L^2-MODULI SPACES OF SYMPLECTIC VORTICES ON RIEMANN SURFACES WITH CYLINDRICAL END METRICS

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ABSTRACT. Let (X, ω) be a compact symplectic manifold with a Hamiltonian action of a compact Lie group G and $\mu : X \to \mathfrak{g}$ be its moment map. In this paper, we study the L^2-moduli spaces of symplectic vortices on Riemann surfaces with cylindrical ends (that is, a punctured Riemann surface with a Riemannian metric of cylindrical type at each puncture). We studied a circle-valued action functional whose gradient flow equation corresponds to the symplectic vortex equations on a cylinder $S^1 \times \mathbb{R}$. Assume that 0 is a regular value of the moment map μ, we show that the functional is of Bott-Morse type and its critical points of the functional form twisted sectors of the symplectic reduction (the symplecitc orbifold $[\mu^{-1}(0)/G]$). We show that any gradient flow line approaches its limit point exponentially fast. Fredholm theory and compactness property are then established for the L^2-moduli spaces of symplectic vortices on Riemann surfaces with cylindrical ends.

CONTENTS

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1. INTRODUCTION AND STATEMENTS OF MAIN THEOREMS

The symplectic vortex equations on a Riemann surface Σ associated a principal G-bundle P and a $2n$-dimensional Hamiltonian G-manifold (X, ω) with a G-invariant ω-compatible almost complex structure J, originally discovered by K. Cieliebak, A. R. Gaio, and D. A. Salamon [12], and independently by I. Mundet i Riera [32], is a system of first order partial differential equations

$$\begin{cases} \bar{\partial}_{J,A}(u) = 0 \\ *_\Sigma F_A + \mu(u) = 0 \end{cases} \tag{1.1}$$

for a connection A on P and a G-equivariant map $u : P \to X$. See Section 2 for an explanation of the notations involved. They are natural generalisations of the J-holomorphic equation in a symplectic manifold for $G = \{e\}$, and of the well-known Ginzburg-Landau vortices in a mathematical model of superconductors for $\Sigma = \mathbb{C}$ and $X = \mathbb{C}^n$ as the standard Hamiltonian $U(1)$-space. Ginzburg-Landau vortices have been studied both from mathematicians and physicists' viewpoints. They are two-dimensional solitons, as time-independent solutions with finite energy to certain classical field equations in the Abelian Higgs model, see [26] for a complete account of Ginzburg-Landau vortices.

Since the inception of these symplectic vortices, there have been steady developments in the study of the moduli spaces of symplectic vortices and their associated invariants, the so-called Hamitonian Gromov-Witten (GW) invariants. Many fascinating conjectures have been proposed, for example see [12], [23] and [41].

As in Gromov-Witten theory, there are several main technical issues in the definition of invariants from symplectic vortices such as compactification, gluing analysis and regularization for the moduli spaces of symplectic vortices. There have been many works focused on the compactification issue ([13],[32],[34],[41],[37]). On one hand, when Σ is closed, X is symplectically aspherical and satisfies some convexity condition, A. R. Gaio, I. Mundet i Riera and D. A. Salamon in [13] proved compactness of the moduli space of symplectic vortices with compact support and bounded energy. On the other hand, when $G = U(1)$ and X is closed, with strong monotone conditions, I. Mundet i Riera in [32] compactified the moduli space of bounded energy symplectic vortices over a fixed closed Riemann surface. When $G = U(1)$ and X is a general compact symplectic manifold, I. Mundet i Riera and G. Tian in [34] compactified the moduli space of symplectic vortices with bounded energy over smooth Riemann surfaces degenerating to nodal Riemann surfaces. In particular, they discovered a new feature in the bubbling off phenomena near nodal points in the sense that energy may be lost and there are gradient flows of the moment map appearing in the compactification instead. This is not present in the usual Gromov-Witten theory, and was elegantly and carefully presented in [34]. Also, there are some studies on special models such as on the affine vortices ([41]). Based on their compactification, Mundet i Riera and Tian have a long project on defining Hamiltonian GW invariants ([35]). On the other hand, Woodward, following Mundet i Riera's approach([33]), gave an algebraic geometry approach to define gauged Gromov-Witten invariants ([36]), and established its relation to Gromov-Witten invariants of the symplectic reduction $X /\!\!/ G$ via quantum Kirwan morphisms ([36]).

In this paper, we study the moduli spaces of symplectic vortices on a Riemann surface with a cylindrical end metric. In particular, for a genus g Riemann surface with n-marked points, we will study the L^2-moduli space of symplectic vortices on a Riemann surface Σ with a cylindrical end metric near each marked points. Here the energy of (A, u), defined to be the Yang-Mills-Higgs energy functional

$$E(A,u) = \int_\Sigma \frac{1}{2}(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2)\nu_\Sigma, \tag{1.2}$$

is finite. It turns out that the Hamiltonian GW type invariants are very sensitive to the volume forms used near punctured points. Readers may refer to §6 for further discussions.

In Section 2, we briefly review the moduli spaces of symplectic vortices on a closed Riemann surface as developed in [12], [13] and [34]. In Section 3, we investigate the asymptotic behaviour of symplectic vortices on a half cylinder $S^1 \times \mathbb{R}^{\geq 0}$ with finite energy. For this we adapt the action functional in [19] and [20] to get a circle-valued functional whose L^2-gradient flow equation realizes the symplectic vortex equations (1.1) on $S^1 \times \mathbb{R}^{\geq 0}$ in temporal gauge. The critical point set of this functional modulo gauge transformations, denoted by Crit, can be identified with

$$\left(\bigsqcup_{g\in G} \left(\mu^{-1}(0)\right)^g \right) /G,$$

as a topological space. Here that the action of G on $\bigsqcup_{g\in G} \left(\mu^{-1}(0)\right)^g$ is given by $h \cdot (x, g) = (h \cdot x, hgh^{-1})$. When 0 is a regular value of the moment map μ, the symplectic reduced space is a symplectic orbifold

$$\mathcal{X}_0 = [\mu^{-1}(0)/G],$$

where we use the square bracket to denote the orbifold structure arising from the locally free action of G on $\mu^{-1}(0)$. Then Crit is diffeomorphic to the inertia orbifold of $\mathcal{X}_0$

$$I\mathcal{X}_0 = \bigsqcup_{(g)} \mathcal{X}_0^{(g)}$$

where (g) runs over the conjugacy class in G with non-empty fixed points in $\mu^{-1}(0)$. Note that for a non-trivial conjugacy class (g), $\mathcal{X}_0^{(g)}$ is often called a twisted sector of $\mathcal{X}_0$, which is diffeomorphic to the orbifold arising from the action of $C(g)$ on $\mu^{-1}(0)^g$ (the g-fixed points in $\mu^{-1}(0)$. Here $C(g)$ is the centralizer of g in G for a representative g in the conjugacy class (g).

