $w = 0$,

$$g(w) = Q(w) + h(w),$$

where $Q(w)$ is the principal part of the Laurent expansion of g at 0, and $h(w)$ is holomorphic near 0. Since ∞ is not an essential singularity, $Q(w)$ has only finitely many negative-power terms:

$$Q(w) = \sum_{m=1}^{N} b_{-m} w^{-m}$$

for some $N \ge 0$ (with $Q = 0$ if ∞ is removable).

Substituting $w = 1/z$, we obtain a polynomial

$$Q(1/z) = \sum_{m=1}^{N} b_{-m} z^m.$$

Define

$$F(z) := f(z) - \sum_{j=1}^{k} P_j(z) - Q(1/z).$$

We claim that F is entire. Indeed, away from $z_1, \dots, z_k$, this is clear. Near each z_j, the Laurent expansion of f has principal part $P_j(z)$, so $f - P_j$ is holomorphic near z_j, while all other P_i $(i \neq j)$ and the polynomial $Q(1/z)$ are holomorphic near z_j. Hence F is holomorphic near every z_j, and therefore F is entire.

Next we examine F at ∞. In terms of $w = 1/z$,

$$F(1/w) = g(w) - \sum_{j=1}^{k} P_j(1/w) - Q(w).$$

Now each $P_j(1/w)$ is holomorphic near $w = 0$, since $P_j(z)$ has its only pole at the finite point z_j. Also, by construction,

$$g(w) - Q(w) = h(w)$$

is holomorphic near $w = 0$. Therefore $F(1/w)$ is holomorphic near $w = 0$. Equivalently, F has a removable singularity at ∞.

Thus F is entire and removable at ∞, so F is bounded for $|z|$ sufficiently large. Since F is entire, it follows from Liouville's theorem that F is constant.

Therefore

$$f(z) = \sum_{j=1}^{k} P_j(z) + Q(1/z) + C$$

for some constant C. The right-hand side is a rational function, since each $P_j(z)$ is rational and $Q(1/z)$ is a polynomial. Hence f is rational.

□

As a corollary, we obtain the following result, whose proof is left to the reader as an exercise.

COROLLARY 2.14. *A map*

$$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

is biholomorphic if and only if it is a linear fractional transformation, i.e. of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0.$$

REMARK 2.15. The set of all biholomorphic self-maps of $\widehat{\mathbb{C}}$ forms a group under composition. By the corollary, this is exactly the group of linear fractional transformations

$$z \mapsto \frac{az+b}{cz+d}, \qquad ad - bc \neq 0.$$

This group is naturally identified with

$$PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}.$$

Indeed, the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad -\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

define the same linear fractional transformation, and every such transformation arises in this way.

3. Laurent expansion in general

We know that if z_0 is a pole of f, then near z_0, the function f can be written as a Laurent series

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n.$$

In that case, only finitely many negative-power terms appear. In fact, such an expansion exists for any isolated singularity, and more generally for any holomorphic function on an annulus. This is the content of Laurent's theorem.

THEOREM 3.1 (Laurent's theorem). *Let*

$$A = \{z \in \mathbb{C} : r < |z - z_0| < R\}, \qquad 0 \le r < R \le \infty,$$

and let f be holomorphic on A. Then f admits a Laurent expansion about z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad z \in A.$$

Moreover, the series converges absolutely and uniformly on every compact subset of A, and the coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta, \qquad r < \rho < R.$$

PROOF. Fix $z \in A$. Choose radii ρ_1, ρ_2 such that

$$r < \rho_1 < |z - z_0| < \rho_2 < R.$$

Consider the annular region

$$\Omega = \{\zeta \in \mathbb{C} : \rho_1 < |\zeta - z_0| < \rho_2\}.$$

Since f is holomorphic on a neighborhood of $\overline{\Omega}$, Cauchy's integral formula gives

insert a picture here.

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We expand the kernel $\frac{1}{\zeta - z}$ differently on the two circles.

For $|\zeta - z_0| = \rho_2$, since $|z - z_0| < \rho_2$, we have

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

This series converges uniformly on the circle $|\zeta - z_0| = \rho_2$. Hence we may integrate term by term:

$$\frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \right) (z - z_0)^n.$$

For $|\zeta - z_0| = \rho_1$, since $\rho_1 < |z - z_0|$, we write

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\sum_{n=0}^{\infty} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}}.$$

Again this converges uniformly on $|\zeta - z_0| = \rho_1$, so

$$-\frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho_1} f(\zeta)(\zeta - z_0)^n\, d\zeta \right) (z - z_0)^{-n-1}.$$

Combining the two expansions, we obtain

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n},$$

where

$$a_n = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho_2} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta, \qquad n \ge 0,$$

and

$$a_{-n} = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho_1} f(\zeta)(\zeta-z_0)^{n-1}\,d\zeta, \qquad n \ge 1.$$

These formulas can be written uniformly as

$$a_m = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho} \frac{f(\zeta)}{(\zeta-z_0)^{m+1}}\,d\zeta, \qquad m \in \mathbb{Z},$$

for any $r < \rho < R$. Indeed, this follows from Cauchy's theorem, since the integrand is holomorphic in ζ on the annulus $r < |\zeta - z_0| < R$.

Finally, because the two geometric series above converge uniformly on compact subsets of A, the resulting Laurent series converges absolutely and uniformly on every compact subset of A. □

COROLLARY 3.2. *Let f be holomorphic on a punctured disk*

$$0 < |z - z_0| < R,$$

and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

be its Laurent expansion at z_0. Then:

(1) *z_0 is removable if and only if $a_n = 0$ for all $n < 0$;*
(2) *z_0 is a pole if and only if $a_n = 0$ for all sufficiently negative n;*
(3) *z_0 is essential if and only if $a_n \neq 0$ for infinitely many negative integers n.*

The coefficient a_{-1} is called the **residue** of f at z_0, and is denoted by

$$\operatorname{Res}_{z=z_0} f.$$

Thus

$$\operatorname{Res}_{z=z_0} f = a_{-1}, \qquad \int_{|\zeta-z_0|=\rho} f(\zeta)\,d\zeta = 2\pi i \operatorname{Res}_{z=z_0} f \qquad (0 < \rho < R),$$

since all other Laurent terms integrate to 0.

EXAMPLE 3.3. Consider the function

$$f(z) = \frac{1}{z(z-1)}.$$

We compute its Laurent expansions about $z = 0$ in different annuli.

(1) If $0 < |z| < 1$, then

$$\frac{1}{z(z-1)} = -\frac{1}{z(1-z)} = -\frac{1}{z}\sum_{n=0}^{\infty} z^n = -\sum_{n=0}^{\infty} z^{n-1}.$$

Hence

$$\frac{1}{z(z-1)} = -\frac{1}{z} - 1 - z - z^2 - \cdots, \qquad 0 < |z| < 1.$$

(2) If $|z| > 1$, then

$$\frac{1}{z(z-1)} = \frac{1}{z^2} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} z^{-n-2}.$$

Hence

$$\frac{1}{z(z-1)} = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots, \qquad |z| > 1.$$

This example shows that the Laurent expansion depends on the annulus of expansion.

EXAMPLE 3.4. Let

$$f(z) = e^{1/z}.$$

Using the exponential series,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots.$$

There are infinitely many negative-power terms, so 0 is an essential singularity. Moreover,

$$\mathrm{Res}_{z=0} e^{1/z} = 1.$$

CHAPTER 7

Residue formula and its applications

1. The residue formula

We have seen the residue formula for several special cases, and we now give the basic global form of the residue theorem. The key point is that although a meromorphic function may fail to have a primitive globally, its contour integral is determined entirely by the local contributions from its singularities.

THEOREM 1.1 (Residue formula). *Let $\Omega \subset \mathbb{C}$ be open, and let γ be a positively oriented, piecewise C^1, simple closed curve in Ω. Assume that f is meromorphic on an open neighborhood of*

$$\gamma \cup \operatorname{Int}(\gamma),$$

and that the only pole singularities of f in $\operatorname{Int}(\gamma)$ are the finitely many points

$$z_1, \dots, z_n.$$

Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z=z_j} f.$$

PROOF. Choose pairwise disjoint closed disks

$$\overline{D(z_j;\varepsilon_j)} \subset \operatorname{Int}(\gamma), \qquad j = 1, \dots, n,$$

so small that each disk contains no singularity of f other than z_j.

Set

$$U := \operatorname{Int}(\gamma) \setminus \bigcup_{j=1}^{n} \overline{D(z_j;\varepsilon_j)}.$$

Then U is a bounded region with piecewise C^1 boundary, and f is holomorphic on an open neighborhood of $\overline{U}$. Hence, by Cauchy's theorem applied to U,

$$\int_{\partial U} f(z)\, dz = 0.$$

Now the boundary of U consists of the outer boundary γ, oriented positively, together with the circles

$$\partial D(z_j;\varepsilon_j), \qquad j = 1, \dots, n,$$

each with the negative orientation. Therefore

$$0 = \int_{\partial U} f(z)\, dz = \int_\gamma f(z)\, dz - \sum_{j=1}^{n} \int_{\partial D(z_j;\varepsilon_j)} f(z)\, dz.$$

Thus

$$\int_\gamma f(z)\, dz = \sum_{j=1}^{n} \int_{\partial D(z_j;\varepsilon_j)} f(z)\, dz.$$

For each j, since z_j is an isolated singularity of f, the residue at z_j is defined by

$$\operatorname{Res}_{z=z_j} f = \frac{1}{2\pi i}\int_{\partial D(z_j;\varepsilon_j)} f(z)\,dz,$$

where the circle is positively oriented. Hence

$$\int_{\partial D(z_j;\varepsilon_j)} f(z)\,dz = 2\pi i \operatorname{Res}_{z=z_j} f.$$

Substituting this into the previous identity gives

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z=z_j} f.$$

This proves the theorem. □

REMARK 1.2. In fact, the same residue formula remains valid if the singularities inside γ are arbitrary isolated singularities, not necessarily poles, since the residue is defined for every isolated singularity. However, in general the residue at an essential singularity is not easy to compute explicitly, so this more general form has fewer direct applications in contour integral computations.

2. Applications

In this section, we go over some typical applications of the residue formula. Such applications arise naturally in physics and engineering, as well as in Fourier analysis, partial differential equations, asymptotic analysis, and number theory.

2.1. Improper integrals of the form $\int_{-\infty}^{+\infty} f(x)\,dx$. One of the most common applications of the residue formula is the evaluation of improper real integrals of the form

$$\int_{-\infty}^{+\infty} f(x)\,dx,$$

where $f(x)$ is a rational function and has no poles on the real axis.

The basic idea is to regard $f(x)$ as the restriction of a rational function $f(z)$ to the real axis, and then integrate $f(z)$ over a large semicircular contour in the upper half-plane. If the contribution from the semicircle tends to 0 as the radius goes to infinity, then the residue formula shows that the real integral is equal to $2\pi i$ times the sum of the residues of f at the poles in the upper half-plane.

We begin with a simple example.

EXAMPLE 2.1. Calculate the improper integral

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^2}\,dx.$$

SOLUTION. Consider the rational function

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2}.$$

Its poles are at $z = \pm i$, both of order 2. In the upper half-plane, the only pole is $z = i$.

Let C_R be the positively oriented contour consisting of the interval $[-R, R]$ together with the upper semicircle

$$\Gamma_R := \{Re^{it} : 0 \le t \le \pi\}.$$

By the residue formula,

$$\int_{C_R} f(z)\,dz = 2\pi i \operatorname{Res}_{z=i} f.$$

Hence

$$\int_{-R}^{R} \frac{1}{(x^2+1)^2}\,dx + \int_{\Gamma_R} \frac{1}{(z^2+1)^2}\,dz = 2\pi i \operatorname{Res}_{z=i} f.$$

We first compute the residue at $z = i$. Since $z = i$ is a pole of order 2,

$$\operatorname{Res}_{z=i} f = \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\Bigg|_{z=i} = -\frac{2}{(z+i)^3}\Bigg|_{z=i} = -\frac{2}{(2i)^3} = \frac{1}{4i}.$$

Therefore

$$2\pi i \operatorname{Res}_{z=i} f = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}.$$

Next we show that

$$\int_{\Gamma_R} \frac{1}{(z^2+1)^2}\,dz \to 0 \qquad (R \to \infty).$$

Indeed, for $z \in \Gamma_R$, we have $|z| = R$, so

$$|z^2+1| \geq \big||z|^2 - 1\big| = R^2 - 1.$$

Hence

$$\left|\frac{1}{(z^2+1)^2}\right| \leq \frac{1}{(R^2-1)^2}.$$

Since the length of Γ_R is πR, the ML-estimate gives

$$\left|\int_{\Gamma_R} \frac{1}{(z^2+1)^2}\,dz\right| \leq \pi R \cdot \frac{1}{(R^2-1)^2} \to 0.$$

Taking $R \to \infty$, we obtain

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{2}.$$

□

In fact, if

$$f(z) = \frac{p(z)}{q(z)}$$

is a rational function, where p, q are polynomials, and if $q(x) \neq 0$ for all $x \in \mathbb{R}$, then the condition

$$\deg q \geq \deg p + 2$$

guarantees that the integral over the upper semicircle Γ_R tends to 0 as $R \to +\infty$. Therefore,

$$\int_{-\infty}^{+\infty} f(x)\,dx$$

can be computed as

$$2\pi i \sum_{\Im z_j > 0} \operatorname{Res}_{z=z_j} f,$$

where the sum is over all poles of f in the upper half-plane.

The following more examples are left to the reader to solve.

(1) $\int_0^{+\infty} \frac{1}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}$.
(2) $\int_0^{+\infty} \frac{1}{(x^2+1)^2(x^2+4)}\,dx = \frac{\pi}{6}$.

2.2. Improper integrals of the form $\int_{-\infty}^{+\infty} f(x)\cos(ax)\,dx$ or $\int_{-\infty}^{+\infty} f(x)\sin(ax)\,dx$. We can generalize the previous examples to integrals in which a rational function $f(x)$, with no singularities on $\mathbb{R}$, is multiplied by a sine or cosine factor. The basic idea is to consider the auxiliary function

$$f(z)e^{iaz}$$

and then take the real or imaginary part. Here is a typical example.

EXAMPLE 2.2. Compute the integral

$$\int_0^{+\infty} \frac{\cos(ax)}{x^2+1}\,dx, \qquad a > 0.$$

Notice $\frac{\cos(ax)}{x^2+1}$ is an even function, we can compute it using

$$\int_0^{+\infty} \frac{\cos(ax)}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos(ax)}{x^2+1}\,dx.$$

For this, if we try to mimic the previous example and consider the function

$$F(z) = \frac{\cos(az)}{z^2+1}$$

together with the contour C_R, where C_R is the positively oriented contour consisting of the interval $[-R, R]$ and the upper semicircle

$$\Gamma_R = \{Re^{it} : 0 \le t \le \pi\},$$

then we need to estimate $|\cos(az)|$ on Γ_R. By Euler's identity,

$$\cos(az) = \frac{e^{iaz} + e^{-iaz}}{2}.$$

Now if $z = x + iy$ with $y \ge 0$, then

$$|e^{iaz}| = e^{-ay} \le 1,$$

but

$$|e^{-iaz}| = e^{ay},$$

which may grow exponentially on the upper semicircle. Thus $\cos(az)$ itself is not well suited for direct estimation on Γ_R, and this naive approach does not work.

Instead, we consider the modified function

$$F(z) = \frac{e^{iaz}}{z^2+1}.$$

This works well on the upper semicircle because $|e^{iaz}| \le 1$ there. After evaluating

$$\int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2+1}\,dx,$$

we can recover the desired cosine integral by taking the real part, since

$$\frac{e^{iax}}{x^2+1} = \frac{\cos(ax)}{x^2+1} + i\,\frac{\sin(ax)}{x^2+1}.$$

SOLUTION: Consider

$$F(z) = \frac{e^{iaz}}{z^2+1} = \frac{e^{iaz}}{(z-i)(z+i)}, \qquad a > 0.$$

We integrate F over the contour C_R. The only pole of F in the upper half-plane is at $z = i$. Hence, by the residue formula,

$$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}\,dx + \int_{\Gamma_R} \frac{e^{iaz}}{z^2+1}\,dz = 2\pi i \operatorname{Res}_{z=i} \frac{e^{iaz}}{z^2+1}.$$

We first show that

$$\int_{\Gamma_R} \frac{e^{iaz}}{z^2+1}\,dz \to 0 \qquad (R\to\infty).$$

Indeed, if $z = x + iy \in \Gamma_R$, then $y \geq 0$, so

$$|e^{iaz}| = e^{-ay} \leq 1.$$

Also, for R sufficiently large,

$$|z^2+1| \geq R^2 - 1.$$

Therefore

$$\left|\frac{e^{iaz}}{z^2+1}\right| \leq \frac{1}{R^2-1} \qquad (z\in\Gamma_R).$$

Since L$ength(\Gamma_R) = \pi R$, it follows that

$$\left|\int_{\Gamma_R} \frac{e^{iaz}}{z^2+1}\,dz\right| \leq \frac{\pi R}{R^2-1} \to 0.$$

Next,

$$\mathrm{Res}_{z=i}\frac{e^{iaz}}{z^2+1} = \lim_{z\to i}\frac{e^{iaz}}{z+i} = \frac{e^{iai}}{2i} = \frac{e^{-a}}{2i}.$$

Hence

$$\int_{-\infty}^{+\infty} \frac{e^{iax}}{x^2+1}\,dx = 2\pi i \cdot \frac{e^{-a}}{2i} = \pi e^{-a}.$$

Finally, taking the real part, we obtain

$$\int_0^{+\infty} \frac{\cos(ax)}{x^2+1}\,dx = \frac{1}{2}\Re\left(\int_{-\infty}^{+\infty}\frac{e^{iax}}{x^2+1}\,dx\right) = \frac{\pi e^{-a}}{2}.$$

□

A similar method applies to integrals of the form

$$\int_{-\infty}^{+\infty} f(x)\cos(ax)\,dx \qquad \text{or} \qquad \int_{-\infty}^{+\infty} f(x)\sin(ax)\,dx,$$

where $f(x)$ is a rational function with no poles on $\mathbb{R}$. In such cases, one may consider the function

$$f(z)e^{iaz},$$

choose the upper or lower half-plane according to the sign of a, and then recover the desired integral by taking the real or imaginary part.

2.3. Integrals of the form $\int_0^{2\pi} F(\cos\theta,\sin\theta)\,d\theta$. For integrals involving sine and cosine over a full period $[0,2\pi]$, one can often translate the problem into a contour integral over the unit circle. The basic substitution is

$$z = e^{i\theta},$$

so that as θ runs from 0 to 2π, the point z travels once counterclockwise around the unit circle. Under this substitution,

$$d\theta = \frac{dz}{iz}, \qquad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right), \qquad \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right).$$

Thus many trigonometric integrals can be rewritten as contour integrals of rational functions on $|z| = 1$, and the residue formula may then be applied.

We illustrate this method with the following example.