Throughout this paper, we assume that 0 is a regular value of the moment map μ. Then we show that this circle-valued functional is actually of Bott-Morse type. We also establish a crucial inequality (Proposition 3.12) near each critical point. This inequality enables us to establish an exponential decay result for a symplectic vortex on $S^1 \times \mathbb{R}^{\geq 0}$ with finite energy, Cf. Theorem 3.14.

In Section 4, we study the L^2-moduli space $\mathcal{N}_\Sigma(X, P)$ of symplectic vortices on a Riemann surface Σ with k-cylindrical ends, associated to a principal G-bundle and a closed Hamiltonian

G-manifold (X, ω). Applying the asymptotic analysis in Section 3 to the cylindrical ends, we get a continuous asymptotic limit map (Proposition 4.1 and Subsection 4.2)

$$\partial_\infty : \mathcal{N}_\Sigma(X, P) \longrightarrow (\text{Crit})^k \cong (I\mathcal{X}_0)^k.$$

Note that the Yang-Mills-Higgs energy functional takes discrete values on $\mathcal{N}_\Sigma(X, P)$ depending on homology classes in $H_2^G(X, \mathbb{Z})$.

Fix a homology class $B \in H_2^G(X, \mathbb{Z})$, denote by $\mathcal{N}_\Sigma(X, P, B)$ the L^2-moduli space of symplectic vortices on a Riemann surface Σ with the topological type defined by B. Then in Section 4, we develop the Fredholm theory for $\mathcal{N}_\Sigma(X, P, B)$ and calculate the expected dimension of the L^2-moduli space of symplectic vortices with prescribed asymptotic behaviours. The main result in this paper is summarized in the following theorem. Here $\mathcal{M}$ is said to admit an *orbifold Fredhom system* with its virtual dimension d, we mean that, there exists a triple,

$$(\mathcal{B}, \mathcal{E}, S)$$

consisting of an orbifold Banach manifold $\mathcal{B}$, and an orbifold Banach bundle $\mathcal{E}$ over $\mathcal{B}$ with a section S such that the zero set $S^{-1}(0)$ is $\mathcal{M}$, and the vertical differentiation of S at any $x \in \mathcal{M}$

$$(\mathcal{D}S)_x : T_x\mathcal{B} \longrightarrow \mathcal{E}_x$$

is a Fredholm operator of index d.

Theorem A (Theorem 4.3) *Let* $\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1,\cdots,k})$ *be the subset of* $\mathcal{N}_\Sigma(X, P, B)$ *consisting of symplectic vortices* $[(A, u)]$ *such that*

$$\partial_\infty(A, u) \in \left(\mathcal{X}_0^{(g_1)} \times \cdots \times \mathcal{X}_0^{(g_k)}\right) \subset (\mathit{Crit})^k$$

Then $\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1,\cdots,k})$ *admits an orbifold Fredhom system with its virtual dimension given by*

$$2\langle c_1^G(TX), B\rangle + 2(n - \dim G)(1 - g_\Sigma) - 2\sum_{i=1}^{k} \iota_{CR}(\mathcal{X}_0^{(g_i)}, \mathcal{X}_0)$$

where g_Σ *is the genus of the Riemann surface* Σ, *and* $\iota_{CR}(\mathcal{X}_0^{(g_i)}, \mathcal{X}_0)$ *is the degree shift as introduced in* [11].

In Section 5, we also establish the compactness property for these L^2-moduli spaces of symplectic vortices on Σ with prescribed asymptotic behaviours. We show that there are two types of limiting vortices appearing in the compactification. The first type occurs as the bubbling phenomenon of pseudo-holomorphic spheres at interior points just as in the Gromov-Witten theory. To describe this type of limiting vortices, we introduce the usual weighted trees to classify the resulting topological type. The second type is due to the sliding-off of the Yang-Mills-Higgs energy along the cylindrical ends as happened in the instanton Floer theory. The combination of these two types of convergence sequences is called the weak chain convergence in instanton Floer theory in [14]. The choice of cylindrical metric on Σ is crucial in our study the compactness property in the sense that these are the only two types of limiting vortices appearing in the compactification of the L^2-moduli spaces of symplectic vortices on a cylindrical Riemann surface.

To describe the topological types appearing in the compactification, we introduce a notion of web of stable weighted trees of the type $(\Sigma; B)$ which consists of a principal tree Γ_0 with k-tails and a collections of ordered sequence of trees of finite length

$$\Gamma_i = \bigsqcup_{j=1}^{m} T_i(j)$$

for each tail $i = 1, \cdots, k$. See Definition 5.1 for a precise definition. Let $\mathcal{S}_{\Sigma;B}$ be the set of webs of stable weighted trees of the type $(\Sigma; B)$, which is a partially ordered finite set. For each $\Gamma \in \mathcal{S}_{\Sigma;B}$, we associate an L^2-moduli space $\mathcal{N}_\Gamma$ of symplectic vortices of type Γ. Let $\mathcal{N}_\Gamma((g_1), \cdots, (g_k))$ be the corresponding L^2-moduli space of symplectic vortices of type Γ with prescribed asymptotic data in

$$\mathcal{X}_0^{(g_1)} \times \cdots \times \mathcal{X}_0^{(g_k)} \subset (\text{Crit})^k.$$

Then the second main theorem of this paper is to show that the coarse L^2-moduli space of symplectic vortices on Σ can be compactified into a stratified topological space whose strata are labelled by a web of stable weighted trees in $\mathcal{S}_{\Sigma;B}$. In the following theorem, we use the notation $|\mathcal{N}|$ to denote the coarse space of an orbifold topological space $\mathcal{N}$.

Theorem B (Theorem 5.5) *Let Σ be a Riemann surface of genus g with k-cylindrical ends. The coarse L^2-moduli space $|\mathcal{N}_\Sigma(X, P, B)|$ can be compactified to a stratified topological space*

$$|\overline{\mathcal{N}}_\Sigma(X, P, B)| = \bigsqcup_{\Gamma \in \mathcal{S}_{\Sigma;B}} |\mathcal{N}_\Gamma|$$

such that the top stratum is $|\mathcal{N}_\Sigma(X, P, B)|$. Moreover, the coarse moduli space

$$|\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1,\cdots,k})|$$

with a specified asymptotic datum can be compactified to a stratified topological space

$$|\overline{\mathcal{N}}_\Sigma(X, P, B; \{(g_i)\}_{i=1,\cdots,k})| = \bigsqcup_{\Gamma \in \mathcal{S}_{\Sigma;B}} |\mathcal{N}_\Gamma((g_1), \cdots, (g_k))|.$$

Remark 1.1. Note that the evaluation map has its image in $I\mathcal{X}_0$, motivated by the definition of the usual Gromov-Witten invariants, our invariants will be defined on $H^*_{CR}(\mathcal{X}_0)$ in the sequel [9]. This is different from the Hamiltonian Gromov-Witten invariants defined earlier, as the invariants are defined on $H^*_G(X)$ in [13] and [36] . Hence the invariants we will define in [9] is essentially different from the usual HGW invariants. One may refer to §6 for further discussion.

We remark that the compactness properties of the moduli spaces of symplectic vortices have been studied earlier in [34], [37], [41], [42] and [40]. Under the assumption that X is a Kähler Hamiltonian G-manifold with semi-free action, the above compactness theorem has also been obtained by Venugopalan in [40] using a different approach.

We finish this paper in section 6 about an outlook of the future work in the sequels.

2. Review of symplectic vortices

In this section, we review some of basic facts for the symplectic vortices following [13] [32] and [34].

2.1. Symplectic vortex equations.

Let (X, ω) be a $2n$-dimensional symplectic manifold with a Hamiltonian action of a connected compact Lie group G

$$G \times X \longrightarrow X, \qquad (g, x) \mapsto gx,$$

and J is a G-invariant ω-compatible almost complex structure. Let $\mathfrak{g}$ be the Lie algebra of G with a G-invariant inner product $\langle \cdot, \cdot \rangle$. Recall that an action of G on X is Hamiltonian if there exists an equivariant map, called the moment map,

$$\mu : X \longrightarrow \mathfrak{g}$$

satisfying the defining property

$$d\mu_\xi = \omega(\tilde{\xi}, \cdot), \qquad \text{for any } \xi \in \mathfrak{g}.$$

Here the function μ_ξ is given by $\mu_\xi(x) = \langle \mu(x), \xi \rangle$, and $\tilde{\xi}$ is the vector field on X defined by the infinitesimal action of $\xi \in \mathfrak{g}$ on X

$$(\tilde{\xi} f)(x) = \frac{d}{dt}\Big|_{t=0} f\left(\exp(t\xi)x\right), \qquad \text{for} \quad f \in C^\infty(X).$$

Note that under this definition, $\widetilde{[\xi_1, \xi_2]} = -[\tilde{\xi}_1, \tilde{\xi}_2]$, that is, the infinitesimal action of the Lie bracket $[\xi_1, \xi_2]$ in $\mathfrak{g}$ is the negative of the Lie bracket of the vector fields $\tilde{\xi}_1$ and $\tilde{\xi}_2$. Note that the moment map is unique up to a shift by an element in $Z(\mathfrak{g})$ (the centre Lie subalgebra of $\mathfrak{g}$). See Chapter 2 in [25] for a detailed discussion on the geometry of moment maps.

Let $P \to \Sigma$ be a smooth (principal) G-bundle over a Riemann surface $(\Sigma, \mathrm{j}_\Sigma)$ (not necessarily compact). Let g_Σ be a Riemannian metric on Σ, $*_\Sigma$ be the associated Hodge star operator and ν_Σ be the volume form of (Σ, g_Σ) . Denote by $C^\infty_G(P, X)$ be the space of smooth G-equivariant maps $u : P \to X$ and by $\mathcal{A}(P)$ the space of smooth connections on P which is an affine space modelled $\Omega^1(\Sigma, ad\ P)$. Here $ad\ P = P \times_G \mathfrak{g}$ is the bundle of Lie algebras associated to the adjoint representation $ad : G \to GL(\mathfrak{g})$.

Denote the fiber bundle of P associated to the action of G on X by

$$\pi : Y = P \times_G X \longrightarrow \Sigma.$$

Then a smooth G-equivariant map $u : P \to X$ yields a section $\tilde{u} : \Sigma \to Y$. Note that any connection A on P induces splittings

$$TY \cong \pi^* T\Sigma \oplus T^{\text{vert}} Y. \tag{2.1}$$

Here $T^{\text{vert}}Y$ is the vertical tangent bundle of Y. The covariant derivative $d_A\tilde{u} \in \Omega^1(\Sigma, \tilde{u}^* T^{\text{vert}} Y)$ is derived from $d\tilde{u}$ as follows:

$$d_A \tilde{u} : T\Sigma \xrightarrow{d\tilde{u}} TY \xrightarrow{projection} T^{\text{vert}} Y.$$

For simplicity, we denote $d_A\tilde{u}$ by $d_A u$ as well, so $d_A u$ is a 1-form over Σ with values in $u^*T^{\mathrm{vert}}Y$.

The **symplectic vortex equations** on Σ are the following first order partial differential equations for pairs $(A, u) \in \mathcal{A}(P) \times C^\infty_G(P, X)$

$$\begin{cases} \bar{\partial}_{J,A}(u) = 0 \\ *_\Sigma F_A + \mu(u) = 0 \end{cases} \tag{2.2}$$

where F_A is the curvature of the connection A. The almost complex structures j_Σ and J define an almost complex structure J_A on TY under the decomposition (2.1). The first equation in (2.2) implies that u is a J_A-holomorphic section, as $\bar{\partial}_{J,A}(u)$ is the complex anti-linear part of $d_A u$,

$$\bar{\partial}_{J,A}(u) = \frac{1}{2}\left(d_A u + J \circ d_A u \circ \mathrm{j}_\Sigma\right) = 0 \tag{2.3}$$

in $\Omega^{0,1}(\Sigma, u^*T^{\mathrm{vert}}Y)$. For the second equation in (2.2), we remark that $\mu \circ u$ is a section of $ad\ P$ and the Hodge star operator defines a map

$$*_\Sigma : \Omega^2(\Sigma, ad\ P) \longrightarrow \Omega^0(\Sigma, ad\ P).$$

Using the Riemannian volume ν_Σ, the second equation in (2.2) is equivalent to

$$F_A + \mu(u)\nu_\Sigma = 0. \tag{2.4}$$

A solution (A, u) to (2.2) is called a symplectic vortex on Σ associated to a principal G-bundle P and a Hamiltonian G-space X. Two elements $w = (P, A, u)$ and $w' = (P', A', u')$ are called equivalent if there is a bundle isomorphism

$$\Phi : P' \to P$$

such that

$$\Phi^*(A, u) = (\Phi^* A, u \circ \Phi) = (A', u').$$

When P is evident in the context, we will omit P from the notation and simply call (A, u) for a symplectic vortex on Σ. As the symplectic vortex equations (2.2) on Σ for a fixed P is invariant under the action of gauge group $\mathcal{G}(P) = Aut(P)$, the moduli space of symplectic vortices on Σ is the set of solutions to (2.2) modulo the gauge transformations. We remark that P is an essential part of symplectic vortices, in particularly in the study of the compactifications of the moduli spaces of vortices.