EXAMPLE 2.3. Assume $a, b \in \mathbb{R}$ and $a > |b|$. Compute

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta.$$

SOLUTION. Let

$$I := \int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta.$$

Set

$$z = e^{i\theta}.$$

Then

$$d\theta = \frac{dz}{iz}, \qquad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Hence

$$I = \int_{|z|=1} \frac{1}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}.$$

Multiplying numerator and denominator by $2z$, we obtain

$$I = \int_{|z|=1} \frac{2}{i(bz^2 + 2az + b)}\, dz.$$

So we are reduced to a contour integral of the rational function

$$f(z) = \frac{2}{i(bz^2 + 2az + b)}.$$

Its poles are the roots of

$$bz^2 + 2az + b = 0,$$

namely

$$z_{\pm} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}.$$

Since $a > |b|$, we have $a^2 - b^2 > 0$, so the roots are real. Also,

$$z_+ z_- = 1.$$

Thus exactly one of them lies inside the unit circle. It is

$$z_0 = \frac{-a + \sqrt{a^2 - b^2}}{b}.$$

Therefore, by the residue formula,

$$I = 2\pi i \operatorname{Res}_{z=z_0} \frac{2}{i(bz^2 + 2az + b)}.$$

Since the pole is simple,

$$\operatorname{Res}_{z=z_0} \frac{2}{i(bz^2 + 2az + b)} = \frac{2}{i(2bz_0 + 2a)} = \frac{1}{i(bz_0 + a)}.$$

Now from the definition of z_0,

$$bz_0 + a = \sqrt{a^2 - b^2}.$$

Hence

$$\operatorname{Res}_{z=z_0} \frac{2}{i(bz^2 + 2az + b)} = \frac{1}{i\sqrt{a^2 - b^2}}.$$

It follows that

$$I = 2\pi i \cdot \frac{1}{i\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

Therefore

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

□

2.4. Improper integrals with a singularity on the contour. In the previous examples, the contour did not pass through any singularity of the integrand. In some important improper integrals, however, the auxiliary complex function naturally used in the contour method has a singularity on the contour itself, typically at 0. In such cases, we modify the contour by removing a small semicircle around the singular point, and then let its radius tend to 0.

A basic example is the following classical integral.

EXAMPLE 2.4. Compute the improper integral

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx.$$

Following the experience from the previous subsection, we should not directly consider

$$\frac{\sin z}{z}.$$

Indeed,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

and although e^{iz} decays in the upper half-plane, e^{-iz} grows exponentially there. Thus $\sin z$ itself is not well suited for estimation on the upper semicircle.

Instead, we consider

$$F(z) = \frac{e^{iz}}{z},$$

and later recover the desired integral by taking the imaginary part.

SOLUTION. The function

$$F(z) = \frac{e^{iz}}{z}$$

has a singularity at $z = 0$, so unlike the previous examples, we must avoid this point. For $R > \varepsilon > 0$, let $C_{R,\varepsilon}$ be the contour consisting of

- the interval $[\varepsilon, R]$ on the real axis;
- the upper semicircle
$$\Gamma_R = \{Re^{it} : 0 \le t \le \pi\};$$
- the interval $[-R, -\varepsilon]$ on the real axis;
- the small upper semicircle
$$\gamma_\varepsilon = \{\varepsilon e^{it} : \pi \ge t \ge 0\},$$
oriented clockwise.

Since F is holomorphic on and inside this contour, Cauchy's theorem gives

$$\int_{C_{R,\varepsilon}} \frac{e^{iz}}{z}\, dz = 0.$$

Therefore,

$$\int_\varepsilon^R \frac{e^{ix}}{x}\, dx + \int_{\Gamma_R} \frac{e^{iz}}{z}\, dz + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\, dz = 0.$$

We first show that
$$\int_{\Gamma_R} \frac{e^{iz}}{z}\, dz \to 0 \qquad (R \to \infty).$$
This requires a more careful estimate. If we only write $z = x + iy$ on Γ_R, then we get
$$|e^{iz}| = e^{-y} \le 1.$$
However, this is not enough, since the denominator z has size R, while the length of Γ_R is also of order R. Thus this only gives a bounded estimate, not one tending to 0.

To obtain a sharper estimate, parametrize Γ_R by
$$z(t) = Re^{it}, \qquad 0 \le t \le \pi.$$
Then
$$dz = iRe^{it}\, dt,$$
so
$$\int_{\Gamma_R} \frac{e^{iz}}{z}\, dz = i \int_0^\pi e^{iRe^{it}}\, dt.$$
Hence
$$\left| \int_{\Gamma_R} \frac{e^{iz}}{z}\, dz \right| \le \int_0^\pi \left| e^{iR(\cos t + i \sin t)} \right| dt = \int_0^\pi e^{-R\sin t}\, dt.$$
Fix $\delta \in (0, \pi/2)$. Split the integral into three parts:
$$\int_0^\pi e^{-R\sin t}\, dt = \int_0^\delta e^{-R\sin t}\, dt + \int_\delta^{\pi-\delta} e^{-R\sin t}\, dt + \int_{\pi-\delta}^\pi e^{-R\sin t}\, dt.$$
On the middle interval $[\delta, \pi - \delta]$, we have
$$\sin t \ge \sin \delta,$$
so
$$\int_\delta^{\pi-\delta} e^{-R\sin t}\, dt \le (\pi - 2\delta) e^{-R\sin\delta} \to 0 \qquad (R \to \infty).$$
For the two end pieces, note that by symmetry,
$$\int_{\pi-\delta}^\pi e^{-R\sin t}\, dt = \int_0^\delta e^{-R\sin t}\, dt,$$
so it suffices to estimate the first one. On $[0, \delta]$, we use the elementary bound
$$\sin t \ge kt$$
for some constant $k > 0$; for instance, one may take
$$k = \frac{2}{\pi}.$$
Therefore
$$\int_0^\delta e^{-R\sin t}\, dt \le \int_0^\delta e^{-kRt}\, dt = \frac{1 - e^{-kR\delta}}{kR} \to 0 \qquad (R \to \infty).$$
The same estimate holds for the interval $[\pi - \delta, \pi]$. Combining the three estimates, we conclude that
$$\int_0^\pi e^{-R\sin t}\, dt \to 0,$$
and hence
$$\int_{\Gamma_R} \frac{e^{iz}}{z}\, dz \to 0 \qquad (R \to \infty).$$

Next we deal with the small semicircle γ_ε. Parametrize it by

$$z = \varepsilon e^{it}, \qquad \pi \geq t \geq 0.$$

Then

$$dz = i\varepsilon e^{it}\, dt,$$

and so

$$\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\, dz = \int_\pi^0 \frac{e^{i\varepsilon e^{it}}}{\varepsilon e^{it}} i\varepsilon e^{it}\, dt = -i\int_0^\pi e^{i\varepsilon e^{it}}\, dt.$$

Letting $\varepsilon \to 0$, we obtain

$$\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\, dz \to -i\pi.$$

Now combine the two integrals along the real axis:

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\, dx = \int_\varepsilon^R \frac{e^{-ix}}{-x}\, dx = -\int_\varepsilon^R \frac{e^{-ix}}{x}\, dx.$$

Therefore

$$\int_\varepsilon^R \frac{e^{ix}}{x}\, dx + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\, dx = \int_\varepsilon^R \frac{e^{ix} - e^{-ix}}{x}\, dx = 2i\int_\varepsilon^R \frac{\sin x}{x}\, dx.$$

Now let $R \to \infty$ and $\varepsilon \to 0$. Using the three limits above, the contour identity becomes

$$2i\int_0^{+\infty} \frac{\sin x}{x}\, dx - i\pi = 0.$$

Hence

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

□

2.5. Integrals involving multivalued functions. Some improper integrals involve expressions such as x^α ($\alpha \notin \mathbb{Z}$) or $\log x$, and have singularities at 0 or at infinity. From the complex point of view, the corresponding functions z^α and $\mathrm{Log}z$ are multivalued. To use contour integration, one first chooses a branch on a slit domain.

A standard contour in this setting is the keyhole contour around the positive real axis. The essential point is that, once a branch is fixed on the slit plane, its boundary values on the two sides of the branch cut differ by an explicit factor or additive constant. This jump is what allows us to extract the desired real integral.

We illustrate the method with the following two examples:

$$\int_0^\infty \frac{\sqrt{x}}{x^2+2x+2}\, dx, \qquad \int_0^\infty \frac{\ln x}{x^2+2x+2}\, dx.$$

EXAMPLE 2.5. Compute the improper integral

$$\int_0^\infty \frac{\sqrt{x}}{x^2+2x+2}\, dx.$$

SOLUTION. Let

$$I := \int_0^\infty \frac{\sqrt{x}}{x^2+2x+2}\, dx.$$

We choose the branch

$$\mathrm{Log}z = \ln|z| + i\mathrm{Arg}z, \qquad 0 < \mathrm{Arg}z < 2\pi,$$

on

$$\mathbb{C} \setminus [0, \infty),$$

and define

$$z^{1/2} := e^{\frac{1}{2}\operatorname{Log} z}.$$

Thus $z^{1/2}$ is holomorphic on the slit plane.

REMARK 2.6. Our branch choice here is different from the principal branch used earlier. There we usually took

$$-\pi < \operatorname{Arg} z \leq \pi,$$

with branch cut along the negative real axis. Here, since the real integral is taken over $(0, \infty)$, it is more convenient to place the branch cut along the positive real axis and choose

$$0 < \operatorname{Arg} z < 2\pi.$$

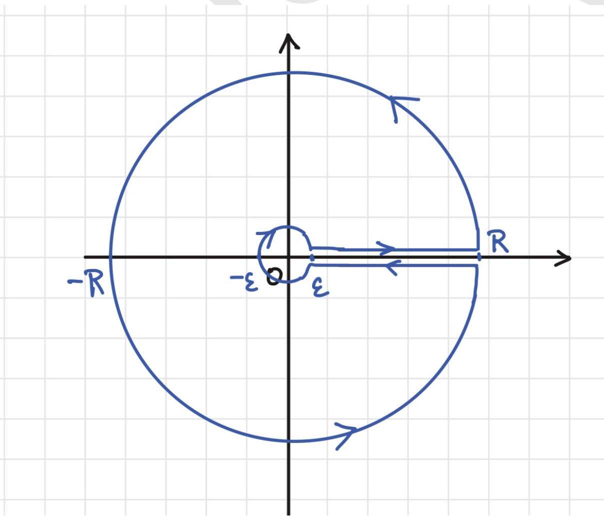
This allows the keyhole contour to run along the interval on which the integrand is defined.

Consider

$$f(z) = \frac{z^{1/2}}{z^2 + 2z + 2}$$

and integrate it over the keyhole contour $C_{R,\varepsilon}$, consisting of

- the interval $[\varepsilon, R]$ along the upper side of the positive real axis;
- the circle $|z| = R$, traversed counterclockwise;
- the interval $[R, \varepsilon]$ along the lower side of the positive real axis;
- the circle $|z| = \varepsilon$, traversed clockwise.



We first show that the contributions from the large and small circles tend to 0.

On the large circle $|z| = R$, we have

$$|z^{1/2}| = R^{1/2},$$

and

$$|z^2 + 2z + 2| \geq R^2 - 2R - 2.$$

Hence

$$\left| \frac{z^{1/2}}{z^2 + 2z + 2} \right| \leq \frac{R^{1/2}}{R^2 - 2R - 2}.$$

Since the length of the circle is $2\pi R$, the contribution from the large circle is bounded by

$$2\pi R \cdot \frac{R^{1/2}}{R^2 - 2R - 2} \to 0 \qquad (R \to \infty).$$

On the small circle $|z| = \varepsilon$, we have

$$|z^{1/2}| = \varepsilon^{1/2}.$$

Also,

$$z^2 + 2z + 2 \to 2 \qquad (z \to 0),$$

so for ε sufficiently small, the denominator stays bounded away from 0. Thus

$$\left|\frac{z^{1/2}}{z^2 + 2z + 2}\right| \le C\varepsilon^{1/2}$$

for some constant $C > 0$. Since the length of the small circle is $2\pi\varepsilon$, its contribution is bounded by

$$2\pi\varepsilon \cdot C\varepsilon^{1/2} = 2\pi C\, \varepsilon^{3/2} \to 0 \qquad (\varepsilon \to 0).$$

Next we analyze the integrals along the two sides of the branch cut. On the upper side, $z = x + i\delta$ with $\Im z = \delta > 0$ small,

$$z^{\frac{1}{2}} \to \sqrt{x}, \quad \text{as} \delta \to 0+.$$

and the contribution tends to

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 2x + 2}\, dx = I.$$

On the lower side, $z = x - i\delta$ with $\Im z = -\delta < 0$ small,

$$z^{\frac{1}{2}} \to -\sqrt{x}, \quad \text{as} \delta \to 0+.$$

Since this side is traversed from R to ε, its contribution is

$$\int_R^\varepsilon \frac{-\sqrt{x}}{x^2 + 2x + 2}\, dx = \int_\varepsilon^R \frac{\sqrt{x}}{x^2 + 2x + 2}\, dx,$$

which tends to I. Therefore the two line segments together contribute $2I$.

It remains to compute the residues. The poles are the roots of

$$z^2 + 2z + 2 = 0,$$

namely

$$-1 + i, \qquad -1 - i.$$

Both lie inside the contour, and both are simple poles. Then

$$\operatorname{Res}_{z=z_0} f = \frac{z_0^{1/2}}{2z_0 + 2}.$$

Therefore

$$\operatorname{Res}_{z=-1+i} f = \frac{(-1+i)^{1/2}}{2i}, \qquad \operatorname{Res}_{z=-1-i} f = \frac{(-1-i)^{1/2}}{-2i}.$$

Adding them gives

$$\operatorname{Res}_{z=-1+i} f + \operatorname{Res}_{z=-1-i} f = \frac{1}{2i}\Big((-1+i)^{1/2} - (-1-i)^{1/2}\Big) = -\, i\, 2^{1/4} \sin\frac{\pi}{8}.$$

By the residue formula,

$$\int_{C_{R,\varepsilon}} f(z)\, dz = 2\pi i \left(\operatorname{Res}_{z=-1+i} f + \operatorname{Res}_{z=-1-i} f\right) = 2\pi\, 2^{1/4} \sin\frac{\pi}{8}.$$

Letting $R \to \infty$ and $\varepsilon \to 0$, and using the estimates above, we obtain

$$2I = 2\pi\, 2^{1/4} \sin\frac{\pi}{8}.$$

Therefore

$$I = \pi\, 2^{1/4} \sin\frac{\pi}{8} = \frac{\pi}{2}\sqrt{2\sqrt{2} - 2}.$$

□

EXAMPLE 2.7. Compute the improper integral

$$\int_0^\infty \frac{\ln x}{x^2 + 2x + 2}\, dx.$$

SOLUTION. Let

$$I := \int_0^\infty \frac{\ln x}{x^2 + 2x + 2}\, dx.$$

As discussed before, if we directly integrate

$$\frac{\mathrm{Log} z}{z^2 + 2z + 2}$$

around a keyhole contour, then the logarithmic terms on the two sides of the branch cut cancel after taking orientation into account. So instead, we consider

$$f(z) = \frac{(\mathrm{Log} z)^2}{z^2 + 2z + 2},$$

where

$$\mathrm{Log} z = \ln|z| + i\mathrm{Arg} z, \qquad 0 < \mathrm{Arg} z < 2\pi,$$

on

$$\mathbb{C} \setminus [0, \infty).$$

We integrate $f(z)$ over the keyhole contour $C_{R,\varepsilon}$ the same as the previous example.

We first show that the contributions from the large and small circles tend to 0.

On the large circle $|z| = R$, we have

$$|\mathrm{Log} z| \le \ln R + 2\pi,$$

and

$$|z^2 + 2z + 2| \ge R^2 - 2R - 2.$$

Hence

$$\left| \frac{(\mathrm{Log} z)^2}{z^2 + 2z + 2} \right| \le \frac{(\ln R + 2\pi)^2}{R^2 - 2R - 2}.$$

Since the length of the circle is $2\pi R$, the contribution from the large circle is bounded by

$$2\pi R \cdot \frac{(\ln R + 2\pi)^2}{R^2 - 2R - 2} \to 0 \qquad (R \to \infty).$$

On the small circle $|z| = \varepsilon$, we have

$$|\mathrm{Log} z| \le |\ln \varepsilon| + 2\pi.$$

Also,

$$z^2 + 2z + 2 \to 2 \qquad (z \to 0),$$

so for ε sufficiently small, the denominator stays bounded away from 0. Thus

$$\left|\frac{(\mathrm{Log} z)^2}{z^2+2z+2}\right| \le C\big(|\ln\varepsilon| + 2\pi\big)^2$$

for some constant $C > 0$. Since the length of the small circle is $2\pi\varepsilon$, its contribution is bounded by

$$2\pi\varepsilon \cdot C\big(|\ln\varepsilon| + 2\pi\big)^2 \to 0 \qquad (\varepsilon \to 0).$$

Next we analyze the two sides of the branch cut. On the upper side, $z = x + i\delta$ with $\Im z = \delta > 0$ small, so

$$\mathrm{Log} z \to \ln x, \quad \text{as} \delta \to 0+.$$

Hence the contribution tends to

$$\int_0^\infty \frac{(\ln x)^2}{x^2+2x+2}\,dx.$$

On the lower side, $z = x - i\delta$ with $\Im z = -\delta < 0$ small,

$$\mathrm{Log} z = \ln x + 2\pi i, \quad \text{as} \delta \to 0+.$$

Since this side is traversed from R to ε, its contribution is

$$\int_R^\varepsilon \frac{(\ln x + 2\pi i)^2}{x^2+2x+2}\,dx = -\int_\varepsilon^R \frac{(\ln x + 2\pi i)^2}{x^2+2x+2}\,dx.$$

Therefore the total contribution from the two line segments tends to

$$\int_0^\infty \frac{(\ln x)^2 - (\ln x + 2\pi i)^2}{x^2+2x+2}\,dx.$$

Expanding,

$$(\ln x)^2 - (\ln x + 2\pi i)^2 = -4\pi i \ln x + 4\pi^2.$$

Hence the two line segments together contribute

$$-4\pi i \int_0^\infty \frac{\ln x}{x^2+2x+2}\,dx + 4\pi^2 \int_0^\infty \frac{dx}{x^2+2x+2}.$$

That is,

$$-4\pi i\, I + 4\pi^2 \int_0^\infty \frac{dx}{(x+1)^2+1}.$$

Since

$$\int_0^\infty \frac{dx}{(x+1)^2+1} = [\arctan(x+1)]_0^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

the total contribution from the two line segments is

$$-4\pi i\, I + \pi^3.$$

It remains to compute the residues. The poles are again

$$-1+i, \qquad -1-i,$$

and both are simple. Then

$$\mathrm{Res}_{z=z_0} f = \frac{(\mathrm{Log} z_0)^2}{2z_0+2}.$$

Therefore

$$\mathrm{Res}_{z=-1+i} f = \frac{(\mathrm{Log}(-1+i))^2}{2i}, \qquad \mathrm{Res}_{z=-1-i} f = \frac{(\mathrm{Log}(-1-i))^2}{-2i}.$$

Now

$$-1+i = \sqrt{2}\, e^{3\pi i/4}, \qquad -1-i = \sqrt{2}\, e^{5\pi i/4},$$

so with our choice of branch,

$$\mathrm{Log}(-1+i) = \frac{1}{2}\ln 2 + \frac{3\pi i}{4}, \qquad \mathrm{Log}(-1-i) = \frac{1}{2}\ln 2 + \frac{5\pi i}{4}.$$

Hence

$$\begin{aligned} &\mathrm{Res}_{z=-1+i} f + \mathrm{Res}_{z=-1-i} f \\ = \ & \frac{1}{2i}\left[\left(\frac{1}{2}\ln 2 + \frac{3\pi i}{4}\right)^2 - \left(\frac{1}{2}\ln 2 + \frac{5\pi i}{4}\right)^2\right] \\ = \ & -\frac{\pi}{4}\ln 2 - \frac{\pi^2}{2} i. \end{aligned}$$

By the residue formula,

$$\begin{aligned} &\int_{C_{R,\varepsilon}} f(z)\,dz \\ = \ & 2\pi i \left(\mathrm{Res}_{z=-1+i} f + \mathrm{Res}_{z=-1-i} f\right) \\ = \ & 2\pi i \left(-\frac{\pi}{4}\ln 2 - \frac{\pi^2}{2} i\right) \\ = \ & \pi^3 - \frac{\pi^2 i}{2}\ln 2. \end{aligned}$$

Now let $R \to \infty$ and $\varepsilon \to 0$. Using the estimates above, we obtain

$$-4\pi i\, I + \pi^3 = \pi^3 - \frac{\pi^2 i}{2}\ln 2.$$

Hence

$$\int_0^\infty \frac{\ln x}{x^2 + 2x + 2}\,dx = I = \frac{\pi}{8}\ln 2.$$

□

REMARK 2.8. For this particular problem, the integral can also be treated by a different contour argument, for example by using a keyhole contour in the upper half-plane. However, the method used here is more systematic and adapts more readily to other logarithmic integrals. In particular, for integrals of the form

$$\int_0^\infty F(\ln x)\,R(x)\,dx,$$

where F is a polynomial and $R(x)$ is a rational function, the method above is often more flexible and more systematic.