There is an equivariant map $P \to EG$ classifying the principal G-bundle P. Together with the section $u : \Sigma \to P \times_G X$, they define to a continuous map

$$u_G : \Sigma \to X_G := EG \times_G X,$$

which in turn determines a degree 2 equivariant homology class $[u_G]$ in $H^G_2(X, \mathbb{Z})$ when Σ is closed. Denote by $\widetilde{\mathcal{M}}_\Sigma(X, B)$ the space of symplectic vortices on Σ associated to (P, X) with a fixed equivariant homology class in $B \in H^G_2(X, \mathbb{Z})$, that means,

$$\widetilde{\mathcal{M}}_\Sigma(X, B) = \{(A, u) | [u_G] = B, (A, u) \text{ satisfies the equations (2.2)}\}.$$

The quotient of $\widetilde{\mathcal{M}}_\Sigma(X, B)$ under the gauge group $\mathcal{G}(P)$-action

$$\mathcal{M}_\Sigma(X, B) = \widetilde{\mathcal{M}}_\Sigma(X, B)/\mathcal{G}(P)$$

is called the moduli space of symplectic vortices with a fixed homology class B.

A solution to (2.2) with a fixed $B \in H_2^G(X, \mathbb{Z})$ is an absolute minimizer (hence, a critical point) of the Yang-Mills-Higgs energy functional

$$E(A, u) = \int_\Sigma \frac{1}{2}(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2)\nu_\Sigma. \tag{2.5}$$

This is due to the fact (Proposition 3.1 in [12]) that for any $(A, u) \in \mathcal{A}(P) \times C_G^\infty(P, X)$,

$$E(A, u) = \int_\Sigma \left(|\bar{\partial}_{J,A}(u)|^2 + \frac{1}{2}| *_\Sigma F_A + \mu(u)|^2 \right) \nu_\Sigma + \int_\Sigma (u^*\omega - d\langle \mu(u), A\rangle). \tag{2.6}$$

Here, $u^*\omega - d\langle\mu(u), A\rangle$ is a horizontal and G-equivaraint 2-form on P and descends to a 2-form Σ, denoted by the same notation. On the other hand, $[\omega - \mu] \in H^2_G(X)$ is the equivariant cohomology class defined by the equivariant closed 2-form $\omega - \mu \in \Omega^2_G(X)$. The pairing $\langle [\omega - \mu], [u_G]\rangle$ is computed by

$$\langle [\omega - \mu], [u_G]\rangle = \int_\Sigma \big((d_A u)^*\omega - \langle \mu(u), F_A\rangle\big) = \int_\Sigma (u^*\omega - d\langle \mu(u), A\rangle).$$

Here $d_A u$ is a horizontal and G-equivaraint one-form on P with values in u^*TX and descends to a $u^*T^{\text{vert}}Y$-valued one form on Σ, see Proposition 3.1 in [12].

Remark 2.1. We remark that (2.6) is true for any surface Σ. In particular, when (A, u) is a symplectic vortex on Σ,

$$E(A, u) = \int_\Sigma (u^*\omega - d\langle \mu(u), A\rangle). \tag{2.7}$$

This is the crucial identity for us to define the action functional $\mathcal{L}$ in Section 3.

Remark 2.2. (1) If $G = U(1)$ the unit circle in $\mathbb{C}$ and $X = (\mathbb{C}^n, \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j)$ with the moment map $\mu : \mathbb{C}^n \to i\mathbb{R}$ given by

$$\mu(z_1, z_2, \cdots, z_n) = -\frac{i}{2}\sum_{j=1}^n |z_j|^2 + \frac{i}{2},$$

then symplectic vortex equation is a generalisation of the well-studied vortex equations (Cf. [26]). In particular, when Σ is compact and $X = \mathbb{C}$, Bradlow ([3]) showed that the moduli space of vortices on Σ with vortex number

$$d = \langle c_1(P \times_{U(1)} \mathbb{C}), [\Sigma]\rangle > 0$$

is non-empty if and only if $d <$ Vol$(\Sigma)/4\pi$, and is $Sym^d(\Sigma)$ (the d-th symmetric product of Σ).

(2) As observed in [12], the space $\mathcal{A}(P) \times C^\infty_G(P, X)$ is an infinite dimensional Fréchet manifold with a natural symplectic structure. The action of gauge group $\mathcal{G}(P)$ is Hamiltonian with a moment map

$$\mathcal{A}(P) \times C^\infty_G(P, X) \to C^\infty_G(\Sigma, ad\ P)$$

defined by $(A, u) \mapsto *F_A + \mu(u)$. Hence, the moduli space of symplectic vortices can be thought as a symplectic quotient if the space

$$\mathcal{S} = \{(A, u) | \bar{\partial}_{J,A}(u) = 0\}$$

is a symplectic submanifold of $\mathcal{A}(P) \times C^\infty_G(P, X)$. In practice, the space $\mathcal{S}$ is not a smooth submanifold in general. It still provides a good guiding principle for the development of Hamiltonian Gromov-Witten theory. See [1] [4] for some applications of this principle in similar contexts.

When $\Sigma = S^1 \times \mathbb{R}$ with the flat metric $(dt)^2 + (d\theta)^2$ and the standard complex structure $\mathrm{j}(\partial_t) = \partial_\theta$, with respect to a fixed trivialization of P, we can use the temporal gauge

$$A = d + \xi(\theta, t) d\theta, \qquad \text{for } \xi : S^1 \times \mathbb{R} \to \mathfrak{g},$$

to write the symplectic vortex equations (2.2) as

$$\begin{cases} \dfrac{\partial u}{\partial t} + J\left(\dfrac{\partial u}{\partial \theta} + \widetilde{\xi}(\theta, t)(u(x))\right) = 0 \\ \dfrac{\partial \xi}{\partial t} + \mu(u) = 0. \end{cases} \tag{2.8}$$

This is the downward gradient flow equation for a particular function on $C^\infty(S^1, X \times \mathfrak{g})$ defined in Section 3, where we will study this function in more details.

2.2. **Moduli spaces of symplectic vortices on a closed Riemann surface.**

In the study of the moduli space $\mathcal{M}_\Sigma(X, B)$, we need to develop certain Fredholm theory. This requires some Sobolev completion of the space

$$\mathcal{A}(P) \times C^\infty_{G,B}(P, X)$$

where $C^\infty_{G,B}(P, X) = \{u \in C^\infty_G(P, X) | [u_G] = B\}$. The Sobolev embedding theorem in dimension 2 leads to the $W^{1,p}$-Sobolev space for $p > 2$ so that the connections and maps involved are continuous. We denote the resulting Banach manifold by $\widetilde{\mathcal{B}}$ and the $W^{2,p}$-gauge transformation group by $\mathcal{G}(P)$.

Let $\widetilde{\mathcal{E}} \to \widetilde{\mathcal{B}}$ be the $\mathcal{G}(P)$-equivariant vector bundle whose fiber over (A, u) is given by

$$\widetilde{\mathcal{E}}_{(A,u)} = \Omega^{0,1}_{L^p}(\Sigma, u^*T^{\text{vert}}Y) \oplus \Omega^0_{L^p}(\Sigma, ad\ P).$$

Then the symplectic vortex equations (2.2) define a $\mathcal{G}(P)$-equivariant section

$$S(A, u) = \left(\bar{\partial}_{J,A}(u), *_\Sigma F_A + \mu(u)\right)$$

such that $\widetilde{\mathcal{M}}_\Sigma(X, B)$ is the zero set of this section. The vertical differential of this section

$$\mathcal{D}_{A,u} : \Omega^1_{W^{1,p}}(\Sigma, ad\ P) \oplus \Omega^0_{W^{1,p}}(\Sigma, u^*T^{\text{vert}}Y) \to \Omega^{0,1}_{L^p}(\Sigma, u^*T^{\text{vert}}Y) \oplus \Omega^0_{L^p}(\Sigma, ad\ P)$$

is given by the linearization operator of the symplectic vortex equations (2.2) at (A, u). Note that the linearization of the $\mathcal{G}(P)$-action at (A, u) is

$$L_{A,u} : \Omega^0_{W^{2,p}}(\Sigma, ad\ P) \to \Omega^1_{W^{1,p}}(\Sigma, ad\ P) \oplus \Omega^0_{W^{1,p}}(\Sigma, u^*T^{\text{vert}}Y)$$

is given by $L_{A,u}(\eta) = (-d_A\eta, \tilde{\eta}(u))$. It was shown in [13] that the operator $\mathcal{D}_{A,u} \oplus L^*_{A,u}$ is a Fredholm operator with real index given by

$$(n - \dim G)\chi(\Sigma) + 2\langle u^* \left(c_1(T^{\text{vert}}Y)\right), [\Sigma]\rangle.$$

Hence the triple $(\widetilde{\mathcal{B}}, \widetilde{\mathcal{E}}, S)$, modulo $\mathcal{G}(P)$, defines a Fredholm system

$$(\mathcal{B}, \mathcal{E}, S)$$

in the sense of [6]. The zero set of S is the moduli space $\mathcal{M}_\Sigma(X, B)$. The central issue in extracting invariants from $\mathcal{M}_\Sigma(X, B)$ is to establish virtual fundamental cycles as in [21] or a virtual system as in [8] for a compactified version of the moduli space $\mathcal{M}_\Sigma(X, B)$. This will be studied in [9].

3. Symplectic vortices on a cylinder $S^1 \times \mathbb{R}$

The symplectic vortex equations (2.8) on $S^1 \times \mathbb{R}$ in temporal gauge suggests that it is a gradient flow equation for an action functional on an infinite dimensional space. This functional has been studied in [12], [19] and [42]. After we describe the critical point set and the Hessian of this functional at critical points, we establish an inequality (Proposition 3.12) for this functional which plays a crucial role in analysing the asymptotic behaviour of an L^2 symplectic vortex on $S^1 \times [0, \infty)$. This crucial inequality is applied to show that a gradient flow line γ with a finite energy condition

$$E(\gamma) = \int_0^\infty \|\frac{\partial \gamma(t)}{\partial t}\|^2 dt < \infty$$

has a well-defined limit point, and converges exponentially fast to the limit point. Similar exponential decay estimates has also been obtained by in [34] and [42] using different methods.

3.1. **Action functional for symplectic vortices.** Let P_{S^1} be a principal G-bundle over S^1, and $\mathcal{A}_{S^1}$ be the space of smooth connections on P_{S^1} which is an affine space over $\Omega^1(S^1, \mathfrak{g})$. Since $C^\infty_G(P_{S^1}, X) \cong C^\infty(S^1, X)$, the set of connected components of $\mathcal{C}$ is identified with $\pi_1(X)$. For each $c \in \pi_1(X)$ we denote the component by $\mathcal{C}^c$.

Now choosing a trivialization $P_{S^1} \to S^1 \times G$ and the standard metric from $S^1 \cong \mathbb{R}/\mathbb{Z}$, we have the identification

$$\mathcal{C} = C^\infty_G(P_{S^1}, X) \times \mathcal{A}_{S^1} \cong C^\infty(S^1, X \times \mathfrak{g}).$$

We sometimes use the same notation to denote a map u in $C^\infty_G(P_{S^1}, X)$ and in $C^\infty(S^1, X)$ which should be clear in the context. We remark that the identification of $\mathcal{A}_{S^1}$ with $C^\infty(S^1, \mathfrak{g})$ is with respect to the trivial connection on P_{S^1}.

With respect to the Fréchet topology, $\mathcal{C}$ is a smooth manifold whose tangent space at (x, η) is

$$T_{(x,\eta)}\mathcal{C} = \Omega^0(S^1, x^*TX \times \mathfrak{g}),$$

the space of smooth sections of the bundle $x^*TX \times \mathfrak{g}$. Under the identification $\mathcal{C} = C^\infty(S^1, X \times \mathfrak{g})$, the full gauge group $LG = C^\infty(S^1, G)$ acts on $\mathcal{C}$ by

$$g \cdot (x, \eta) = (g^{-1}x, g^{-1}\frac{dg}{d\theta} + Ad_{g^{-1}}\eta). \tag{3.1}$$

Here we denote by $Ad : G \to GL(\mathfrak{g})$ the adjoint action of G on its Lie algebra $\mathfrak{g}$.