We now look at another example where the logarithmic trick is useful.

EXAMPLE 2.9. Compute the improper integral

$$\int_0^\infty \frac{x}{x^3 + x^2 + x + 1}\,dx.$$

This problem is different from the earlier contour integrals over the upper half-disk. There, we worked with integrals over $(-\infty, \infty)$, or could reduce the problem to that case by symmetry. Here there is no simple way to rewrite it as an integral over the whole real line. The logarithmic keyhole method can be used to compute it.

SOLUTION. We use the branch

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z, \qquad 0 < \operatorname{Arg} z < 2\pi,$$

on

$$\mathbb{C} \setminus [0, \infty)$$

again, and consider the function

$$F(z) = \frac{z\operatorname{Log} z}{z^3 + z^2 + z + 1}.$$

As in the previous two examples, we integrate $F(z)$ over the keyhole contour $C_{R,\varepsilon}$.

We first show that the contributions from the large and small circles tend to 0.

On the large circle $|z| = R$, we have

$$|z\operatorname{Log} z| \le R(\ln R + 2\pi),$$

and

$$|z^3 + z^2 + z + 1| \ge R^3 - R^2 - R - 1.$$

Hence

$$\left|\frac{z\operatorname{Log} z}{z^3 + z^2 + z + 1}\right| \le \frac{R(\ln R + 2\pi)}{R^3 - R^2 - R - 1}.$$

Since the length of the circle is $2\pi R$, the contribution from the large circle is bounded by

$$2\pi R \cdot \frac{R(\ln R + 2\pi)}{R^3 - R^2 - R - 1} \to 0 \qquad (R \to \infty).$$

On the small circle $|z| = \varepsilon$, we have

$$|z\operatorname{Log} z| \le \varepsilon\big(|\ln\varepsilon| + 2\pi\big).$$

Also,

$$z^3 + z^2 + z + 1 \to 1 \qquad (z \to 0),$$

so for ε sufficiently small, the denominator stays bounded away from 0. Thus

$$\left|\frac{z\operatorname{Log} z}{z^3 + z^2 + z + 1}\right| \le C\varepsilon\big(|\ln\varepsilon| + 2\pi\big)$$

for some constant $C > 0$. Since the length of the small circle is $2\pi\varepsilon$, its contribution is bounded by

$$2\pi\varepsilon \cdot C\varepsilon\big(|\ln\varepsilon| + 2\pi\big) \to 0 \qquad (\varepsilon \to 0).$$

Next we analyze the two sides of the branch cut. On the upper side, approximately, $z = x$ with $\operatorname{Arg} z = 0$, so

$$\operatorname{Log} z \to \ln x$$

as it approaches to $\mathbb{R}^+$. Hence the contribution tends to

$$\int_0^\infty \frac{x\ln x}{x^3 + x^2 + x + 1}\,dx.$$

On the lower side, approximately, $z = x$ with $\operatorname{Arg} z = 2\pi$, so

$$\operatorname{Log} z \to \ln x + 2\pi i$$

as it approaches to $\mathbb{R}^+$. Since this side is traversed from R to ε, its contribution is

$$\int_R^\varepsilon \frac{x(\ln x + 2\pi i)}{x^3 + x^2 + x + 1}\,dx = -\int_\varepsilon^R \frac{x(\ln x + 2\pi i)}{x^3 + x^2 + x + 1}\,dx.$$

Therefore the total contribution from the two line segments tends to

$$\int_0^\infty \frac{x\ln x - x(\ln x + 2\pi i)}{x^3+x^2+x+1}\,dx = -2\pi i\int_0^\infty \frac{x}{x^3+x^2+x+1}\,dx = -2\pi i\, I.$$

It remains to compute the residues. We first solve

$$z^3+z^2+z+1=0.$$

Notice that

$$z^3+z^2+z+1=\frac{z^4-1}{z-1}.$$

Since

$$z^4-1=(z-1)(z+1)(z^2+1),$$

we get

$$z^3+z^2+z+1=(z+1)(z^2+1).$$

Hence the poles are

$$-1, \qquad i, \qquad -i,$$

all simple.

We compute

$$\operatorname{Res}_{z=-1}F = \left.\frac{z\mathrm{Log}z}{z^2+1}\right|_{z=-1} = \frac{(-1)(i\pi)}{2} = -\frac{i\pi}{2}.$$

Also,

$$\operatorname{Res}_{z=i}F = \left.\frac{z\mathrm{Log}z}{(z+1)(z+i)}\right|_{z=i} = \frac{i\cdot(\pi i/2)}{(1+i)(2i)} = \frac{\pi}{8}(1+i),$$

and

$$\operatorname{Res}_{z=-i}F = \left.\frac{z\mathrm{Log}z}{(z+1)(z-i)}\right|_{z=-i} = \frac{(-i)\cdot(3\pi i/2)}{(1-i)(-2i)} = -\frac{3\pi}{8}(1-i).$$

Therefore

$$\operatorname{Res}_{z=-1}F + \operatorname{Res}_{z=i}F + \operatorname{Res}_{z=-i}F = -\frac{\pi}{4}.$$

By the residue formula,

$$\int_{C_{R,\varepsilon}} F(z)\,dz = 2\pi i\left(-\frac{\pi}{4}\right) = -\frac{\pi^2 i}{2}.$$

Now let $R\to\infty$ and $\varepsilon\to 0$. Using the estimates above, we obtain

$$-2\pi i\, I = -\frac{\pi^2 i}{2}.$$

Hence

$$\int_0^\infty \frac{x}{x^3+x^2+x+1}\,dx = I = \frac{\pi}{4}.$$

□

2.6. Infinite series. The residue formula can also be used to evaluate certain infinite series. The basic idea is to construct a meromorphic function whose poles occur at the integers, so that summing residues at those poles reproduces the terms of the series.

We begin by considering the function

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}.$$

It has simple poles at

$$z = n, \qquad n \in \mathbb{Z}.$$

Moreover,

$$\operatorname{Res}_{z=n} \cot(\pi z) = \lim_{z \to n} \frac{(z-n)\cos(\pi z)}{\sin(\pi z)} = \frac{1}{\pi}.$$

Hence the function

$$\pi \cot(\pi z)$$

has simple poles at every integer, each with residue 1.

Now let $f(z)$ be a meromorphic function which is holomorphic at every integer. (This assumption can be relaxed in some cases; exceptional integer poles can sometimes be handled separately, as in the example of $\sum \frac{1}{n^2}$ later.) Then for each integer $n \in \mathbb{Z}$,

$$\operatorname{Res}_{z=n}\big(\pi \cot(\pi z) f(z)\big) = f(n).$$

Suppose that for a sequence $R_N \to +\infty$, with

$$N < R_N < N+1,$$

and denote by C_N some simple C^1 regular curve that intersects the real axis at $\pm R_N$. Suppose that C_N contains no pole of f on it. Then by the residue formula,

$$\int_{C_N} \pi \cot(\pi z) f(z)\, dz = 2\pi i \left(\sum_{n=-N}^{N} f(n) + \sum_{\substack{z_0 \text{ a pole of } f \\ \text{inside } C_N}} \operatorname{Res}_{z=z_0}\big(\pi \cot(\pi z) f(z)\big) \right).$$

Therefore,

$$\sum_{n=-N}^{N} f(n) = \frac{1}{2\pi i} \int_{C_N} \pi \cot(\pi z) f(z)\, dz - \sum_{\substack{z_0 \text{ a pole of } f \\ |z_0| < R_N}} \operatorname{Res}_{z=z_0}\big(\pi \cot(\pi z) f(z)\big).$$

If, as $N \to +\infty$, the right-hand side converges and can be evaluated, then we obtain the value of

$$\sum_{-\infty}^{+\infty} f(n).$$

Let's see a concrete example.

EXAMPLE 2.10. Compute the series

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+1}.$$

SOLUTION. Let

$$f(z) = \frac{1}{z^2+1}, \qquad F(z) = \pi\cot(\pi z)\, f(z) = \frac{\pi\cot(\pi z)}{z^2+1}.$$

Since f is holomorphic at every integer $n \in \mathbb{Z}$, the function $F(z)$ has simple poles at all integers, and

$$\mathrm{Res}_{z=n} F(z) = f(n) = \frac{1}{n^2+1}.$$

The function $f(z)$ also has simple poles at

$$z = i, \qquad z = -i.$$

Hence $F(z)$ has poles at all integers and also at $z = \pm i$.

Let R_N be the square contour with vertices

$$\left(N+\frac{1}{2}\right) \pm i\left(N+\frac{1}{2}\right), \qquad -\left(N+\frac{1}{2}\right) \pm i\left(N+\frac{1}{2}\right),$$

oriented counterclockwise. Then by the residue formula,

$$\int_{R_N} F(z)\,dz = 2\pi i\left(\sum_{n=-N}^{N} \frac{1}{n^2+1} + \mathrm{Res}_{z=i} F(z) + \mathrm{Res}_{z=-i} F(z)\right).$$

We claim that

$$\int_{C_N} F(z)\,dz \to 0 \qquad (N \to \infty).$$

Indeed, notice that

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$$

is uniformly bounded on C_N, independently of N, whose proof is left to the reader. On the other hand,

$$\frac{1}{z^2+1} = O\left(\frac{1}{|z|^2}\right)$$

on C_N, while the length of C_N is of order N. Therefore

$$\int_{C_N} F(z)\,dz \to 0.$$

Letting $N \to \infty$, we obtain

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+1} = -\mathrm{Res}_{z=i} F(z) - \mathrm{Res}_{z=-i} F(z).$$

It remains to compute the residues at $z = \pm i$. Since $1/(z^2+1)$ has simple poles at $z = i$ and $z = -i$, direct calculation shows

$$\mathrm{Res}_{z=i} F(z) = -\frac{\pi}{2}\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} = \mathrm{Res}_{z=-i} F(z) = -\frac{\pi}{2}\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}.$$

Therefore

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+1} = \pi\,\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} = \pi\,\coth(\pi).$$

□

EXAMPLE 2.11. Compute the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

SOLUTION. Consider

$$f(z)=\frac{1}{z^2},\qquad F(z)=\pi\cot(\pi z)\,f(z)=\frac{\pi\cot(\pi z)}{z^2}.$$

For every nonzero integer $n\in\mathbb{Z}$, the function $f(z)$ is holomorphic at $z=n$, so

$$\mathrm{Res}_{z=n}F(z)=f(n)=\frac{1}{n^2}.$$

The point $z=0$ requires separate consideration, since both $\pi\cot(\pi z)$ and $1/z^2$ are singular there.

Let R_N be the square contour with vertices

$$\left(N+\frac{1}{2}\right)\pm i\left(N+\frac{1}{2}\right),\qquad -\left(N+\frac{1}{2}\right)\pm i\left(N+\frac{1}{2}\right),$$

oriented counterclockwise. By the residue formula,

$$\int_{R_N}F(z)\,dz=2\pi i\left(\sum_{\substack{-N\le n\le N\\ n\ne 0}}\frac{1}{n^2}+\mathrm{Res}_{z=0}F(z)\right).$$

As in the previous example, $\cot(\pi z)$ is uniformly bounded on R_N, while

$$\frac{1}{z^2}=O\left(\frac{1}{|z|^2}\right)$$

on the contour. Since the length of R_N is of order N, it follows that

$$\int_{R_N}F(z)\,dz\to 0\qquad (N\to\infty).$$

Therefore

$$\sum_{n\ne 0}\frac{1}{n^2}=-\mathrm{Res}_{z=0}F(z).$$

It remains to compute the residue at $z=0$. Using the Laurent expansion of $\cot z$ at 0,

$$\cot z=\frac{1}{z}-\frac{z}{3}+O(z^3),$$

we get

$$\pi\cot(\pi z)=\frac{1}{z}-\frac{\pi^2}{3}z+O(z^3).$$

Hence

$$F(z)=\frac{\pi\cot(\pi z)}{z^2}=\frac{1}{z^3}-\frac{\pi^2}{3}\frac{1}{z}+O(z).$$

Therefore

$$\mathrm{Res}_{z=0}F(z)=-\frac{\pi^2}{3}.$$

Thus

$$\sum_{n\ne 0}\frac{1}{n^2}=\frac{\pi^2}{3}.$$

Since

$$\sum_{n\ne 0}\frac{1}{n^2}=2\sum_{n=1}^{\infty}\frac{1}{n^2},$$

we conclude that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

□

CHAPTER 8

The Argument Principle and Some Fundamental Theorems

In this chapter we develop several fundamental tools in complex analysis, centered around the argument principle. Although these results are basic, they are among the most useful theorems in the subject.

We begin with the argument principle, which relates contour integrals to the zeros and poles of a meromorphic function.

Next, we prove Rouché's theorem, a powerful method for comparing the zeros of two holomorphic functions. As applications, we prove the fundamental theorem of algebra, a holomorphic version of the implicit function theorem, the open mapping theorem, and the maximum modulus principle.

1. The argument principle

We begin with two motivating examples.

EXAMPLE 1.1. (1) Consider the function

$$f(z) = z^n.$$

It has 0 as its only zero in $\mathbb{C}$, and the order of this zero is n. We now see that this integer n can be recovered from a complex integral.

Indeed,

$$f'(z) = nz^{n-1}, \qquad \frac{f'(z)}{f(z)} = \frac{n}{z}, \qquad z \neq 0.$$

If γ is any positively oriented, piecewise C^1, regular, simple closed curve enclosing 0, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{n}{z}\,dz = n.$$

(2) Similarly, consider

$$f(z) = \frac{1}{z^n}.$$

It has 0 as its only pole in $\mathbb{C}$, and the order of this pole is n. Again, the contour integral of $\frac{f'(z)}{f(z)}$ over a loop enclosing 0 recovers this order, but now with a minus sign:

$$f'(z) = -nz^{-n-1}, \qquad \frac{f'(z)}{f(z)} = -\frac{n}{z}, \qquad z \neq 0.$$

Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \left(-\frac{n}{z}\right)\,dz = -n.$$

The negative sign appears because 0 is a pole rather than a zero.

The phenomenon illustrated by these examples is not a coincidence. In general, the integral of $\frac{f'}{f}$ counts zeros positively and poles negatively.

THEOREM 1.2 (The argument principle). *Let $\Omega \subset \mathbb{C}$ be open, and let γ be a positively oriented, piecewise $\mathbf{C}^1$, regular, simple closed curve in Ω. Suppose that f is meromorphic on a neighborhood of γ and its interior, and that f has neither zeros nor poles on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = n_0 - n_p,$$

where n_0 is the number of zeros of f inside γ, counted with multiplicity, and n_p is the number of poles of f inside γ, counted with multiplicity.

PROOF. Since f is meromorphic on a neighborhood of γ and its interior, and since the interior of γ together with γ is compact, f has only finitely many zeros and poles inside γ. Let

$$z_1, \dots, z_r$$

be all the zeros and poles of f inside γ.

Choose $\varepsilon > 0$ sufficiently small so that the closed disks

$$\overline{D_\varepsilon(z_j)}$$

are pairwise disjoint, lie entirely inside the interior of γ, and each contains exactly one zero or pole of f.

We first compute the local contribution near a zero. Suppose z_0 is a zero of order n. Then on a neighborhood of z_0,

$$f(z) = (z - z_0)^n g(z),$$

where g is holomorphic and nowhere vanishing near z_0. Hence

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Since g'/g is holomorphic near z_0, its integral over the circle $\partial D_\varepsilon(z_0)$ vanishes. Therefore

$$\frac{1}{2\pi i}\int_{\partial D_\varepsilon(z_0)} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\partial D_\varepsilon(z_0)} \frac{n}{z - z_0}\,dz = n.$$

Next, suppose z_0 is a pole of order m. Then on a neighborhood of z_0,

$$f(z) = \frac{h(z)}{(z - z_0)^m},$$

where h is holomorphic and nowhere vanishing near z_0. Hence

$$\frac{f'(z)}{f(z)} = -\frac{m}{z - z_0} + \frac{h'(z)}{h(z)}.$$

Since h'/h is holomorphic near z_0, we obtain

$$\frac{1}{2\pi i}\int_{\partial D_\varepsilon(z_0)} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{\partial D_\varepsilon(z_0)} \left(-\frac{m}{z - z_0}\right) dz = -m.$$

Now remove these small disks from the interior of γ. On the region

$$\mathrm{Int}(\gamma) \setminus \bigcup_{j=1}^{r} \overline{D_\varepsilon(z_j)},$$

the function $\frac{f'}{f}$ is holomorphic. Hence, by Cauchy's theorem,

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{z_0 \text{ a zero}} \int_{\partial D_\varepsilon(z_0)} \frac{f'(z)}{f(z)}\,dz + \sum_{z_0 \text{ a pole}} \int_{\partial D_\varepsilon(z_0)} \frac{f'(z)}{f(z)}\,dz.$$

Dividing by $2\pi i$, and using the local computations above, we obtain

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{z_0 \text{ a zero}} \operatorname{ord}_{z_0}(f) - \sum_{z_0 \text{ a pole}} \operatorname{ord}^{\text{pole}}_{z_0}(f).$$

This is exactly

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = n_0 - n_p.$$

□

The same idea gives a weighted version and we leave the proof to the reader.

THEOREM 1.3. *Let $\Omega \subset \mathbb{C}$ be open, and let γ be a positively oriented, piecewise C^1, regular, simple closed curve in Ω. Suppose that f is meromorphic on a neighborhood of γ and its interior, and that f has neither zeros nor poles on γ. Let g be holomorphic on a neighborhood of γ and its interior. Then*

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}\,dz = \sum_a m_a\, g(a) - \sum_b n_b\, g(b),$$

where the first sum is over the zeros a of f inside γ, counted with multiplicity m_a, and the second sum is over the poles b of f inside γ, counted with multiplicity n_b.

2. Rouché's theorem

Rouché's theorem is an important application of the argument principle.