Let (x_0, η_0) and (x_1, η_1) be in a connected component of $\mathcal{C}$ and γ be a path

$$\gamma(t) = (x(t), \eta(t)) : I = [0, 1] \to \mathcal{C}$$

connecting (x_0, η_0) and (x_1, η_1). Then γ determines a pair

$$(u_\gamma, A_\gamma) \in C^\infty_G(P_{S^1} \times I, X) \times \mathcal{A}(P_{S^1} \times I).$$

Define the energy functional for this path γ as

$$E(\gamma) = \int_{S^1 \times I} \big((d_{A_\gamma} u_\gamma)^* \omega - \langle \mu(u_\gamma), F_{A_\gamma} \rangle\big). \tag{3.2}$$

Note that if the path γ satisfies the symplectic vortex equations (2.8) on $[0, 1] \times S^1$, then $E(\gamma)$ agrees with its Yang-Mills-Higgs energy. Using the coordinate (θ, t) for $S^1 \times I$, we can compute (cf. (2.7))

$$E(\gamma) = \int_{S^1 \times I} (x(t))^* \omega + \int_{S^1} (\langle \mu(x_0), \eta_0 \rangle - \langle \mu(x_1), \eta_1 \rangle) d\theta.$$

Lemma 3.1. *Under the identification $C^\infty_G(P_{S^1} \times I, X) \times \mathcal{A}(P_{S^1} \times I) \cong C^\infty(S^1 \times I, X \times \mathfrak{g})$, the energy function defined in (3.2) enjoys the following properties.*

(1) *For any $g \in LG$, let $g \cdot \gamma$ be the path obtained from the action of g, then*

$$E(\gamma) = E(g \cdot \gamma).$$

(2) *If γ_1 and γ_2 are homotopic paths relative to the boundary point (x_0, η_0) and (x_1, η_1), then $E(\gamma_1) = E(\gamma_2)$.*

Proof. (1) is obvious. We explain (2). The path $\gamma_1 \sharp (-\gamma_2)$ defines a pair (u, A) on a bundle P over $S^1 \times S^1$, then

$$E(\gamma_1) - E(\gamma_2) = \langle [\omega - \mu], [u_G] \rangle.$$

Since $\gamma_1 \sim \gamma_2$, $[u_G] = 0$. Hence $E(\gamma_1) = E(\gamma_2)$. □

We now define a (circle-valued) function on $\mathcal{C}$ as follows. For each component $\mathcal{C}^c$ we fix a based point (x_c, η_c). Given a point $(x, \eta) \in \mathcal{C}^c$, let $\gamma : [0, 1] \to \mathcal{C}^c$ be a path connecting (x_c, η_c) and (x, η). As above, this path can be written as a pair

$$(\tilde{x}, \tilde{\eta}) \in C^\infty_G(P_\Sigma, X) \times \mathcal{A}_\Sigma,$$

where $\Sigma = [0, 1] \times S^1$ and $\mathcal{A}_\Sigma$ is the space of connections on a principal G-bundle $P_\Sigma = P_{S^1} \times [0, 1]$. Then we define

$$\mathcal{L}_\Sigma(\tilde{x}, \tilde{\eta}) = -E(\tilde{x}, \tilde{\eta}). \tag{3.3}$$

For a different extension $(\Sigma, \tilde{x}', \tilde{\eta}')$, by the same argument in the proof of Lemma 3.1, we know that

$$\mathcal{L}_\Sigma(\tilde{x}', \tilde{\eta}') - \mathcal{L}_\Sigma(\tilde{x}, \tilde{\eta}) = \langle [\omega - \mu], [u_G] \rangle,$$

for some $[u_G] \in H_2^G(X, \mathbb{Z})$ defined by $(\tilde{x}\sharp(-\tilde{x}'), \eta\sharp(-\tilde{\eta}'))$. Recall that $\langle[\omega-\mu], \cdot\rangle$ is the homomorphism

$$\langle[\omega-\mu], \cdot\rangle : H_2^G(X, \mathbb{Z}) \longrightarrow \mathbb{R}.$$

The image of $\langle[\omega-\mu], \cdot\rangle$ consists of integer multiples of a fixed positive real number $N_{[\omega-\mu]}$. Hence, modulo $\mathbb{Z}N_{[\omega-\mu]}$, $\mathcal{L}_\Sigma(\tilde{x}, \tilde{\eta})$ descends to a well-defined function

$$\mathcal{L}(x, \eta) = \mathcal{L}_\Sigma(\tilde{x}, \tilde{\eta}) \mod (\mathbb{Z}N_{[\omega-\mu]}). \tag{3.4}$$

We denote by $\mathcal{L} : \mathcal{C} \to \mathbb{R}/\mathbb{Z}N_{[\omega-\mu]}$ the resulting circle-valued function.

Lemma 3.1 implies that the following action functional on $\mathcal{C}$ is well-defined.

Definition 3.2. Given a collection of based points $\{(x_c, \eta_c) | c \in \pi_1(X)\}$ for the connected components $\mathcal{C}$ labelled by $\pi_1(X)$, let $\tilde{\mathcal{C}}_{uni}$ be the associated universal cover of $\mathcal{C}$ defined by the homotopy paths to the based point. The action functional on $\tilde{\mathcal{L}} : \tilde{\mathcal{C}}_{uni} \to \mathbb{R}$ is defined by (3.3) for a homotopy path from $(x, \eta) \in \mathcal{C}$ to the based point for the connected component. The induced function

$$\mathcal{L} : \mathcal{C} \longrightarrow \mathbb{R}/\mathbb{Z}N_{[\omega-\mu]}$$

is called the action functional on $\mathcal{C}$.

Remark 3.3. There is a minimal covering space of $\mathcal{C}$, denoted by $\tilde{\mathcal{C}}$, such that the action functional $\mathcal{L}$ can be lifted to a $\mathbb{R}$-valued function $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{C}}$ and the following diagram commutes

$$\begin{array}{ccc} \tilde{\mathcal{C}}_{uni} & \xrightarrow{\tilde{\mathcal{L}}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{C}} & \xrightarrow{\tilde{\mathcal{L}}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\mathcal{L}} & \mathbb{R}/\mathbb{Z}N_{[\omega-\mu]}. \end{array} \tag{3.5}$$

We write an element of $\tilde{\mathcal{C}}$ in the fiber over $(x, \eta) \in \mathcal{C}$ as an equivalent class a path connecting (x, η) to the based point of the connected component.

As the covering map $\tilde{\mathcal{C}}_{uni} \to \mathcal{C}$ is a local diffeomorphism, the differential and the Hessian operator of $\mathcal{L}$ can be calculated by the Fréchet derivatives of $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{C}}_{uni}$ or $\tilde{\mathcal{C}}$. For this purpose, we introduce an L^2-inner product on the tangent bundle $T\mathcal{C}$, that is, for $(v_1, \xi_1), (v_2, \xi_2) \in T_{(x,\eta)}\mathcal{C}$,

$$\langle (v_1, \xi_1), (v_2, \xi_2) \rangle = \int_{S^1} (\omega(v_1, Jv_2) + \langle \xi_1, \xi_2 \rangle)\, d\theta. \tag{3.6}$$

Proposition 3.4. *With respect to the L^2-inner product, the L^2-gradient of $\mathcal{L}$ is given by*

$$\nabla \mathcal{L}(x, \eta) = \left(J(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x), \mu(x) \right). \tag{3.7}$$

Hence, the critical point set is define by the equations

$$\frac{\partial x}{\partial \theta} + \tilde{\eta}_x = 0, \qquad \mu(x) = 0. \tag{3.8}$$

Proof. Let $(D\mathcal{L})_{(x,\eta)}$ be the first Fréchet derivative of $\mathcal{L}$, that is, for any $(v,\xi) \in T_{(x,\eta)}\mathcal{C}$,

$$\begin{aligned}(D\mathcal{L})_{(x,\eta)}(v,\xi) &= \frac{\partial}{\partial s}\Big|_{s=0}\mathcal{L}(\exp_x(sv), \eta + s\xi)\\ &= -\int_{S^1}\omega(v, \frac{\partial x}{\partial\theta})d\theta + \int_{S^1}\left(\langle d\mu_x(v), \eta\rangle + \langle\mu(x),\xi\rangle\right)d\theta\\ &= \int_{S^1}\left(\omega(\frac{\partial x}{\partial\theta} + \tilde{\eta}_x, v) + \langle\mu(x),\xi\rangle\right)d\theta \qquad (3.9)\\ &= \int_{S^1}\left(\omega(J(\frac{\partial x}{\partial\theta} + \tilde{\eta}_x), Jv) + \langle\mu(x),\xi\rangle\right)d\theta\\ &= \langle(J(\frac{\partial x}{\partial\theta} + \tilde{\eta}_x), \mu(x)), (v,\xi)\rangle.\end{aligned}$$

Hence, L^2-gradient of $\mathcal{L}$ at (x,η) is given by (3.7). The proposition is proved. □

Remark 3.5. The gradient equation $\nabla\mathcal{L}(x,\eta) = 0$ can be thought as the Euler-Lagrange equation for the action functional $\mathcal{L}$. Moreover, the downward gradient flow equation of $\mathcal{L}$ on $\mathcal{C}$

$$\frac{\partial}{\partial t}\left(x(t),\eta(t)\right) = -\left(J(\frac{\partial x}{\partial\theta} + \tilde{\eta}_x), \mu(x)\right) \qquad (3.10)$$

is exactly the symplectic vortex equation (2.8) on $S^1 \times \mathbb{R}$ in temporal gauge.

Before we proceed further, let us investigate the gauge invariance of the action functional $\tilde{\mathcal{L}}$.

Lemma 3.6. *The action functional $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{C}}$ is invariant under the action of L_0G, the connected component of LG of the identity.*

Proof. We show that $\tilde{\mathcal{L}}$ is constant on any orbit of L_0G, equivalently, for any path $\gamma(t)$ in $\tilde{\mathcal{C}}$ through $\gamma(0) = [x,\eta,[\tilde{x}]]$ along the L_0G-orbit, we need to prove

$$\frac{\partial}{\partial t}\Big|_{t=0}\tilde{\mathcal{L}}(\gamma(t)) = 0.$$

We can assume that the tangent vector defined by $\gamma(t)$ is $(-\tilde{\xi}_x, \frac{\partial\xi}{\partial\theta} + [\eta,\xi])$ for $\xi \in L\mathfrak{g} = C^\infty(S^1,\mathfrak{g})$. Then the calculation in (3.9) implies that

$$\begin{aligned}&\frac{\partial}{\partial t}\Big|_{t=0}\tilde{\mathcal{L}}(\gamma(t))\\ =\;& \langle\nabla\tilde{\mathcal{L}}(x,\eta), (-\tilde{\xi}_x, \frac{\partial\xi}{\partial\theta} + [\eta,\xi])\rangle\\ =\;& \int_{S^1}\left(\omega(\frac{\partial x}{\partial\theta} + \tilde{\eta}_x, -\tilde{\xi}_x) + \langle\mu(x), \frac{\partial\xi}{\partial\theta} + [\eta,\xi]\rangle\right)d\theta\\ =\;& \int_{S^1}\left(\omega(\frac{\partial x}{\partial\theta}, -\tilde{\xi}_x) - \omega(\tilde{\eta}_x,\tilde{\xi}_x) + \langle\mu(x), \frac{\partial\xi}{\partial\theta}\rangle + \omega(\tilde{\eta}_x,\tilde{\xi}_x)\right)d\theta\\ =\;& \int_{S^1}\left(\langle d\mu_x(\frac{\partial x}{\partial\theta}),\xi\rangle + \langle\mu(x),\frac{\partial\xi}{\partial\theta}\rangle\right)d\theta\\ =\;& \int_{S^1} d\langle\mu(x),\xi\rangle = 0.\end{aligned}$$

Here we applied the equality: $\omega(\tilde{\eta}_x,\tilde{\xi}_x) = \langle\mu(x),[\eta,\xi]\rangle$. This completes the proof. □

Given $(x,\eta) \in$ Crit$(\mathcal{L})$ and $g \in LG$, by property (1) in Lemma 3.1, $g\cdot(x,\eta)$ is also a critical point. That means, the critical point set Crit$(\mathcal{L})$ is LG-invariant. Note that the based gauge group

$$\Omega G = \{g \in LG | g(1) = e, \text{the identity element in } G\}$$

acts on $\mathcal{C}$ freely. In the next lemma, we provide a description of the critical point set modulo the group ΩG on the set theoretical level. For this purpose, we consider $C^\infty(S^1, \mathfrak{g})$ as the space of connections on the trivial bundle $S^1 \times G$, where we treat $\xi \in C^\infty(S^1,\mathfrak{g})$ as a $\mathfrak{g}$-valued 1-form $\xi d\theta$ on S^1. Then there is a holonomy map

$$Hol : C^\infty(S^1,\mathfrak{g}) \longrightarrow G.$$

Note that $Hol : C^\infty(S^1,\mathfrak{g}) \longrightarrow G$ is the universal principal ΩG-bundle with ΩG-action on $C^\infty(S^1,\mathfrak{g})$ given by the gauge transformation.