THEOREM 2.1 (Rouché's theorem). *Let $\Omega \subset \mathbb{C}$ be open, and suppose that $\overline{D} \subset \Omega$ for some disk D. Let f and g be holomorphic on Ω. If*

$$|f(z)| > |g(z)|, \qquad z \in \partial D,$$

then f and $f+g$ have the same number of zeros in D, counted with multiplicity.

PROOF. For $t \in [0,1]$, define

$$f_t(z) := f(z) + t g(z).$$

Then $f_0 = f$ and $f_1 = f + g$.

For $z \in \partial D$, we have

$$|f_t(z)| \ge |f(z)| - t|g(z)| > |f(z)| - |g(z)| > 0.$$

Thus f_t has no zeros on ∂D for any $t \in [0,1]$.

By the argument principle,

$$N(t) := \frac{1}{2\pi i}\int_{\partial D} \frac{f_t'(z)}{f_t(z)}\,dz$$

is the number of zeros of f_t in D, counted with multiplicity. Hence $N(t)$ is integer-valued.

On the other hand, the integrand depends continuously on t, and $f_t(z) \neq 0$ on ∂D, so $N(t)$ is continuous in t. Since $[0,1]$ is connected and $N(t)$ is integer-valued, $N(t)$ is constant. Therefore

$$N(0) = N(1),$$

which says that f and $f+g$ have the same number of zeros in D. □

As an application, we now use Rouché's theorem to give another proof of the fundamental theorem of algebra.

THEOREM 2.2 (Fundamental theorem of algebra). *Let*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial with $a_n \neq 0$ and $n \geq 1$. Then p has exactly n zeros in $\mathbb{C}$, counted with multiplicity.

PROOF. Dividing by a_n, we may assume $a_n = 1$. Write

$$p(z) = z^n + g(z),$$

where

$$g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0.$$

Since $\deg g < n$, for sufficiently large R, we have

$$|g(z)| < |z^n| = R^n, \qquad |z| = R.$$

By Rouché's theorem, $p(z) = z^n + g(z)$ and z^n have the same number of zeros in $D(0; R)$, counted with multiplicity. Since z^n has exactly n zeros in $D(0; R)$, counted with multiplicity, so does p.

Finally, R was chosen large enough so that all zeros counted by Rouché's theorem lie in $D(0; R)$. Hence p has exactly n zeros in $\mathbb{C}$, counted with multiplicity. □

3. The open mapping theorem

We will use Rouché's theorem to prove an important property of nonconstant holomorphic functions: they send open sets to open sets.

THEOREM 3.1 (The open mapping theorem). *Let $\Omega \subset \mathbb{C}$ be open, and let f be a nonconstant holomorphic function on Ω. Then f is an open map. That is, if $U \subset \Omega$ is open, then $f(U)$ is open in $\mathbb{C}$.*

PROOF. Let $U \subset \Omega$ be open. We show that every point of $f(U)$ is an interior point of $f(U)$.

Take $w_0 \in f(U)$. Then there exists $z_0 \in U$ such that

$$f(z_0) = w_0.$$

Since U is open, we may choose $\delta > 0$ such that

$$\overline{D(z_0; \delta)} \subset U.$$

Since f is nonconstant, the zeros of $f(z) - w_0$ are isolated. By choosing δ smaller if necessary, we may assume that z_0 is the only zero of $f(z) - w_0$ in $\overline{D(z_0; \delta)}$.

Define

$$\epsilon := \frac{1}{2} \min_{z \in \partial D(z_0; \delta)} |f(z) - w_0|.$$

This number is positive, since $f(z) - w_0$ has no zeros on $\partial D(z_0; \delta)$.

Now let $w \in D(w_0; \epsilon)$. Then on $\partial D(z_0; \delta)$, we have

$$|w - w_0| < \epsilon$$

and

$$|f(z) - w_0| \geq 2\epsilon.$$

Hence

$$|f(z) - w_0| > |w - w_0|.$$

By Rouché's theorem, the functions

$$f(z) - w_0 \qquad \text{and} \qquad f(z) - w = (f(z) - w_0) + (w_0 - w)$$

have the same number of zeros in $D(z_0; \delta)$, counted with multiplicity.

Since $f(z) - w_0$ has at least one zero in $D(z_0; \delta)$, namely z_0, it follows that $f(z) - w$ has a zero in $D(z_0; \delta)$. Thus there exists $z \in D(z_0; \delta) \subset U$ such that

$$f(z) = w.$$

Therefore

$$D(w_0; \epsilon) \subset f(U).$$

So every point of $f(U)$ is an interior point, and hence $f(U)$ is open. □

4. Maximum principle

An immediate consequence of the open mapping theorem is the following maximum principle.

THEOREM 4.1 (Maximum modulus principle). *Let $\Omega \subset \mathbb{C}$ be open, and let f be holomorphic on Ω. If $|f|$ attains a local maximum at some point $z_0 \in \Omega$, then f is constant on the connected component of Ω containing z_0.*

In particular, if Ω is connected and f is nonconstant, then $|f|$ has no local maximum in Ω.

PROOF. Suppose that $|f|$ attains a local maximum at $z_0 \in \Omega$. Let Ω_0 be the connected component of Ω containing z_0. We will show that f is constant on Ω_0.

Assume, for contradiction, that f is nonconstant on Ω_0. Since z_0 is a local maximum point of $|f|$, there exists $\delta > 0$ such that

$$\overline{D(z_0; \delta)} \subset \Omega_0$$

and

$$|f(z)| \leq |f(z_0)| \qquad \text{for all } z \in D(z_0; \delta).$$

Hence

$$f(D(z_0; \delta)) \subset \{w \in \mathbb{C} : |w| \leq |f(z_0)|\}.$$

But f is nonconstant on $D(z_0; \delta)$, so by the open mapping theorem the image $f(D(z_0; \delta))$ is open in $\mathbb{C}$. Since it contains $f(z_0)$, it must contain points w with

$$|w| > |f(z_0)|.$$

This contradicts the inclusion above. Therefore f is constant on Ω_0.

The final statement follows immediately: if Ω is connected and f is nonconstant, then $|f|$ cannot attain a local maximum at any point of Ω. □

THEOREM 4.2 (Boundary maximum principle). *Let $\Omega \subset \mathbb{C}$ be a bounded domain, and suppose f is holomorphic on Ω and continuous on $\overline{\Omega}$. Then*

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\Omega} |f(z)|.$$

PROOF. Since $\overline{\Omega}$ is compact and $|f|$ is continuous on $\overline{\Omega}$, the function $|f|$ attains a maximum on $\overline{\Omega}$.

If the maximum is attained on $\partial\Omega$, we are done. If it is attained at a point in Ω, then by the maximum modulus principle, f is constant on Ω. In that case, the same maximum is also attained on $\partial\Omega$. □

We can apply it to harmonic functions.

PROPOSITION 4.3 (Maximum principle for harmonic functions). *Let $\Omega \subset \mathbb{C}$ be a domain, and let u be harmonic on Ω. If u attains a maximum at some point $z_0 \in \Omega$, then u is constant on Ω.*

PROOF. Since harmonicity is a local notion, we may work in a small disk $D \subset \Omega$ centered at z_0. Because D is simply connected and u is harmonic, there exists a holomorphic function F on D such that

$$u = \Re F.$$

Consider

$$h(z) := e^{F(z)}.$$

Then h is holomorphic on D, and

$$|h(z)| = e^{\Re F(z)} = e^{u(z)}.$$

Since u has a maximum at z_0, the function $|h|$ also has a maximum at z_0. By the maximum modulus principle, h is constant on D. Hence F is constant on D, so $u = \Re F$ is constant on D.

Thus the set of points of Ω near which u is constant is nonempty. It is clearly open, and it is also closed in Ω. Since Ω is connected, it follows that u is constant on all of Ω. □

COROLLARY 4.4 (Boundary maximum principle). *Let $\Omega \subset \mathbb{C}$ be a bounded domain, and let $u \in C(\overline{\Omega})$ be harmonic on Ω. Then*

$$\max_{z\in\overline{\Omega}} u(z) = \max_{z\in\partial\Omega} u(z).$$

Similarly,

$$\min_{z\in\overline{\Omega}} u(z) = \min_{z\in\partial\Omega} u(z).$$

PROOF. Since $\overline{\Omega}$ is compact and u is continuous on $\overline{\Omega}$, the function u attains a maximum at some point of $\overline{\Omega}$. If this maximum were attained at an interior point of Ω, then by the maximum principle u would be constant on Ω, and hence the same maximum would also be attained on $\partial\Omega$. Therefore

$$\max_{z\in\overline{\Omega}} u(z) = \max_{z\in\partial\Omega} u(z).$$

The statement for the minimum follows by applying the same argument to $-u$. □

CHAPTER 9

Conformal Mappings

In this chapter we study biholomorphic maps between planar domains. Such maps are called conformal maps. They form the basic objects in geometric complex analysis.

1. Definition and first properties

DEFINITION 1.1. Let $U, V \subset \mathbb{C}$ be open sets. A map

$$f : U \to V$$

is called a **conformal mapping** if f is biholomorphic, that is,

(1) f is holomorphic,
(2) f is bijective,
(3) $f^{-1} : V \to U$ is holomorphic.

When there is a conformal mapping between U and V, we say U, V are **conformally equivalent**.

In fact, the proposition below shows that, for maps between domains in $\mathbb{C}$, injective holomorphic maps are automatically conformal onto their images.

PROPOSITION 1.2. *Let $U \subset \mathbb{C}$ be open, and let*

$$f : U \to \mathbb{C}$$

be a holomorphic injective map. Set

$$V := f(U).$$

Then V is open, $f'(z) \neq 0$ for every $z \in U$, and the inverse map

$$f^{-1} : V \to U$$

is holomorphic. In particular, $f : U \to V$ is biholomorphic.

PROOF. Since f is nonconstant and holomorphic, the open mapping theorem implies that $V = f(U)$ is open.

We first show that

$$f'(z) \neq 0 \qquad \text{for all } z \in U.$$

Suppose, toward a contradiction, that $f'(z_0) = 0$ for some $z_0 \in U$. Write

$$w_0 := f(z_0).$$

Since f is holomorphic, there exists an integer $k \geq 2$ and a holomorphic function g defined near z_0, with $g(z_0) \neq 0$, such that

$$f(z) = w_0 + (z - z_0)^k g(z)$$

near z_0.

Choose $\varepsilon > 0$ so small that

$$\overline{D(z_0;\varepsilon)} \subset U$$

and

$$g(z) \neq 0 \qquad \text{for all } z \in \overline{D(z_0;\varepsilon)}.$$

Then z_0 is the only zero of $f(z) - w_0$ in $\overline{D(z_0;\varepsilon)}$, and that zero has multiplicity $k \geq 2$.

Define

$$\delta := \frac{1}{2} \min_{z \in \partial D(z_0;\varepsilon)} |f(z) - w_0|.$$

Since $f(z) - w_0 \neq 0$ on $\partial D(z_0;\varepsilon)$, we have $\delta > 0$.

Now let $w \in D(w_0;\delta)$. On $\partial D(z_0;\varepsilon)$,

$$|w - w_0| < \delta < |f(z) - w_0|.$$

Hence by Rouché's theorem, the functions

$$f(z) - w_0 \qquad \text{and} \qquad f(z) - w = (f(z) - w_0) + (w_0 - w)$$

have the same number of zeros in $D(z_0;\varepsilon)$, counted with multiplicity.

Since $f(z) - w_0$ has a zero of multiplicity $k \geq 2$ at z_0, it follows that $f(z) - w$ has exactly k zeros in $D(z_0;\varepsilon)$, counted with multiplicity. For $w \neq w_0$, these zeros are distinct, because z_0 is the only point where $f'(z) = 0$ in a sufficiently small neighborhood of z_0. This contradicts the injectivity of f. Therefore $f'(z) \neq 0$ for all $z \in U$.

It remains to show that f^{-1} is holomorphic. First notice that the open mapping property of f guarantees that f^{-1} is continuous on V.

Fix $w_0 \in V$, and let

$$z_0 := f^{-1}(w_0).$$

Then for each $w \neq w_0$, $z := f^{-1}(w) \neq z_0 = f^{-1}(w_0)$, and

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}},$$

As $w \to w_0$ ($w \neq w_0$), by the continuity of f^{-1}, we have $z \to z_0$ and $z \neq z_0$. Since

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \neq 0,$$

by the continuity of the function $\frac{1}{z}$ at $z \neq 0$, there is

$$\lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} \to \frac{1}{f'(z_0)}.$$

Hence f^{-1} is complex differentiable at w_0, with derivative $\frac{1}{f'(z_0)}$.

As $w_0 \in V$ was arbitrary, f^{-1} is holomorphic on V. □

In fact, the term *conformal mapping* is used in a broader sense in other contexts, where the main emphasis is on the **angle-preserving property**. Now, let's prove that a biholomorphic map is indeed angle-preserving.

PROPOSITION 1.3 (Angle-preserving property). *Let $U, V \subset \mathbb{C}$ be open, and let*

$$f : U \to V$$

be biholomorphic. Let $z_0 \in U$, and let γ_1, γ_2 be C^1-curves in U such that

$$\gamma_1(0) = \gamma_2(0) = z_0, \qquad \gamma_1'(0) \neq 0, \qquad \gamma_2'(0) \neq 0.$$

Then the oriented angle between γ_1 and γ_2 at z_0 is equal to the oriented angle between $f \circ \gamma_1$ and $f \circ \gamma_2$ at $f(z_0)$.

PROOF. Set

$$w_0 := f(z_0).$$

By the chain rule,

$$(f \circ \gamma_j)'(0) = f'(z_0)\gamma_j'(0), \qquad j = 1, 2.$$

Hence

$$\frac{(f \circ \gamma_2)'(0)}{(f \circ \gamma_1)'(0)} = \frac{f'(z_0)\gamma_2'(0)}{f'(z_0)\gamma_1'(0)} = \frac{\gamma_2'(0)}{\gamma_1'(0)}.$$

Taking arguments, we obtain

$$\arg(f \circ \gamma_2)'(0) - \arg(f \circ \gamma_1)'(0) = \arg \gamma_2'(0) - \arg \gamma_1'(0).$$

Thus the oriented angle is preserved. □

EXAMPLE 1.4. The map

$$f(z) = z^2$$

is holomorphic on $\mathbb{C}$, but it is not conformal at $z = 0$, because

$$f'(0) = 0.$$

Indeed, near 0, the map $z \mapsto z^2$ is not locally injective. Geometrically, it doubles arguments and so does not preserve angles at 0.

On the other hand, if we restrict f to a sector

$$S_{\alpha,\beta} := \{re^{i\theta} : r > 0,\ \alpha < \theta < \beta\}, \qquad \beta - \alpha < \pi,$$

then f is injective on $S_{\alpha,\beta}$, and maps it biholomorphically onto the sector

$$S_{2\alpha,2\beta} = \{re^{i\theta} : r > 0,\ 2\alpha < \theta < 2\beta\}.$$

Hence f is conformal on $S_{\alpha,\beta}$.

EXAMPLE 1.5 (A conformal map from the upper half-plane to the unit disk). Let

$$\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$$

be the upper half-plane, and let

$$\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$$

be the unit disk. Consider the linear fractional map (also called a Möbius transformation)

$$\phi : \mathbb{H} \to \mathbb{C}, \qquad \phi(z) = \frac{z - i}{z + i}.$$

It maps $\mathbb{H}$ bijectively onto $\mathbb{D}$, with i mapped to 0.

Indeed, solving

$$w = \frac{z - i}{z + i}$$

for z, we obtain

$$z = i\,\frac{1+w}{1-w},$$

which is the inverse map. Since the inverse is again a Möbius transformation, ϕ is a conformal mapping from $\mathbb{H}$ onto $\mathbb{D}$.

Furthermore, ϕ maps $\mathbb{R} \cup \{\infty\}$ onto the unit circle $\partial\mathbb{D}$. In particular,

$$\phi(-1) = i, \qquad \phi(1) = -i, \qquad \phi(0) = -1, \qquad \phi(\infty) = 1.$$

The positive real axis is mapped onto the lower semicircle, while the negative real axis is mapped onto the upper semicircle. More precisely, as x runs from 0 to $+\infty$, the point $\phi(x)$ moves from -1 to 1 along the lower semicircle; and as x runs from 0 to $-\infty$, the point $\phi(x)$ moves from -1 to 1 along the upper semicircle.

EXAMPLE 1.6. The previous example can be combined with simple holomorphic maps to produce many further conformal mappings.

For instance, consider the sector

$$S := \{z \in \mathbb{C} : 0 < \arg z < \tfrac{\pi}{4}\}.$$

The map

$$z \longmapsto z^4$$

maps S conformally onto the upper half-plane $\mathbb{H}$, since it multiplies arguments by 4. Composing with the conformal map

$$\phi(w) = \frac{w-i}{w+i}$$

from $\mathbb{H}$ onto $\mathbb{D}$, we obtain a conformal mapping

$$S \to \mathbb{D}, \qquad z \longmapsto \frac{z^4 - i}{z^4 + i}.$$

Thus, once a few basic conformal maps are known, many others can be constructed by composition. We will see many further examples later.

2. Automorphisms of the upper half-plane

Given an open set $\Omega \subset \mathbb{C}$, we use

$$\mathrm{Aut}(\Omega)$$

to denote the set of all biholomorphic maps from Ω to itself. This is indeed a group under composition: if

$$f, g \in \mathrm{Aut}(\Omega),$$

then $f \circ g$ is again biholomorphic from Ω to itself; the identity map

$$\mathrm{id}_\Omega : \Omega \to \Omega$$

is biholomorphic; and if $f \in \mathrm{Aut}(\Omega)$, then by definition its inverse

$$f^{-1} : \Omega \to \Omega$$

is also biholomorphic. Thus $\mathrm{Aut}(\Omega)$ is a group.

In this section we determine the automorphism group of the upper half-plane

$$\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}.$$

Since we have seen that $\mathbb{H}$ and $\mathbb{D}$ are conformally equivalent via

$$f : \mathbb{H} \to \mathbb{D}, \qquad f(z) = \frac{z-i}{z+i},$$

every automorphism of $\mathbb{H}$ gives rise to an automorphism of $\mathbb{D}$. Indeed, if

$$\phi \in \mathrm{Aut}(\mathbb{H}),$$

then

$$f \circ \phi \circ f^{-1} \in \mathrm{Aut}(\mathbb{D}).$$

This defines a group isomorphism

$$\mathrm{Aut}(\mathbb{H}) \longrightarrow \mathrm{Aut}(\mathbb{D}), \qquad \phi \longmapsto f \circ \phi \circ f^{-1},$$

whose inverse is given by

$$\psi \longmapsto f^{-1} \circ \psi \circ f.$$

Therefore, $\mathrm{Aut}(\mathbb{H})$ and $\mathrm{Aut}(\mathbb{D})$ are naturally identified. Since it is slightly easier to understand $\mathrm{Aut}(\mathbb{D})$, we will first determine $\mathrm{Aut}(\mathbb{D})$, and then transfer the result to $\mathbb{H}$.

We first look at some basic examples of holomorphic maps from $\mathbb{D}$ to itself.

EXAMPLE 2.1. (1) For each $\theta \in \mathbb{R}$, the rotation

$$z \longmapsto e^{i\theta} z$$

is an automorphism of $\mathbb{D}$.

(2) The map

$$z \longmapsto z^2$$

is holomorphic on $\mathbb{D}$, but it is not injective, hence it does not belong to $\mathrm{Aut}(\mathbb{D})$.

(3) The map

$$z \longmapsto \overline{z}$$

maps $\mathbb{D}$ bijectively to itself, but it is not holomorphic, hence it does not belong to $\mathrm{Aut}(\mathbb{D})$.

(4) For $\alpha \in \mathbb{D}$, the map

$$\psi_\alpha(z) := \frac{z-\alpha}{\overline{\alpha} z - 1}$$

is called a **Blaschke factor**. The following lemma shows it is an automorphism of $\mathbb{D}$.

We now record some basic properties of the Blaschke factor.

LEMMA 2.2. *Let $\alpha \in \mathbb{D}$, and define*

$$\psi_\alpha(z) := \frac{z-\alpha}{\overline{\alpha} z - 1}.$$

Then:

(1) *ψ_α is a Möbius transformation of $\widehat{\mathbb{C}}$;*
(2) *$\psi_\alpha(0) = \alpha$ and $\psi_\alpha(\alpha) = 0$;*
(3) *$\psi_\alpha \circ \psi_\alpha = \mathrm{id}$;*
(4) *if $|z| = 1$, then $|\psi_\alpha(z)| = 1$;*
(5) *$\psi_\alpha \in \mathrm{Aut}(\mathbb{D})$.*

PROOF. (1) We may write

$$\psi_\alpha(z) = \frac{az+b}{cz+d}$$

with

$$a = 1, \qquad b = -\alpha, \qquad c = \overline{\alpha}, \qquad d = -1.$$

Its determinant is

$$ad - bc = -1 + |\alpha|^2 \neq 0$$

since $|\alpha| < 1$. Hence ψ_α is a Möbius transformation.

(2) Direct computation gives

$$\psi_\alpha(0) = \frac{-\alpha}{-1} = \alpha, \qquad \psi_\alpha(\alpha) = 0.$$

(3) By direct calculation,

$$\psi_\alpha(\psi_\alpha(z)) = \frac{\dfrac{z-\alpha}{\overline{\alpha}z-1} - \alpha}{\overline{\alpha}\dfrac{z-\alpha}{\overline{\alpha}z-1} - 1}.$$

Simplifying the numerator,

$$\frac{z-\alpha-\alpha(\overline{\alpha}z-1)}{\overline{\alpha}z-1} = \frac{(1-|\alpha|^2)z}{\overline{\alpha}z-1}.$$

Simplifying the denominator,

$$\frac{\overline{\alpha}(z-\alpha)-(\overline{\alpha}z-1)}{\overline{\alpha}z-1} = \frac{1-|\alpha|^2}{\overline{\alpha}z-1}.$$

Therefore

$$\psi_\alpha(\psi_\alpha(z)) = \frac{\dfrac{(1-|\alpha|^2)z}{\overline{\alpha}z-1}}{\dfrac{1-|\alpha|^2}{\overline{\alpha}z-1}} = z.$$

Hence

$$\psi_\alpha \circ \psi_\alpha = \mathrm{id}.$$

(4) Suppose $|z| = 1$. Then $|\overline{z}| = 1$, and

$$|\psi_\alpha(z)| = \left|\frac{z-\alpha}{\overline{z}(\overline{\alpha}z-1)}\right| = \left|\frac{z-\alpha}{\overline{\alpha}-\overline{z}}\right| = \left|\frac{z-\alpha}{-\overline{(z-\alpha)}}\right| = 1.$$

(5) Since ψ_α is holomorphic on $\mathbb{D}$, it is enough to show that it maps $\mathbb{D}$ bijectively to itself. By part (4), ψ_α maps the unit circle to itself. Hence, by the maximum modulus principle,

$$\psi_\alpha(\mathbb{D}) \subset \mathbb{D}.$$

Since $\psi_\alpha \circ \psi_\alpha = \mathrm{id}$, the map ψ_α is bijective and its inverse is itself. Thus

$$\psi_\alpha \in \mathrm{Aut}(\mathbb{D}).$$

□

We will prove the following result.

THEOREM 2.3. *Any $\phi \in \mathrm{Aut}(\mathbb{D})$ can be written in the form*

$$\phi(z) = e^{i\theta}\psi_\alpha(z)$$

for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$.

The proof will use the Schwarz lemma.

PROPOSITION 2.4 (Schwarz lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, and suppose that*

$$f(0) = 0.$$

Then

$$|f(z)| \leq |z| \qquad \textit{for all } z \in \mathbb{D},$$

and

$$|f'(0)| \leq 1.$$

Moreover, if equality holds at some nonzero point, that is, if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ such that

$$|f(z_0)| = |z_0|,$$

or if

$$|f'(0)| = 1,$$

then there exists $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} z \qquad \textit{for all } z \in \mathbb{D}.$$

PROOF. Define

$$g(z) := \begin{cases} \dfrac{f(z)}{z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

Since $f(0) = 0$, the function g is holomorphic on $\mathbb{D}$.

Fix $0 < r < 1$. Since g is holomorphic on $D(0; r)$ and continuous on $\overline{D(0; r)}$, the maximum modulus principle gives

$$\max_{|z| \leq r} |g(z)| = \max_{|z|=r} |g(z)|.$$

Now if $|z| = r$, then

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r},$$

because $f(\mathbb{D}) \subset \mathbb{D}$. Hence

$$|g(z)| \leq \frac{1}{r} \qquad \text{for all } |z| \leq r.$$

Letting $r \to 1-$, we obtain

$$|g(z)| \leq 1 \qquad \text{for all } z \in \mathbb{D}.$$

Therefore

$$|f(z)| = |z|\,|g(z)| \leq |z| \qquad \text{for all } z \in \mathbb{D},$$

and in particular

$$|f'(0)| = |g(0)| \leq 1.$$

If there exists $z_0 \in \mathbb{D} \setminus \{0\}$ such that

$$|f(z_0)| = |z_0|,$$

then

$$|g(z_0)| = 1.$$

Since $|g| \leq 1$ on $\mathbb{D}$, the maximum modulus principle implies that g is constant. Thus

$$g(z) \equiv e^{i\theta}$$

for some $\theta \in \mathbb{R}$, and hence

$$f(z) = e^{i\theta} z.$$

Similarly, if

$$|f'(0)| = 1,$$

then

$$|g(0)| = 1,$$

so again the maximum modulus principle implies that g is constant of modulus 1. Hence

$$f(z) = e^{i\theta} z$$

for some $\theta \in \mathbb{R}$. □

PROOF OF THE THEOREM. Let $\phi \in \mathrm{Aut}(\mathbb{D})$, and set

$$\alpha := \phi^{-1}(0) \in \mathbb{D}.$$

Then

$$\psi_\alpha(\alpha) = 0.$$

Consider the map

$$f := \phi \circ \psi_\alpha.$$

Since both ϕ and ψ_α belong to $\mathrm{Aut}(\mathbb{D})$, the map f is an automorphism of $\mathbb{D}$. Moreover,

$$f(0) = \phi(\psi_\alpha(0)) = \phi(\alpha) = 0.$$

By the Schwarz lemma, there exists $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} z.$$

Therefore

$$\phi(z) = e^{i\theta} \psi_\alpha(z),$$

as required. □

COROLLARY 2.5. *Let $\phi \in \mathrm{Aut}(\mathbb{D})$ and suppose that*

$$\phi(0) = 0.$$

Then

$$\phi(z) = e^{i\theta} z$$

for some $\theta \in \mathbb{R}$.

PROOF. By the theorem, we may write

$$\phi(z) = e^{i\theta} \psi_\alpha(z)$$

for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$. Evaluating at $z = 0$, we obtain

$$0 = \phi(0) = e^{i\theta} \psi_\alpha(0) = e^{i\theta} \alpha.$$

Hence $\alpha = 0$. Therefore

$$\phi(z) = e^{i\theta} \psi_0(z).$$

Since

$$\psi_0(z) = \frac{z}{-1} = -z,$$

it follows that
$$\phi(z) = -e^{i\theta} z = e^{i(\theta+\pi)} z.$$
Renaming the angle, we conclude that
$$\phi(z) = e^{i\vartheta} z$$
for some $\vartheta \in \mathbb{R}$. $\square$

COROLLARY 2.6. *The action of* $\mathrm{Aut}(\mathbb{D})$ *on* $\mathbb{D}$ *is transitive. That is, for any* $a, b \in \mathbb{D}$, *there exists* $\phi \in \mathrm{Aut}(\mathbb{D})$ *such that*
$$\phi(a) = b.$$

PROOF. It is enough to show that for every $a \in \mathbb{D}$, there exists an automorphism sending a to 0. But this is exactly the Blaschke factor ψ_a, since
$$\psi_a(a) = 0.$$
Now given $a, b \in \mathbb{D}$, the maps $\psi_a, \psi_b \in \mathrm{Aut}(\mathbb{D})$ satisfy
$$\psi_a(a) = 0, \qquad \psi_b(0) = b.$$
Hence
$$\phi := \psi_b \circ \psi_a \in \mathrm{Aut}(\mathbb{D})$$
satisfies
$$\phi(a) = \psi_b(\psi_a(a)) = \psi_b(0) = b.$$
Thus the action is transitive. $\square$

We can also understand $\mathrm{Aut}(\mathbb{D})$ from a more geometric point of view.

We already know that every $\phi \in \mathrm{Aut}(\mathbb{D})$ can be written in the form
$$\phi(z) = e^{i\theta}\psi_\alpha(z), \qquad \psi_\alpha(z) = \frac{z-\alpha}{\overline{\alpha}z - 1},$$
for some $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$.

Thus ϕ is represented by the matrix
$$\begin{pmatrix} e^{i\theta} & -e^{i\theta}\alpha \\ \overline{\alpha} & -1 \end{pmatrix}.$$
Its determinant is
$$-e^{i\theta} + e^{i\theta}|\alpha|^2 = -e^{i\theta}(1 - |\alpha|^2).$$
If we set
$$\vartheta := \theta + \pi,$$
then
$$-e^{i\theta} = e^{i\vartheta},$$
so the determinant becomes
$$e^{i\vartheta}(1 - |\alpha|^2).$$

Since multiplying a matrix by a nonzero scalar does not change the induced Möbius transformation, we may rescale the above matrix by
$$\frac{e^{-i\vartheta/2}}{\sqrt{1 - |\alpha|^2}}.$$

This gives

$$A_{\alpha,\vartheta} = \frac{1}{\sqrt{1-|\alpha|^2}} \begin{pmatrix} e^{i\vartheta/2} & -e^{i\vartheta/2}\alpha \\ -e^{-i\vartheta/2}\overline{\alpha} & e^{-i\vartheta/2} \end{pmatrix},$$

and now

$$\det A_{\alpha,\vartheta} = 1.$$

Notice that $A_{\alpha,\vartheta}$ has the form

$$A_{\alpha,\vartheta} = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \qquad |a|^2 - |b|^2 = 1.$$

Now let

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This defines a Hermitian form on $\mathbb{C}^2$,

$$h(v,w) = v^* J w = v_1\overline{w_1} - v_2\overline{w_2}.$$

We define

$$SU(1,1) := \{A \in GL_2(\mathbb{C}) : A^* J A = J,\ \det A = 1\}.$$

A direct calculation shows that

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}.$$

Thus $SU(1,1)$ is the group of determinant-one matrices preserving a Hermitian form of signature $(1,-1)$.

The discussion above shows that every automorphism of $\mathbb{D}$ gives rise to an element of $SU(1,1)$. Conversely, we leave it to the reader to check that every matrix in $SU(1,1)$ induces a Möbius transformation preserving $\mathbb{D}$. Since two matrices differing by a sign induce the same projective transformation, this correspondence is two-to-one. Therefore

$$\mathrm{Aut}(\mathbb{D}) \cong PSU(1,1) := SU(1,1)/\{\pm I\}.$$

REMARK 2.7. This Hermitian form is closely related to the geometry of the disk. If we view $\widehat{\mathbb{C}}$ as the projective line $\mathbb{P}^1(\mathbb{C})$, then $z \in \mathbb{C}$ corresponds to the point $[z : 1]$, and

$$h((z,1),(z,1)) = |z|^2 - 1.$$

Hence

$$|z| < 1 \iff h((z,1),(z,1)) < 0, \qquad |z| = 1 \iff h((z,1),(z,1)) = 0.$$

Thus the unit circle is the projectivization of the null cone of h, and the unit disk is the projectivization of its negative cone. From this point of view, the appearance of $SU(1,1)$ is natural.

We now return to the automorphism group of the upper half-plane

$$\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}.$$

THEOREM 2.8. *A map* $\phi : \mathbb{H} \to \mathbb{H}$ *belongs to* $\mathrm{Aut}(\mathbb{H})$ *if and only if it has the form*

$$\phi(z) = \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{R},\ ad - bc > 0.$$

Equivalently,

$$\mathrm{Aut}(\mathbb{H}) \cong PSL_2(\mathbb{R}).$$

PROOF. Let

$$f(z) = \frac{z-i}{z+i}$$

be the conformal mapping from $\mathbb{H}$ onto $\mathbb{D}$. The matrix corresponding to f is

$$C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

From the previous discussion, we know that

$$\mathrm{Aut}(\mathbb{D}) \cong PSU(1,1).$$

Thus it suffices to conjugate $SU(1,1)$ by C.

Let

$$A = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(1,1), \qquad |\alpha|^2 - |\beta|^2 = 1.$$

A direct computation shows that

$$C^{-1}AC = \begin{pmatrix} \Re(\alpha+\beta) & \Im(\alpha-\beta) \\ -\Im(\alpha+\beta) & \Re(\alpha-\beta) \end{pmatrix} \in SL_2(\mathbb{R}).$$

Hence

$$C^{-1}SU(1,1)C \subset SL_2(\mathbb{R}).$$

Conversely, every matrix in $SL_2(\mathbb{R})$ arises in this way, and therefore

$$C^{-1}SU(1,1)C = SL_2(\mathbb{R}).$$

Passing to projective groups, we obtain

$$PSU(1,1) \cong PSL_2(\mathbb{R}).$$

Since f gives an isomorphism

$$\mathrm{Aut}(\mathbb{H}) \cong \mathrm{Aut}(\mathbb{D}),$$

it follows that

$$\mathrm{Aut}(\mathbb{H}) \cong PSL_2(\mathbb{R}).$$

This means precisely that

$$\mathrm{Aut}(\mathbb{H}) = \left\{ z \mapsto \frac{az+b}{cz+d} \,:\, a,b,c,d \in \mathbb{R},\ ad-bc>0 \right\}.$$

□

3. An application of conformal mappings: the Dirichlet problems

3.1. The Dirichlet problem on the disk. Complex analytic methods are very useful in analysis, especially in the study of PDEs (partial differential equations). We illustrate this with a basic example: the Dirichlet problem on the unit disk. Using Cauchy's integral formula and Blaschke factors, we will derive the Poisson integral formula, and hence obtain the solution of the Dirichlet problem on $\mathbb{D}$.

We begin by formulating the Dirichlet problem on the unit disk. Given a continuous function

$$g : \partial\mathbb{D} \to \mathbb{R},$$

the Dirichlet problem asks whether one can find a harmonic function

$$u : \mathbb{D} \to \mathbb{R}$$

which extends continuously to $\overline{\mathbb{D}}$ and satisfies

$$u|_{\partial\mathbb{D}} = g.$$

We first note that if such a function exists, then it is unique.

PROPOSITION 3.1 (Uniqueness). *Let $g : \partial\mathbb{D} \to \mathbb{R}$ be continuous. There exists at most one function*

$$u \in C(\overline{\mathbb{D}})$$

which is harmonic on $\mathbb{D}$ and satisfies

$$u|_{\partial\mathbb{D}} = g.$$

PROOF. Suppose $u_1, u_2 \in C(\overline{\mathbb{D}})$ are harmonic on $\mathbb{D}$ and satisfy

$$u_1|_{\partial\mathbb{D}} = u_2|_{\partial\mathbb{D}} = g.$$

Set

$$v := u_1 - u_2.$$

Then v is harmonic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and

$$v|_{\partial\mathbb{D}} = 0.$$

By Corollary 4.4,

$$\max_{z\in\overline{\mathbb{D}}} v(z) = \max_{z\in\partial\mathbb{D}} v(z) = 0.$$

Similarly,

$$\min_{z\in\overline{\mathbb{D}}} v(z) = \min_{z\in\partial\mathbb{D}} v(z) = 0.$$

Hence $v \equiv 0$ on $\overline{\mathbb{D}}$. Therefore

$$u_1 = u_2.$$

□

The next task is to construct a solution of the Dirichlet problem.

3.2. Poisson integral formula. To motivate the construction, we first ask the following question: if u is harmonic on an open neighborhood of $\overline{\mathbb{D}}$, how is u related to its boundary values on $\partial\mathbb{D}$? The question is answered by the Poisson integral formula.

PROPOSITION 3.2 (Poisson integral formula). *Suppose Ω is an open set containing $\overline{\mathbb{D}}$, and let*

$$u : \Omega \to \mathbb{R}$$

be harmonic. Then for every $z = re^{i\theta} \in \mathbb{D}$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\, u(e^{it})\, dt.$$

In general, if an operator T is given by

$$Tf(x) = \int K(x,y) f(y)\, dy,$$

then $K(x,y)$ is called the kernel of T. Here the above integral formula sends the boundary values of u on $\partial\mathbb{D}$ to the values of u in $\mathbb{D}$, and the function

$$P_r(\theta-t) = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

plays the role of the kernel. Hence it is called the **Poisson kernel**.

In particular, if $u \equiv 1$, then Proposition 3.2 gives

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\,dt = 1.$$

Thus the Poisson kernel integrates to 1, so the Poisson integral should be regarded as a weighted average of the boundary values.

A natural idea for proving this formula is to use Cauchy's integral formula for holomorphic functions and then take real parts. However, for a general point $\alpha \in \mathbb{D}$, it is not easy to organize the integrand so that its real part involves only the boundary values of u. In other words, it is not immediately clear how to recover $u(\alpha)$ from an integral depending only on $u|_{\partial\mathbb{D}}$. We will change a strategy and overcome this difficulty in two steps: First, we prove the formula at the origin 0, where the symmetry of the disk makes the argument especially simple. Then we use a Blaschke factor to move an arbitrary point of the disk to the origin, and in this way obtain the formula for general $\alpha \in \mathbb{D}$.

The rest of this subsection is devoted to a proof of Proposition 3.2.

LEMMA 3.3 (Mean value property at the origin). *Let u be a harmonic function defined on a disk neighborhood Ω of $\overline{\mathbb{D}}$. Then*

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta.$$

PROOF. Since Ω is simply connected, there exists a holomorphic function $F \in \mathcal{O}(\Omega)$ whose real part is u. By Cauchy's integral formula,

$$F(0) = \frac{1}{2\pi i}\int_{|w|=1} \frac{F(w)}{w}\,dw.$$

Writing $w = e^{i\theta}$, so that $dw = ie^{i\theta}\,d\theta$, we obtain

$$F(0) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta})\,d\theta.$$

Taking real parts gives

$$u(0) = \Re F(0) = \frac{1}{2\pi}\int_0^{2\pi} \Re F(e^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta.$$

□

Then we need the following usful lemma.

LEMMA 3.4. *For any two conformally equivalent open sets U and V with a conformal mapping*

$$\phi : U \to V,$$

if u is harmonic on V, then $u \circ \phi$ is harmonic on U.

In particular, for $\phi \in \mathrm{Aut}(\Omega)$ and u is harmonic on Ω, then $u \circ \phi$ is also harmonic on Ω.

PROOF. Let $z_0 \in \Omega$. Since u is harmonic, there exists a neighborhood V of $\phi(z_0)$ and a holomorphic function F on V such that

$$u = \Re F \qquad \text{on } V.$$

Since ϕ is holomorphic, after shrinking neighborhoods if necessary, there exists a neighborhood U of z_0 such that

$$\phi(U) \subset V.$$

Then on U,

$$u \circ \phi = \Re(F \circ \phi).$$

Since $F \circ \phi$ is holomorphic on U, it follows that $u \circ \phi$ is harmonic on U. As z_0 was arbitrary, $u \circ \phi$ is harmonic on Ω. □

Now for each $\alpha \in \mathbb{D}$, we consider the Blaschke factor

$$\psi_\alpha(z) = \frac{z - \alpha}{\bar{\alpha} z - 1}.$$

Since $|\alpha| < 1$, the only pole of ψ_α is at

$$z = \frac{1}{\bar{\alpha}},$$

which lies outside $\overline{\mathbb{D}}$. Hence ψ_α is holomorphic on a disk neighborhood Ω of $\overline{\mathbb{D}}$. Moreover,

$$\psi_\alpha^{-1} = \psi_\alpha,$$

and it maps α to 0 and 0 to α.

Now consider the function

$$v := u \circ \psi_\alpha.$$

Then v is harmonic on a neighborhood of $\overline{\mathbb{D}}$, and

$$v(0) = u(\alpha).$$

Applying Lemma 3.3 to v, we obtain

$$u(\alpha) = v(0) = \frac{1}{2\pi} \int_0^{2\pi} v(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} u(\psi_\alpha(e^{i\theta}))\, d\theta.$$

Now write

$$\psi_\alpha(e^{i\theta}) = e^{it}.$$

As θ goes from 0 to 2π, t also goes from 0 to 2π.

Using

$$\psi_\alpha(e^{i\theta}) = e^{it} \quad \text{and} \quad e^{i\theta} = \psi_\alpha(e^{it}),$$

a direct calculation gives

$$\frac{d\theta}{dt} = \frac{1 - |\alpha|^2}{|e^{it} - \alpha|^2}.$$

Therefore, by the change of variables,

$$u(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |\alpha|^2}{|e^{it} - \alpha|^2} u(e^{it})\, dt.$$

Writing $\alpha = re^{i\theta}$, we note that

$$|e^{it} - \alpha|^2 = 1 - 2r\cos(\theta - t) + r^2.$$

Hence the formula becomes

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} u(e^{it})\, dt,$$

which is exactly the Poisson integral formula in Proposition 3.2. In particular, the Poisson kernel is

$$\frac{1 - |\alpha|^2}{|e^{it} - \alpha|^2} = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}, \qquad \alpha = re^{i\theta}.$$

We are done with the proof of Proposition 3.2.

3.3. Solutions to the Dirichlet problem on the unit disk. Poisson's integral formula suggests that for a general continuous function g on $\partial\mathbb{D}$, one should define

$$U(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\, g(e^{it})\, dt, \qquad z \in \mathbb{D}.$$

This integral is well-defined since g is continuous.

To see that U is harmonic on $\mathbb{D}$, it is convenient to rewrite the kernel as

$$\frac{1-|z|^2}{|e^{it}-z|^2} = \Re\left(\frac{e^{it}+z}{e^{it}-z}\right).$$

Thus, if we define

$$F(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, g(e^{it})\, dt = \frac{1}{2\pi i}\int_{|w|=1} \frac{w+z}{w-z}\,\frac{g(w)}{w}\, dw,$$

then

$$U(z) = \Re F(z).$$

For each fixed $w \in \partial\mathbb{D}$, the function

$$z \mapsto \frac{w+z}{w-z}$$

is holomorphic on $\mathbb{D}$. Since g is continuous on $\partial\mathbb{D}$, it follows that F is holomorphic on $\mathbb{D}$. Therefore U is harmonic on $\mathbb{D}$.

It remains to understand the boundary behavior of U. To see this, fix $e^{i\theta_0} \in \partial\mathbb{D}$ and we estimate $U(re^{i\theta}) - g(e^{i\theta_0})$.

Since

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\, dt = 1,$$

where $P_r(\theta-t) := \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$, we may write

$$U(re^{i\theta}) - g(e^{i\theta_0}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-t)\big(g(e^{it}) - g(e^{i\theta_0})\big)\, dt.$$

Let $\varepsilon > 0$. Since g is continuous on $\partial\mathbb{D}$, there exists $\delta > 0$ such that

$$|g(e^{it}) - g(e^{i\theta_0})| < \varepsilon$$

whenever $|t - \theta_0| < \delta$ modulo 2π.

We split the integral into two parts:

$$U(re^{i\theta}) - g(e^{i\theta_0}) = I_1 + I_2,$$

where

$$I_1 := \frac{1}{2\pi}\int_{|t-\theta_0|<\delta} P_r(\theta-t)\big(g(e^{it}) - g(e^{i\theta_0})\big)\, dt,$$

and

$$I_2 := \frac{1}{2\pi}\int_{|t-\theta_0|\geq\delta} P_r(\theta-t)\big(g(e^{it}) - g(e^{i\theta_0})\big)\, dt.$$

For I_1, we have

$$|I_1| \leq \varepsilon \cdot \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-t)\, dt = \varepsilon.$$

Now assume $|\theta - \theta_0| < \delta/2$. Then on the set $|t - \theta_0| \geq \delta$, we have

$$|t - \theta| \geq \delta/2.$$

Hence

$$1 - 2r\cos(\theta - t) + r^2$$

is bounded below by a positive constant depending only on δ. Since the numerator $1 - r^2 \to 0$ as $r \to 1$, it follows that

$$P_r(\theta - t) \to 0$$

uniformly on the set $|t - \theta_0| \geq \delta$ as $r \to 1$ and $\theta \to \theta_0$. Since g is bounded on $\partial\mathbb{D}$, we conclude that

$$I_2 \to 0.$$

Therefore,

$$U(re^{i\theta}) \to g(e^{i\theta_0}) \qquad \text{as } re^{i\theta} \to e^{i\theta_0}.$$

Since $e^{i\theta_0}$ was arbitrary, U extends continuously to $\overline{\mathbb{D}}$ and satisfies

$$U|_{\partial\mathbb{D}} = g.$$

As a conclusion, we have proved the following theorem.

THEOREM 3.5. *Let*

$$g : \partial\mathbb{D} \to \mathbb{R}$$

be continuous. Then the function

$$U(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2}\, g(e^{it})\, dt$$

is the unique solution of the Dirichlet problem on $\mathbb{D}$ with boundary values g.

3.4. The Dirichlet problem on the upper half-plane. We now use the conformal mapping

$$f : \mathbb{H} \to \mathbb{D}, \qquad f(z) = \frac{z - i}{z + i},$$

to transfer the Dirichlet problem from the unit disk to the upper half-plane

$$\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}.$$

Recall that f is biholomorphic, with inverse

$$f^{-1}(w) = i\,\frac{1 + w}{1 - w}.$$

Suppose

$$g : \mathbb{R} \to \mathbb{R}$$

is continuous and satisfies

$$\lim_{x \to +\infty} g(x) = \lim_{x \to -\infty} g(x) \in \mathbb{R}.$$

Then g extends continuously to $\mathbb{R} \cup \{\infty\}$. Since f maps

$$\mathbb{R} \cup \{\infty\}$$

homeomorphically onto $\partial\mathbb{D}$, the function

$$\widetilde{g}(\zeta) := g(f^{-1}(\zeta)), \qquad \zeta \in \partial\mathbb{D},$$

is a continuous function on $\partial\mathbb{D}$.

Let v be the unique harmonic function on $\mathbb{D}$ which extends continuously to $\overline{\mathbb{D}}$ and satisfies

$$v|_{\partial\mathbb{D}} = \widetilde{g}.$$

Then

$$u(z) := v(f(z))$$

is harmonic on $\mathbb{H}$, extends continuously to $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$, and satisfies

$$u|_{\mathbb{R}\cup\{\infty\}} = g.$$

Thus the Dirichlet problem on $\mathbb{H}$ is reduced to the Dirichlet problem on $\mathbb{D}$.

Carrying out this construction explicitly leads to the following formula, whose proof is left to the reader as an exercise.

THEOREM 3.6 (Poisson integral formula on the upper half-plane). *Let*

$$g : \mathbb{R} \to \mathbb{R}$$

be continuous, and suppose that

$$\lim_{x\to+\infty} g(x) = \lim_{x\to-\infty} g(x).$$

Define

$$u(x+iy) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, g(t)\, dt, \qquad y > 0.$$

Then u is harmonic on $\mathbb{H}$, extends continuously to $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$, and satisfies

$$u|_{\mathbb{R}\cup\{\infty\}} = g.$$

In particular, u is the unique solution of the Dirichlet problem on $\mathbb{H}$ with boundary values g.

The kernel

$$P_y(x-t) := \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}$$

is called the **Poisson kernel** on the upper half-plane.

In fact, the same idea extends more generally. Whenever a domain Ω is conformally equivalent to the unit disk by a conformal map which extends appropriately to the boundary, the Dirichlet problem on Ω can be transferred to the Dirichlet problem on $\mathbb{D}$. In this way, once the Dirichlet problem is understood on the disk, one obtains solutions on other domains as well.

This naturally leads to the following question: for which domains does such a conformal mapping to the unit disk exist? In the next section, we will introduce and prove a very powerful result, namely the Riemann mapping theorem. It states that for every simply connected proper domain in $\mathbb{C}$, there exists a conformal mapping onto $\mathbb{D}$. Thus, at least for simply connected planar domains, the unit disk provides a universal model domain for conformal mapping problems.

4. The Riemann mapping theorem

In this section, we state and then prove Riemann's remarkable theorem, which is one of the central results in complex analysis.

4.1. Statement of the theorem.

THEOREM 4.1 (Riemann mapping theorem). *Let $\Omega \subset \mathbb{C}$ be a nonempty open set. Assume that $\Omega \neq \mathbb{C}$, and that Ω is connected and simply connected. Then there exists a conformal mapping*

$$F : \Omega \to \mathbb{D}.$$

Moreover, for any fixed point $z_0 \in \Omega$, there exists a unique conformal mapping

$$F : \Omega \to \mathbb{D}$$

such that

$$F(z_0) = 0, \qquad F'(z_0) > 0.$$

The uniqueness part is easy, and uses our description of the automorphism group of the disk.

PROOF OF UNIQUENESS. Suppose

$$F_1, F_2 : \Omega \to \mathbb{D}$$

are conformal mappings such that

$$F_1(z_0) = F_2(z_0) = 0, \qquad F_1'(z_0) > 0, \quad F_2'(z_0) > 0.$$

Consider

$$G := F_1 \circ F_2^{-1} : \mathbb{D} \to \mathbb{D}.$$

Then $G \in \mathrm{Aut}(\mathbb{D})$, and

$$G(0) = 0.$$

By the corollary above, G must be a rotation. Thus

$$G(z) = e^{i\theta} z \qquad \text{for some } \theta \in \mathbb{R}.$$

Differentiating at 0, we obtain

$$G'(0) = e^{i\theta}.$$

On the other hand, by the chain rule,

$$G'(0) = \frac{F_1'(z_0)}{F_2'(z_0)}.$$

Since both $F_1'(z_0)$ and $F_2'(z_0)$ are positive real numbers, it follows that

$$G'(0) > 0.$$

Hence

$$e^{i\theta} = 1,$$

so $G = \mathrm{id}_{\mathbb{D}}$. Therefore

$$F_1 = F_2.$$

□

Before discussing the Riemann mapping theorem, we mention a related boundary regularity result. It says that for simply connected domains whose boundary is sufficiently nice—for example, the unit disk, a rectangle, a triangle, or more generally a bounded polygon—a conformal mapping extends continuously to the boundary.

THEOREM 4.2 (Carathéodory). *Let $\Omega \subset \mathbb{C}$ be a simply connected domain whose boundary $\partial\Omega$ is a Jordan curve. Then any conformal mapping*

$$f : \mathbb{D} \to \Omega$$

extends continuously to a homeomorphism

$$\overline{\mathbb{D}} \to \overline{\Omega}.$$

REMARK 4.3. We will not prove this theorem here. Its proof is more delicate than that of the Riemann mapping theorem, since it involves the boundary behavior of conformal maps together with some nontrivial topology of Jordan curves.

The condition that $\partial\Omega$ be a Jordan curve is essential. Without it, the conclusion need not hold. A standard example is the comb domain

$$\Omega = (0,1) \times (0,1) \setminus \left(\{0\} \times \left[\frac{1}{2}, 1\right] \cup \bigcup_{n=1}^{\infty} \left\{\frac{1}{n}\right\} \times \left[0, \frac{1}{2}\right] \right).$$

This is a simply connected domain, but its boundary is not a Jordan curve, and a conformal map $f : \mathbb{D} \to \Omega$ does not extend continuously to $\overline{\mathbb{D}}$.

4.2. Proof of the Riemann mapping theorem – Part 1. We now turn to the proof of the existence part of the Riemann mapping theorem.

Let $\Omega \subset \mathbb{C}$ be a nonempty, open, connected, simply connected domain, and assume that

$$\Omega \neq \mathbb{C}.$$

Fix a point

$$z_0 \in \Omega.$$

We seek a conformal mapping from Ω to $\mathbb{D}$ from the family

$$\mathcal{M} := \left\{ f : \Omega \to \mathbb{D} \;\middle|\; f \text{ is holomorphic, injective, and } f(z_0) = 0 \right\}.$$

If a conformal mapping

$$F : \Omega \to \mathbb{D}$$

with $F(z_0) = 0$ exists, then it must lie in $\mathcal{M}$.

The first task is to show that the set $\mathcal{M}$ is not empty.

Step 1: Show that $\mathcal{M} \neq \varnothing$.

Since $\Omega \neq \mathbb{C}$, we may choose a point

$$\alpha \in \mathbb{C} \setminus \Omega.$$

Then

$$z - \alpha \neq 0 \qquad \text{for all } z \in \Omega.$$

Because Ω is simply connected, so is

$$\Omega - \alpha := \{z - \alpha : z \in \Omega\}.$$

Since $0 \notin \Omega - \alpha$, there exists a holomorphic branch of the logarithm on $\Omega - \alpha$. Thus

$$g(z) := \operatorname{Log}(z - \alpha)$$

is holomorphic on Ω.

We claim that g is injective. Indeed, if

$$g(z) = g(w),$$

then

$$z - \alpha = e^{g(z)} = e^{g(w)} = w - \alpha,$$

hence

$$z = w.$$

Next, notice that

$$g(z) \neq g(z_0) + 2\pi i \qquad \text{for all } z \in \Omega.$$

For otherwise, if

$$g(z) = g(z_0) + 2\pi i,$$

then

$$e^{g(z)} = e^{g(z_0)},$$

so

$$z - \alpha = z_0 - \alpha,$$

hence $z = z_0$, which would imply

$$g(z) = g(z_0),$$

a contradiction.

We claim moreover that

$$g(z_0) + 2\pi i \notin \overline{g(\Omega)}.$$

Indeed, if there were a sequence $z_n \in \Omega$ such that

$$g(z_n) \to g(z_0) + 2\pi i,$$

then

$$z_n - \alpha = e^{g(z_n)} \to e^{g(z_0)+2\pi i} = e^{g(z_0)} = z_0 - \alpha,$$

so

$$z_n \to z_0.$$

Since g is continuous,

$$g(z_n) \to g(z_0),$$

contradicting

$$g(z_n) \to g(z_0) + 2\pi i.$$

Thus there exists $\varepsilon > 0$ such that

$$D\big(g(z_0) + 2\pi i, \varepsilon\big) \cap g(\Omega) = \varnothing.$$

Now define

$$h(z) := \frac{1}{g(z) - \big(g(z_0) + 2\pi i\big)} + \frac{1}{2\pi i}.$$

Then h is holomorphic and injective on Ω, and

$$h(z_0) = 0.$$

Moreover, since

$$\Big|g(z) - \big(g(z_0) + 2\pi i\big)\Big| \geq \varepsilon \qquad (z \in \Omega),$$

we have

$$|h(z)| \leq \frac{1}{\varepsilon} + \frac{1}{2\pi} \qquad (z \in \Omega).$$

Thus h is bounded on Ω. Choose $M > 0$ such that

$$|h(z)| \leq M \qquad (z \in \Omega).$$

Rescale h and define

$$f(z) := \frac{1}{M} h(z).$$

Then $f \in \mathcal{M}$. Hence

$$\mathcal{M} \neq \varnothing.$$

Step 2: Reduction to the case $\Omega \subset \mathbb{D}$ and $z_0 = 0$.

Suppose that $f \in \mathcal{M}$. Then

$$f : \Omega \to f(\Omega)$$

is a conformal equivalence, and $f(\Omega) \subset \mathbb{D}$ is again a simply connected domain with

$$0 = f(z_0) \in f(\Omega).$$

If we can prove the theorem for the domain $f(\Omega) \subset \mathbb{D}$ with distinguished point 0, then composing the resulting conformal map $f(\Omega) \to \mathbb{D}$ with f, we obtain a conformal map $\Omega \to \mathbb{D}$.

Thus, from now on, it is enough to assume that

$$\Omega \subset \mathbb{D} \qquad \text{and} \qquad z_0 = 0.$$

Step 3: If $F \in \mathcal{M}$ maximizes $|F'(0)|$, then F is surjective.

Assume there exists $F \in \mathcal{M}$ such that

$$|F'(0)| \geq |f'(0)| \qquad \text{for all } f \in \mathcal{M}.$$

We claim that

$$F(\Omega) = \mathbb{D}.$$

Suppose not. Then there exists

$$\alpha \in \mathbb{D} \setminus F(\Omega).$$

Consider the Blaschke factor

$$\psi_\alpha(w) = \frac{w - \alpha}{\bar{\alpha} w - 1}.$$

Then

$$\psi_\alpha \circ F : \Omega \to \mathbb{D}$$

is holomorphic and injective, and since $\alpha \notin F(\Omega)$, its image omits 0. Because Ω is simply connected, there exists a holomorphic branch of the square root, which we denote by

$$h := \left(\psi_\alpha \circ F\right)^{1/2}.$$

Choose the branch so that

$$h(0) = \alpha^{1/2},$$

where $\alpha^{1/2}$ is one of the two square roots of α.

The map h is injective. Indeed, if

$$h(z_1) = h(z_2),$$

then

$$\psi_\alpha(F(z_1)) = h(z_1)^2 = h(z_2)^2 = \psi_\alpha(F(z_2)),$$

and since both ψ_α and F are injective, it follows that

$$z_1 = z_2.$$

Now define

$$G := \psi_{\alpha^{1/2}} \circ h, \qquad Q := \psi_\alpha \circ s \circ \psi_{\alpha^{1/2}},$$

where

$$s(z) := z^2.$$

Since $\psi_{\alpha^{1/2}}$ maps $\alpha^{1/2}$ to 0, we have

$$G(0) = 0.$$

Also, G is holomorphic and injective, so

$$G \in \mathcal{M}.$$

Moreover, since

$$\psi_\alpha \circ F = h^2 = s \circ h$$

and

$$h = \psi_{\alpha^{1/2}} \circ G$$

because $\psi_{\alpha^{1/2}}^{-1} = \psi_{\alpha^{1/2}}$, we obtain

$$\psi_\alpha \circ F = s \circ \psi_{\alpha^{1/2}} \circ G.$$

Applying ψ_α again, we get

$$F = Q \circ G.$$

Now Q is a holomorphic map from $\mathbb{D}$ to itself and maps 0 to 0. By Schwarz's lemma, the non-injectivity of Q results in

$$|Q'(0)| < 1.$$

By the chain rule,

$$F'(0) = Q'(0)\, G'(0).$$

Hence

$$|F'(0)| < |G'(0)|.$$

But $G \in \mathcal{M}$, contradicting the maximality of $|F'(0)|$.

Therefore our assumption was false, and we must have

$$F(\Omega) = \mathbb{D}.$$

The last step is to show the existence of such an extremal map. The underlying idea is quite natural. In finite-dimensional situations, one often proves the existence of a maximizer by showing that a continuous function attains its maximum on a compact set. Here we would like to apply the same philosophy to the functional

$$f \mapsto |f'(0)|$$

on the family $\mathcal{M}$. However, $\mathcal{M}$ is a family of holomorphic functions and is therefore infinite-dimensional, so ordinary compactness is no longer available.

What we shall prove instead is a sequential compactness statement: every sequence in $\mathcal{M}$ has a subsequence converging uniformly on compact subsets. This is exactly what is needed to extract a limit from a maximizing sequence. To establish this, we now develop the basic analytic tools for families of holomorphic functions in the next subsection.

4.3. Proof of the Riemann mapping theorem – Part 2: Analysis. To make this part rigorous we need some preparation and first establish the analytic tools needed for this argument.

DEFINITION 4.4. Let $\Omega \subset \mathbb{C}$ be an open set. A family $\mathcal{F}$ of holomorphic functions on Ω is called a **normal family** if every sequence $(f_n) \subset \mathcal{F}$ has a subsequence which converges uniformly on every compact subset of Ω.

By Theorem 4.7, the limit of such a subsequence is again holomorphic on Ω.

DEFINITION 4.5. Let $\Omega \subset \mathbb{C}$ be an open set, and let $\mathcal{F}$ be a family of functions on Ω.

(1) We say that $\mathcal{F}$ is **uniformly bounded on compact subsets** if for every compact set $K \subset \Omega$, there exists a constant $M_K > 0$ such that

$$|f(z)| \leq M_K \qquad \text{for all } f \in \mathcal{F} \text{ and all } z \in K.$$

(2) We say that $\mathcal{F}$ is *equicontinuous on compact subsets* if for every compact set $K \subset \Omega$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z_1) - f(z_2)| < \varepsilon$$

whenever $f \in \mathcal{F}$, $z_1, z_2 \in K$, and $|z_1 - z_2| < \delta$.

The following lemma is a fundamental result in analysis whose proof can be found from a standard analysis textbook, e.g., Rudin. We will use the result without proving it here.

LEMMA 4.6 (Arzelà–Ascoli). *Let $K \subset \mathbb{C}$ be compact, and let (f_n) be a sequence of complex-valued functions on K. If f_n is uniformly bounded and equicontinuous on K, then it has a subsequence converging uniformly on K.*

COROLLARY 4.7. *Let $\Omega \subset \mathbb{C}$ be an open set, and let $\mathcal{F}$ be a family of functions on Ω. If $\mathcal{F}$ is uniformly bounded and equicontinuous on every compact subset of Ω, then $\mathcal{F}$ is a normal family.*

PROOF. Let f_n be a sequence in $\mathcal{F}$. Choose an increasing sequence of compact sets $K_1 \subset K_2 \subset \cdots$ in Ω such that

$$\Omega = \bigcup_{m=1}^{\infty} K_m,$$

and K_m is contained in the interior of K_{m+1} for every m.

By Lemma 4.6, the sequence (f_n) has a subsequence converging uniformly on K_1. From that subsequence, extract a further subsequence converging uniformly on K_2. Continuing in this way and then taking a diagonal subsequence, we obtain a subsequence of f_n which converges uniformly on every compact subset of
Omega. Thus $\mathcal{F}$ is a normal family. □

We remark that, up to this point, holomorphicity has not been used. We now state and prove the following fundamental compactness theorem for holomorphic functions.

THEOREM 4.8 (Montel's theorem). *Let $\Omega \subset \mathbb{C}$ be an open set, and let $\mathcal{F}$ be a family of holomorphic functions on Ω. If $\mathcal{F}$ is uniformly bounded on every compact subset of Ω, then $\mathcal{F}$ is a normal family.*

PROOF. By Corollary 4.7, it suffices to show that $\mathcal{F}$ is equicontinuous on every compact subset of Ω.

Let $K \subset \Omega$ be compact. Choose an open set U such that

$$K \subset U \subset \overline{U} \subset \Omega.$$

Since K has positive distance from ∂U, there exists $r > 0$ such that

$$\overline{D(z;2r)} \subset U \qquad \text{for all } z \in K.$$

Set

$$K' := \bigcup_{z\in K} \overline{D(z;2r)}.$$

Then K' is a compact subset of Ω. Since $\mathcal{F}$ is uniformly bounded on compact subsets, there exists $B > 0$ such that

$$|f(w)| \le B \qquad \text{for all } f \in \mathcal{F} \text{ and all } w \in K'.$$

Now let $f \in \mathcal{F}$, and let $z_1, z_2 \in K$ with

$$|z_1 - z_2| < r.$$

Then

$$z_1, z_2 \in D(z_2;2r) \subset U.$$

By Cauchy's integral formula on $\partial D(z_2;2r)$,

$$f(z_1) - f(z_2) = \frac{1}{2\pi i}\int_{\partial D(z_2;2r)} f(w)\left(\frac{1}{w-z_1} - \frac{1}{w-z_2}\right) dw.$$

For $w \in \partial D(z_2;2r)$, we have

$$|w - z_2| = 2r \qquad \text{and} \qquad |w - z_1| \ge r,$$

so

$$\left|\frac{1}{w-z_1} - \frac{1}{w-z_2}\right| = \frac{|z_1 - z_2|}{|w-z_1||w-z_2|} \le \frac{|z_1 - z_2|}{2r^2}.$$

Hence

$$|f(z_1) - f(z_2)| \le \frac{1}{2\pi}\cdot(4\pi r)\cdot B \cdot \frac{|z_1 - z_2|}{2r^2} = \frac{B}{r}|z_1 - z_2|.$$

This estimate is independent of $f \in \mathcal{F}$. Therefore $\mathcal{F}$ is equicontinuous on K. Since $K \subset \Omega$ was arbitrary, the theorem follows. □

In Step 4 of the proof of the Riemann mapping theorem, we will also need the following proposition.

PROPOSITION 4.9. *Let $\Omega \subset \mathbb{C}$ be a domain, and let (f_n) be a sequence of injective holomorphic functions on Ω. Suppose that $f_n \to f$ uniformly on every compact subset of Ω, where f is holomorphic on Ω. Then f is either constant or injective.*

PROOF. Assume that f is not injective. We will show that f must be constant.

Choose distinct points $z_1, z_2 \in \Omega$ such that

$$f(z_1) = f(z_2).$$

Define

$$g_n(z) := f_n(z) - f_n(z_1), \qquad g(z) := f(z) - f(z_1).$$

Then $g_n \to g$ uniformly on every compact subset of Ω. Also,

$$g_n(z_1) = 0$$

for every n, and since each f_n is injective, g_n has no zero other than z_1.

Now $g(z_1) = g(z_2) = 0$. If $g \equiv 0$, then f is constant and we are done. So assume that $g \not\equiv 0$. Then the zeros of g are isolated. Choose a closed disk $\overline{D} \subset \Omega$ centered at z_2 such that z_2 is the only zero of g in $\overline{D}$. In particular,

$$g(w) \neq 0 \qquad \text{for all } w \in \partial D.$$

By the argument principle,

$$\frac{1}{2\pi i} \int_{\partial D} \frac{g'(w)}{g(w)}\, dw$$

is the number of zeros of g in D, counted with multiplicity. Since $z_2 \in D$ is a zero of g, this number is at least 1.

On the other hand, for each n, the function g_n has no zeros in D, so again by the argument principle,

$$\frac{1}{2\pi i} \int_{\partial D} \frac{g_n'(w)}{g_n(w)}\, dw = 0.$$

Since $g_n \to g$ uniformly on a neighborhood of ∂D, it follows from Cauchy's integral formula for derivatives that $g_n' \to g'$ uniformly on ∂D. Therefore

$$\frac{g_n'}{g_n} \to \frac{g'}{g}$$

uniformly on ∂D, and hence

$$\frac{1}{2\pi i} \int_{\partial D} \frac{g_n'(w)}{g_n(w)}\, dw \to \frac{1}{2\pi i} \int_{\partial D} \frac{g'(w)}{g(w)}\, dw.$$

This is impossible, since the left-hand side is always 0, while the right-hand side is at least 1. The contradiction shows that $g \equiv 0$, and hence f is constant. $\square$

4.4. Proof of the Riemann mapping theorem – Part 3: The last step. We now complete the proof of the Riemann mapping theorem by establishing the existence of an extremal map.

Step 4: Existence of an extremal map.

Recall that, after Step 2, we may assume that

$$\Omega \subset \mathbb{D} \qquad \text{and} \qquad 0 \in \Omega.$$

We consider the family

$$\mathcal{M} := \{ f : \Omega \to \mathbb{D} \mid f \text{ is holomorphic, injective, and } f(0) = 0 \}.$$

We now show that there exists a function

$$F \in \mathcal{M}$$

such that

$$|F'(0)| = \sup_{f \in \mathcal{M}} |f'(0)|.$$

Define

$$\Phi : \mathcal{M} \to \mathbb{R}, \qquad \Phi(f) := |f'(0)|.$$

We first note that Φ is bounded on $\mathcal{M}$. Indeed, choose $R > 0$ such that

$$\overline{D(0; R)} \subset \Omega.$$

If $f \in \mathcal{M}$, then $f(\Omega) \subset \mathbb{D}$, so $|f| \le 1$ on $D(0; R)$. By Cauchy's estimate,

$$|f'(0)| \le \frac{1}{R}.$$

Hence

$$s := \sup_{f \in \mathcal{M}} |f'(0)| < \infty.$$

Since $\mathcal{M} \ne \emptyset$, we also have $s > 0$.

Choose a sequence $(f_n) \subset \mathcal{M}$ such that

$$|f_n'(0)| \to s.$$

Because each f_n maps Ω into $\mathbb{D}$, the family $\mathcal{M}$ is uniformly bounded on every compact subset of Ω. By Montel's theorem 4.8, after passing to a subsequence, we may assume that

$$f_n \to F$$

uniformly on every compact subset of Ω, where F is holomorphic on Ω.

Because $f_n(0) = 0$ for all n, we have

$$F(0) = 0.$$

Also, since $|f_n(z)| \le 1$ for all $z \in \Omega$, it follows that

$$|F(z)| \le 1 \qquad \text{for all } z \in \Omega.$$

By Cauchy's integral formula for derivatives,

$$f_n'(0) \to F'(0).$$

Therefore

$$|F'(0)| = s > 0.$$

In particular, F is not constant. Since $|F| \le 1$ on Ω, the maximum modulus principle implies that

$$F(\Omega) \subset \mathbb{D}.$$

Finally, by Proposition 4.9, the function F is injective, since it is the locally uniform limit of injective holomorphic functions and is not constant. Thus

$$F \in \mathcal{M}$$

and

$$|F'(0)| = \sup_{f \in \mathcal{M}} |f'(0)|.$$

So the extremal map exists.

By Step 3, this extremal map must be surjective. Therefore

$$F : \Omega \to \mathbb{D}$$

is a conformal mapping.

Finally, by composing with a suitable rotation of $\mathbb{D}$, we may arrange that

$$F'(0) > 0.$$

This completes the proof of the existence part of the Riemann mapping theorem.

Math 185 of Spring 2026 covers up to here.

CHAPTER 10

Problem set

Homework 1

- **Reading assignment:** Stein & Shakarchi, Sections 1.1–1.3; Priestley, Chapter 1 and Sections 2.1–2.12.
- **Problems:**
  (1) Solve the following equations in $\mathbb{C}$ (Express z in polar form or exponential form if it is not easy to compute the real and imaginary parts explicitly.):
    (a) $z^4 + 2 - 2\sqrt{3}\, i = 0$.
    (b) $1 + z + z^2 + \cdots + z^{10} = 0$.
    (c) $1 - z^2 + z^4 - z^6 = 0$.
    (d) $\arg(iz) = \frac{\pi}{4} + 2\pi\mathbb{Z}$.
  (2) Suppose $z, w \in \mathbb{C}$ and $z \neq w$. Prove that
  $$\Re\left(\frac{z+w}{z-w}\right) = \frac{|z|^2 - |w|^2}{|z-w|^2}.$$
  (3) Describe geometrically the subsets of $\mathbb{C}$ specified by the following conditions:
    (a) $|z - 1 + i| \geq |z - 1 - i|$.
    (b) $|z|^2 > z + \overline{z}$.
    (c) $2|z - i| = |z|$.
    (d) $\mathrm{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$.
  (4) Find the images of the following sets under the map $z \mapsto 1/z$:
    (a) the ray $\mathrm{Arg} z = \pi/6$,
    (b) the disk $\{z \in \mathbb{C} : |z| < 2\}$,
    (c) the line $\Im z = 1$.
  (5) Suppose S is an open set in $\mathbb{C}$. Define
  $$T = \{\overline{z} \mid z \in S\}.$$
  Prove that T is also an open set in $\mathbb{C}$.
  (6) Prove that $\lim_{z\to 0} f(z)$ exists and equals 0 for:
    (a) $f(z) = \dfrac{|z|^2}{z}$,
    (b) $f(z) = \dfrac{\Re(z)\Im(z)}{|z|}$.
  (7) Prove that the limit $\lim_{z\to 0} f(z)$ does not exist for:
    (a) $f(z) = \dfrac{\overline{z}}{z}$,
    (b) $f(z) = \dfrac{\Re(z)}{\Im(z)}$.
  (8) On which subsets of $\mathbb{C}$ are the following functions continuous?
    (a) $f(z) = \dfrac{1}{z^2 - 1}$,

   (b) $f(z) = \dfrac{z}{\Re(z)}$.

(9) Define the function $f(z) = \frac{z}{1+|z|}$.
   (a) Prove f is continuous on $\mathbb{C}$.
   (b) Prove $f : \mathbb{C} \to \mathbb{C}$ is an injective map.
   (c) Prove that f maps $\mathbb{C}$ onto the unit disk $D(0; 1)$.

Homework 2

- **Reading assignment:** Stein & Shakarchi, Chapter 1 Sections 2.1-2.2; Priestley, Chapter 5 and 6.1-6.3.
- **Problems:**
  (1) Consider the function defined by $f(x+iy) = \sqrt{|x||y|}$, where $x, y \in \mathbb{R}$. Show that f satisfies the Cauchy–Riemann equation at 0, but f is not complex differentiable at 0.
  (2) Show that
  $$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \Delta,$$
  where
  $$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
  is the Laplacian. (From this, we see again that if f is holomorphic in an open set, then the real and imaginary parts of f are harmonic.)
  (3) Suppose f is holomorphic in a domain Ω (i.e., open and connected). Prove that in any one of the following cases:
  (a) $\mathfrak{R}(f)$ is constant;
  (b) $\mathfrak{I}(f)$ is constant;
  (c) $|f|$ is constant;
  one can conclude that f is a constant function.
  (4) Try to formulate the complex version of the chain rule (think about how to prove it), and use it to calculate
  $$\frac{\partial}{\partial z} e^{z\bar{z}^2}, \qquad \frac{\partial}{\partial \bar{z}} e^{z\bar{z}^2}.$$
  Check your calculation using the real partial derivatives.
  (5) Suppose f is holomorphic on the unit disk $D(0;1)$. Prove that the function
  $$g(z) = \overline{f(\bar{z})}$$
  is also holomorphic on $D(0;1)$.
  (6) Write down a power series expansion of the form $\sum_{n=0}^{\infty} c_n z^n$ (about $z=0$) for each of the following functions:
  (a) $\frac{1}{2z+5}$;
  (b) $\frac{1}{1+z^4}$;
  (c) $\frac{1+iz}{1-iz}$;
  (d) $\frac{1}{1-z+z^2}$;
  (e) $\frac{1}{(z+1)(z+2)}$;
  (f) $\frac{1}{(z^2-1)(z^2-9)}$.
  Further, in each case, specify where the expansion is valid, i.e., where the series converges to the given function.
  (7) Write down a power series expansion of each of the following functions in powers of $z+1$, and also in powers of $z-i$:
  (a) $\frac{1}{1-z}$;
  (b) $\frac{1}{z(z+2)}$.

Homework 3

- **Reading assignment:** Stein & Shakarchi, Chapter 1 Sections 2.3, 3; Priestley, 6.4-6.11, 7.1-7.12, Chapter 10;
- **Problems:**
  (1) Prove that $u(x, y) = e^{-x}(x \sin y - y \cos y)$ is a harmonic function, and then find the harmonic conjugate of u.
  (2) For a complex sequence $(a_n)_{n=0,1,\cdots}$ with every $a_n \neq 0$, show that
  $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$$
  implies
  $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$
  As a consequence, one can use ratio test to calculate convergence radius of a power series.
  (3) Suppose $a, \theta \in \mathbb{R}$ and $|a| < 1$. Prove that
    (a) $1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \cdots = \frac{1-a\cos\theta}{1-2a\cos\theta+a^2}$;
    (b) $a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \cdots = \frac{a\sin\theta}{1-2a\cos\theta+a^2}$.
  (Hint: Euler's identity.)
  (4) Determine for which values of z the following series converge absolutely:
    (a) $\sum_{n=1}^{\infty} \frac{z^n}{n^n}$;
    (b) $\sum_{n=1}^{\infty} \frac{1}{n^2}(z^n + z^{-n})$. (Notice this is not a power series.)
  (5) (a) Obtain a power series expansion for $\frac{1}{(1+z)^3}$ in the disk $D(0;1)$.
    (b) Given a positive integer m, expand $\frac{1}{(1-z)^m}$ in powers of z. If
    $$\frac{1}{(1-z)^m} = \sum_{n=0}^{\infty} a_n z^n,$$
    show that
    $$\lim_{n\to\infty} \frac{(m-1)!a_n}{n^{m-1}} = 1.$$
  (6) Define the hyperbolic cosine and hyperbolic sine by
  $$\cosh z := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \qquad \sinh z := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.$$
    (a) Show that both series have radius of convergence ∞.
    (b) Prove the following identities:
      (i) $\cosh z = \frac{e^z+e^{-z}}{2}$, $\qquad \sinh z = \frac{e^z-e^{-z}}{2}$.
      (ii) $(\cosh z)' = \sinh z$, $\qquad (\sinh z)' = \cosh z$.
      (iii) $\cosh^2 z - \sinh^2 z = 1$.
      (iv) $\cos(iz) = \cosh z$, $\qquad \sin(iz) = i\,\sinh z$.
    (c) Show that both functions are bounded on the imaginary axis $i\mathbb{R}$, but neither is bounded on $\mathbb{C}$.
    (d) Solve the equations (1). $\sinh z = 0$. (2). $\cosh z = 0$.
  (7) If for constants $a, b \in \mathbb{C}$, there is $|a| < r < |b|$. Prove that
  $$\int_\gamma \frac{1}{(z-a)(z-b)}\,dz = \frac{2\pi i}{a-b},$$
  where γ is a circle centered at the origin, of radius r, with counterclockwise orientation.

Homework 4

- **Reading assignment:** Stein & Shakarchi, Chapter 2 Section 1, 2; Priestley, Chapter 11, 12, 9.
- **Problems:**
  (1) Prove the inequality
  $$\left|\int_{\partial D(0;R)} \frac{z-1}{z+1}\,dz\right| \le \frac{2\pi R(R+1)}{R-1}.$$
  Here assume $R > 1$.
  (2) By integrating $\frac{R+z}{z(R-z)}$ around a suitable loop, prove the identity
  $$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2rR\cos\theta+r^2}\,d\theta = 1.$$
  (3) Prove that for any $R > 0$, the function $1/z$ cannot be uniformly approximated on the circle
  $$C_R = \{z \in \mathbb{C} : |z| = R\}$$
  by polynomials.
  (4) Let $\Omega \subset \mathbb{C}$ be an open set and let $z_0, z_1 \in \Omega$. Consider the set
  $$\mathcal{P}(z_0, z_1; \Omega) := \{\gamma : [0,1] \to \Omega \text{ continuous } : \gamma(0) = z_0,\ \gamma(1) = z_1\}.$$
  Consider the relation defined by homotopy of paths relative to endpoints z_0 and z_1. Prove that this relation is an equivalence relation on $\mathcal{P}(z_0, z_1; \Omega)$.
  (5) Continue the previous problem. Let $\Omega = \mathbb{C}^*$ and fix the basepoint 1, i.e., take $z_0 = z_1 = 1$. For each $n \in \mathbb{Z}$, define the based loop
  $$\gamma_n(t) := e^{2\pi i n t}, \qquad 0 \le t \le 1.$$
  (a) Show that if $m \neq n$, then γ_m and γ_n are not homotopic in $\mathbb{C}^*$ relative to the basepoint 1.
  (b) Show that every based loop $\gamma : [0,1] \to \mathbb{C}^*$ with $\gamma(0) = \gamma(1) = 1$ is homotopic in $\mathbb{C}^*$ relative to the basepoint 1 to γ_n for a unique integer n.
  (6) Let $\gamma_+(t) = e^{i\pi t}$ and $\gamma_-(t) = e^{-i\pi t}$ for $t \in [0,1]$. These are the upper and lower semicircles from 1 to -1.
  (a) Show that γ_+ and γ_- are homotopic relative to endpoints in $\mathbb{C}$.
  (b) Show that γ_+ and γ_- are *not* homotopic relative to endpoints in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
  (7) Let $\gamma = \partial D(0;2)$, i.e.
  $$\gamma(t) = 2e^{it} \quad (0 \le t \le 2\pi).$$
  Suppose values of $f(z) \in [w(z)]$ are selected so that $f(\gamma(t))$ varies continuously as t increases from 0 to 2π, with $f(\gamma(0))$ real. Determine the initial value $f(\gamma(0))$ and final value $f(\gamma(2\pi))$ when $[w(z)]$ is
  (iii) $[\log(1+z)]$,
  (8) Repeat the previous questions for the following multifunctions:
  (i) $\left[(z-1)^{1/3}\right]$,
  (Assume that $f(\gamma(0))$ is real and positive.)
  (9) Verify that the multifunctions below have branch points as indicated and that the cuts suggested outlaw precisely the inadmissible contours. In each case, specify a holomorphic branch.

(ii) $\left[\left(\frac{z-1}{z+1}\right)^{3/4}\right]$ (branch points ± 1; cut along $[-1, 1]$).

(v) $\left[\log(z^2+1)\right]$ (branch points $\pm i$; cuts along $\{\, iy : \ |y| \geq 1 \,\}$).

(10) For each of the following multifunctions, locate the branch points (in $\widetilde{\mathbb{C}}$), specify multi-branches, suggest how the plane should be cut, and specify a holomorphic branch.

(iii) $\left[\left((z-1)(z-\omega)(z-\omega^2)\right)^{1/2}\right]$,

where $\omega = e^{2\pi i/3}$.

Homework 5

- **Reading assignment:** Stein & Shakarchi, Chapter 2 Section 4, 5; Priestley, Chapter 13, 14.
- **Problems:**
  (1) Evaluate
    (a) $\int_{\partial D(0;2)} \frac{z^3+5}{z-i} dz$;
    (b) $\int_{\partial D(0;2)} \frac{1}{z^2+z+1} dz$;
    (c) $\int_{\partial D(0;2)} \frac{1}{(z+1)^2(z^2+9)} dz$.
  (2) Let $a, b \in \mathbb{C}$ with $|a| \neq 1$, $|b| \neq 1$. Evaluate, distinguishing cases,
  $$\int_{\partial D(0;1)} \frac{(z-b)^2}{(z-a)^2} dz.$$
  (3) Evaluate $\int_{\partial D(0;1)} z^n(1-z)^m dz$, where $m \in \mathbb{Z}^{\geq 0}$ and $n \in \mathbb{Z}$.
  (4) Suppose f is an entire function. Use Liouville's theorem to prove that f is a constant function if $\Re f$ is bounded above or bounded below.
  (5) Let $S = \{z \in \mathbb{C} : -1 < \Im z < 1\}$. Suppose f is holomorphic on S and satisfies the growth bound
  $$|f(z)| \leq A(1+|z|)^\alpha \qquad (\forall z \in S),$$
  where $A \geq 0$ and $\alpha \in \mathbb{R}$ are constants. Prove that for each integer $n \geq 0$ there exists a constant $A_n \geq 0$ (depending only on n, A, α) such that
  $$|f^{(n)}(x)| \leq A_n(1+|x|)^\alpha \qquad (\forall x \in \mathbb{R}).$$
  (6) Suppose f is a continuous function on the closed unit disk $\overline{D}$, holomorphic on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, and nowhere vanishing on $\overline{D}$. Assume
  $$|f(z)| = 1 \qquad \text{for all } |z| = 1.$$
  Prove that f is constant.
  (7) Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence ∞. Prove that for any $R > 0$,
  $$\sum_{n=0}^{\infty} |a_n|\, R^n \leq 2\, M(2R),$$
  where
  $$M(r) := \sup\{\, |f(z)| : |z| = r \,\}.$$

Homework 6

- **Reading assignment:** Stein & Shakarchi, Chapter 3 Section 1, 3; Priestley, Chapter 15.1-15.5, 17.8-17.20.
- **Problems:**
  (1) Let f, g be holomorphic functions in $D(z_0; r)$ for some $r > 0$ and assume that

  $$f(z_0) = 0 = g(z_0).$$

  Prove that

  $$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}$$

  if the RHS exists. (This is a complex version of L'Hospital rule.)
  It is a useful tool, for example, use it to compute
  (a) $\lim_{z \to i} \frac{1+e^{\pi z}}{1+z^2}$;
  (b) $\lim_{z \to 0}(\cot z - z^{-1})$.

  (2) Find, with reasoning, the order of the zero at $z_0 = 0$ for each of the following functions.
  (a) $z^2(e^{z^2} - 1)$;
  (b) $6\sin(z^3) + z^3(z^6 - 6)$.

  (3) Determine, with reasoning, all singularities in $\mathbb{C}$ of each of the following functions. Compute the order and residue of each pole.
  (a) $\frac{1}{1+z^4}$;
  (b) $\frac{1}{(1-\cos z)^2}$;
  (c) $\frac{1}{z^4 \sin z}$;
  (d) $\frac{e^{iz}}{\cosh z}$;
  (e) $\frac{e^{\frac{1}{z}}}{z}$;
  (f) $\frac{1}{(\pi+z)\sin z} - \frac{1}{\pi z}$.

  (4) Determine, with reasoning, the nature of the singularity at 0 and at ∞ for each of the following functions. Indicate whether it is removable, a pole, an essential singularity, or not an isolated singularity. Compute the order of the pole if it is a pole.
  (a) $\cot z - z^{-1}$;
  (b) $(z + z^{-1})^{-1}$;
  (c) $e^{z+1/z}$;
  (d) $\csc(\sin z)$.

  (5) (a) Construct a holomorphic function on the extended complex plane (the Riemann sphere) except for poles at $0, 1, \infty$.
  (b) Construct a holomorphic function on the extended complex plane (the Riemann sphere) except for essential singularities at $0, 1, \infty$.

  (6) Suppose $f(z)$ is holomorphic in a punctured disk $D^*(z_0; r)$, with some $r > 0$, and

  $$|f(z)| \le A|z - z_0|^{-2/3}$$

  for some constant $A > 0$ when z is near z_0. Prove that z_0 is a removable singularity of f.

(7) Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Homework 7

- **Reading assignment:** Priestley, Sections 17.1-17.7, 18, 19, 20.
- **Problems:**
  (1) Compute the improper integral
  $$\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\, dx.$$
  Hint: The upper semicircle contour works, but instead of using that method, one can also consider the sector contour C_R consisting of
  - the interval $[0, R]$ on the positive real axis;
  - the circular arc $\{Re^{i\theta} : 0 \le \theta \le \pi/3\}$;
  - the line segment from $Re^{i\pi/3}$ back to 0.

  This can simplify some calculations.

  (2) Compute the improper integral
  $$\int_{-\infty}^{+\infty} \frac{1}{x^{2n}+1}\, dx,$$
  where $n \in \mathbb{Z}^+$.

  (3) Let $a > 0, b > 0$. Compute the improper integral
  $$\int_{-\infty}^{+\infty} \frac{x\sin(ax)}{x^2+b^2}\, dx.$$

  (4) Assume $|a| < 1$. Compute the integral
  $$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}.$$

  (5) Assume $a, b \in \mathbb{R}$ with $a > |b|$, and let $n \ge 1$ be an integer. Compute the integral
  $$\int_0^{2\pi} \frac{\cos(n\theta)}{a+b\cos\theta}\, d\theta.$$

  (6) Let $a > 0$ and $b > 0$. Compute the improper integrals
  $$\int_0^{\infty} \frac{x\sin(ax)}{x^2+b^2}\, dx, \quad \int_0^{\infty} \frac{\cos(ax)}{x^2+b^2}\, dx.$$

  (7) Compute the improper integral
  $$\int_0^{\infty} \frac{1-\cos x}{x^2}\, dx.$$

  (8) Compute the improper integral
  $$\int_0^{\infty} \frac{\sin x}{x(x^2+1)}\, dx.$$

  (9) Let $0 < p < 1$. Compute the improper integral
  $$\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx.$$

  (10) Compute the improper integral
  $$\int_0^{\infty} \frac{(\log x)^2}{1+x^2}\, dx.$$

Homework 8

- **Reading assignment:** Stein & Shakarchi, Chapter 3 Section 4; Priestley, Sections 15.13-15.15, 16.
- **Problems:** (You may skip problems 5 and 6 because they show up in the quiz.)
  (1) For $N \in \mathbb{N}$, let S_N denote the positively oriented boundary of the square
  $$\{z = x + iy \in \mathbb{C} : \ |x| \le N + \tfrac{1}{2},\ |y| \le N + \tfrac{1}{2}\}.$$
  Prove that $\cot(\pi z)$ is uniformly bounded on the contours S_N. That is, prove that there exists a constant $C > 0$, independent of N, such that
  $$|\cot(\pi z)| \le C \qquad \text{for all } z \in S_N, \text{ and all } N \in \mathbb{N}.$$
  (2) Compute $\sum_{n=1}^{\infty} \frac{1}{n^4}$.
  (3) Let $w \in \mathbb{C} \setminus \pi\mathbb{Z}$. Prove that
  $$\frac{1}{\sin^2 w} = \sum_{n\in\mathbb{Z}} \frac{1}{(w - \pi n)^2}.$$
  (4) (Optional.) Integrate the identity in the above problem, and deduce that
  $$\sin w = w \prod_{n=1}^{\infty} \left(1 - \frac{w^2}{\pi^2 n^2}\right).$$
  (5) Prove the following weighted version of the argument principle:
  Let $\Omega \subset \mathbb{C}$ be open, and let γ be a positively oriented, piecewise C^1, regular, simple closed curve in Ω. Suppose that f is meromorphic on a neighborhood of γ and its interior, and that f has neither zeros nor poles on γ. Let g be holomorphic on a neighborhood of γ and its interior. Then
  $$\frac{1}{2\pi i} \int_\gamma g(z) \frac{f'(z)}{f(z)}\, dz = \sum_a m_a\, g(a) - \sum_b n_b\, g(b),$$
  where the first sum is over the zeros a of f inside γ, counted with multiplicity m_a, and the second sum is over the poles b of f inside γ, counted with multiplicity n_b.
  (6) Let
  $$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$
  be a polynomial of degree n, with zeros $\alpha_1, \ldots, \alpha_n$ counted with multiplicity. Let
  $$s_k = \alpha_1^k + \cdots + \alpha_n^k.$$
  Use the argument principle to show that for every integer $k \ge 0$,
  $$s_k = \frac{1}{2\pi i} \int_\gamma z^k \frac{p'(z)}{p(z)}\, dz,$$
  where γ is a sufficiently large positively oriented circle.
  Using this, compute the Laurent expansion of
  $$\frac{p'(z)}{p(z)}$$
  on $\{z \in \mathbb{C} : |z| > R\}$ for R sufficiently large.

REMARK 0.1. If one writes

$$p(z) = a_n\big(z^n + c_1 z^{n-1} + \cdots + c_n\big),$$

then comparing the Laurent expansion above with the expression obtained directly from

$$\frac{p'(z)}{p(z)} = \frac{nz^{n-1} + (n-1)c_1 z^{n-2} + \cdots + c_{n-1}}{z^n + c_1 z^{n-1} + \cdots + c_n}$$

leads to the classical Newton identities relating the power sums s_k to the coefficients $c_1, \ldots, c_n$.

(7) Let

$$F(z, w) = w^n + c_1(z)w^{n-1} + \cdots + c_n(z),$$

where $c_1, \ldots, c_n$ are holomorphic on a disk $D(0; R)$. Assume that w_0 is a simple zero of the polynomial $F(0, w)$. Prove that there exists $r > 0$ and a holomorphic function

$$f : D(0; r) \to \mathbb{C}$$

such that

$$f(0) = w_0 \qquad \text{and} \qquad F(z, f(z)) = 0$$

for all $z \in D(0; r)$.
(First use the ordinary argument principle to show that, for z near 0, the polynomial $F(z, w)$ has exactly one zero in a small disk centered at w_0. Then use the weighted version with $g(w) = w$ to recover that zero as a contour integral.)

(8) Let

$$Y = \{(z, w) \in \mathbb{C}^2 : w^2 = z(z-1)\}.$$

(a) Show that Y has a natural structure of Riemann surface.
Hint: Apply the holomorphic implicit function theorem to

$$F(z, w) = w^2 - z(z-1).$$

(b) Show that the map

$$\pi : Y \to \mathbb{C} \setminus \{0, 1\}, \qquad (z, w) \mapsto z$$

is a surjective holomorphic map, and that each point $z \in \mathbb{C} \setminus \{0, 1\}$ has exactly two preimages in Y.

(9) Prove all zeros of $z^7 - 5z^3 + 12$ are in the annulus

$$\{z : 1 < |z| < 2\}.$$

(10) Suppose $0 < |a| < 1$, and $n \in \mathbb{Z}^+$. Prove that

$$(z-1)^n e^z = a$$

has exactly n distinct solutions in the right half plane $\Re(z) > 0$.

(11) Let $\Omega \subset \mathbb{C}$ be a connected open set, and let $u : \Omega \to \mathbb{R}$ be harmonic.
Show that if u attains a local maximum or local minimum at some point $z_0 \in \Omega$, then u is constant on Ω.

Homework 9

- **Reading assignment:** Stein & Shakarchi, Chapter 8; Priestley, Ch 8, Ch 23.
- **Problems:**

(1) Prove that a holomorphic map $f : U \to V$, where U, V are nonempty open subsets of $\mathbb{C}$, is locally biholomorphic if and only if

$$f'(z) \neq 0 \qquad \text{for all } z \in U.$$

(2) Construct a surjective holomorphic map from $\mathbb{D}$ to $\mathbb{C}$. Is there such a conformal mapping? Why?

(3) Find all Möbius transformations which take the imaginary line (with infinity) to the unit circle, and the point 1 to the origin.

(4) Construct a conformal mapping from the open upper half unit disk to the upper half plane.

(5) Let $f : \mathbb{H} \to \mathbb{H}$ be holomorphic and suppose that f has two distinct fixed points in $\mathbb{H}$. Prove that f must be the identity map.

(6) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Suppose that

$$f(a) = 0$$

for some $a \in \mathbb{D}$.

(a) Prove that

$$|f(z)| \leq \left| \frac{z-a}{1-\bar{a}z} \right| \qquad \text{for all } z \in \mathbb{D}.$$

(b) Prove that

$$|f'(a)| \leq \frac{1}{1-|a|^2}.$$

(7) (Optional.) For $z, w \in \mathbb{D}$, define

$$\rho(z, w) := |\psi_w(z)|,$$

where

$$\psi_w(z) = \frac{z-w}{\overline{w}z - 1}$$

is the Blaschke factor. This function is called the pseudo-hyperbolic distance on $\mathbb{D}$.

(a) Show that $\rho(z, w) = \rho(w, z)$, and that

$$\rho(z, w) = 0 \iff z = w.$$

Remark: In fact, the triangle inequality also holds for ρ (Try to prove it), so ρ defines a distance function on $\mathbb{D}$. The hyperbolic distance on $\mathbb{D}$ is related to the pseudo-hyperbolic distance by

$$d_{\mathbb{D}}(z, w) = 2 \operatorname{arctanh} \rho(z, w) = \log \frac{1+\rho(z, w)}{1-\rho(z, w)}.$$

(For more about the hyperbolic metric, see Stein & Shakarchi Chapter 8 Problem 3.)

(b) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Prove that

$$\rho(f(z), f(w)) \leq \rho(z, w) \qquad \text{for all } z, w \in \mathbb{D}.$$

(c) Show that if $f \in \operatorname{Aut}(\mathbb{D})$, then

$$\rho(f(z), f(w)) = \rho(z, w) \qquad \text{for all } z, w \in \mathbb{D}.$$

(d) Deduce that for every holomorphic map $f : \mathbb{D} \to \mathbb{D}$,

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2} \qquad \text{for all } z \in \mathbb{D}.$$

(8) Suppose that u is harmonic in $\mathbb{C}$. Show that, for $0 \le r < R$,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}\, u(Re^{it})\, dt.$$

Assume in addition that $u \ge 0$. Deduce that

$$\frac{R-r}{R+r}\, u(0) \le u(re^{i\theta}) \le \frac{R+r}{R-r}\, u(0).$$

(This provides another proof that a bounded function which is harmonic in $\mathbb{C}$ is necessarily constant.)

(9) Let

$$\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}.$$

Given a continuous function

$$g : \mathbb{R} \to \mathbb{R}, \quad \text{with} g(-\infty) = g(+\infty) \in \mathbb{R},$$

use the Poisson formula from the unit disk to prove that

$$u(x+iy) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, g(t)\, dt, \qquad y > 0,$$

solves the Dirichlet problem on $\mathbb{H}$ with boundary values g. That is, show that u is harmonic on $\mathbb{H}$, extends continuously to $\mathbb{H} \cup \mathbb{R}$, and satisfies

$$u(x) = g(x) \qquad \text{for all } x \in \mathbb{R}.$$