Lemma 3.7. *Modulo the based gauge group ΩG, the holonomy map Hol : Crit$(\mathcal{L})/\Omega G \longrightarrow G$ defines a fibration over G whose fiber over $g \in G$ is $(\mu^{-1}(0))^g$, the g-fixed point set in $\mu^{-1}(0)$. That is, we have*

$$\text{Crit}(\mathcal{L})/\Omega G = \bigsqcup_{g\in G} (\mu^{-1}(0))^g.$$

Proof. Given $(x(\theta),\eta(\theta)) \in$ Crit$(\mathcal{L})$, then $x(\theta) \in C^\infty(S^1, \mu^{-1}(0))$ and

$$\dot{x}(\theta) = -\tilde{\eta}_{x(\theta)}.$$

Solving the above ordinary differential equation over the interval $x : [0, 2\pi] \to X$ with an initial condition $x(0) = p \in \mu^{-1}(0)$, we get a unique solution. The condition of x being a loop in X is that η satisfies the condition

$$x(2\pi) = Hol(\eta)\cdot p = p.$$

Hence, we get

$$\text{Crit}(\mathcal{L}) \cong \{(p,\eta)|p \in \mu^{-1}(0), \eta \in C^\infty(S^1,\mathfrak{g}), Hol(\eta)\cdot p = p\}.$$

The action of ΩG on the right hand side is given by the gauge transformation on the second component. Note that the holonomy map $Hol : C^\infty(S^1,\mathfrak{g}) \to G$ is a principal ΩG-bundle. Any ΩG-orbit at η is determined by $Hol(\eta)$. So we get the first identification,

$$\text{Crit}(\mathcal{L})/\Omega G \cong \{(p,g)|p \in \mu^{-1}(0), g \in G_p\}.$$

Now it is easy to see that the holonomy map on $\{(p,g)|p \in \mu^{-1}(0), g \in G_p\}$ is just the projection to the second factor, whose fiber at g is $(\mu^{-1}(0))^g$. So the lemma is established. □

Remark 3.8. Set-theoretically, the critical point set Crit$(\mathcal{L})/LG$ can be identified with

$$I[\mu^{-1}(0)/G] \cong \big(\mu^{-1}(0)/G\big) \quad \sqcup \bigsqcup_{(e)\neq(g)\in\mathcal{C}(G)} (\mu^{-1}(0))^g/C(g),$$

the inertia groupoid arising from the action groupoid $[\mu^{-1}(0)/G] = \mu^{-1}(0) \rtimes G$. Here $\mathcal{C}(G)$ is the set of conjugacy class in G with a fixed function $\mathcal{C}(G) \to G$ sending (g) to $g \in (g)$, and $C(g)$ is the centralizer of g in G.

(1) If G acts on $\mu^{-1}(0)$ freely, then Crit$(\mathcal{L})/LG \cong \mu^{-1}(0)/G$ is the symplectic quotient (also called the reduced space) of (X, ω).
(2) If G-action on $\mu^{-1}(0)$ is only locally free, then Crit$(\mathcal{L})/LG$ has an orbifold structure which is the inertia orbifold of the symplectic orbifold $\mathcal{X}_0 = [\mu^{-1}(0)/G]$.
(3) If 0 is not a regular value of μ, then $\mu^{-1}(0)/G$ admits a symplectic orbifold stratified space, labelled by orbit types ([38]).

For the rest of the paper, we assume that 0 is a regular value of μ so the critical point set Crit$(\mathcal{L})/LG$ can be endowed with a symplectic orbifold structure, the inertia orbifold of $\mathcal{X}_0 = [\mu^{-1}(0)/G]$. We shall write the inertia orbifold of $\mathcal{X}_0$ as

$$I\mathcal{X}_0 = \bigsqcup_{(g)} \mathcal{X}_0^{(g)}$$

where (g) runs over the conjugacy class in G with non-empty fixed points in $\mu^{-1}(0)$. Note that for a non-trivial conjugacy class (g), $\mathcal{X}_0^{(g)}$ is often called a twisted sector of $\mathcal{X}_0$, which is diffeomorphic to the orbifold arising from the action of $C(g)$ on $\mu^{-1}(0)^g$ for a representative g in the conjugacy class (g). Here $C(g)$ denotes the centralizer of g in G.

Now we introduce the standard Banach completion of $\mathcal{C}$. This Banach set-up is also crucial for the Fredholm analysis of the gradient flow lines of $\mathcal{L}$, equivalently, the symplectic vortices on $S^1 \times \mathbb{R}$.

Consider the Banach manifold

$$\mathcal{C}_{1,p} = \{(x, \eta) \in W^{1,p}(S^1, X \times \mathfrak{g})\}.$$

Here $p \geq 2$, so (x, η) is a continuous map. The tangent space of $\mathcal{C}_{1,p}$ at (x, η) is

$$T_{(x,\eta)}\mathcal{C}_{1,p} = W^{1,p}(S^1, x^*TX \times \mathfrak{g}),$$

consisting of $W^{1,p}$-sections of the bundle $x^*TX \times \mathfrak{g}$. The gauge group for this Banach manifold is the $W^{2,p}$-loop group

$$\mathcal{G}_{2,p} = W^{2,p}(S^1, G)$$

acting on $\mathcal{C}_{1,p}$ in the way as in (3.1). Denote by $\mathcal{G}^0_{2,p}$ the based $W^{2,p}$-loop group. Then the action of $\mathcal{G}^0_{2,p}$ on $\mathcal{C}_{1,p}$ is free.

By the Sobolev embedding theorem, $T_{(x,\eta)}\mathcal{C}_{1,p}$ is contained in the L^2-tangent space

$$T^{L^2}_{(x,\eta)}\mathcal{C}_{1,p} = L^2(S^1, x^*TX \times \mathfrak{g}),$$

the space of L^2-section of the bundle $x^*TX \times \mathfrak{g}$ on which the L^2-inner product (3.6) is well-defined and the L^2-gradient $\nabla\mathcal{L}$ is a L^2-tangent vector field on $\mathcal{C}_{1,p}$. Modulo $W^{2,p}$ gauge transformation, the equations (3.8) is a first order elliptic equation. By the standard elliptic regularity, we know that modulo gauge transformation, the critical point set Crit$(\mathcal{L})$ consists of smooth loops in $\mathcal{C}_{1,p}$. By the same argument, a solution to the L^2 gradient flow equation (3.10) of $\mathcal{L}$ on $\mathcal{C}_{1,p}$ for

$$(x(t), \eta(t)) : \mathbb{R} \longrightarrow \mathcal{C}_{1,p}$$

is gauge equivalent to a symplectic vortex on $S^1 \times \mathbb{R}$ with finite energy. For simplicity, we assume that $p = 2$.

We can choose a representative for any critical point in Crit($\mathcal{L}$) according to its holonomy. If a critical point has a trivial holonomy, then using a based gauge transformation, it is gauge equivalent to a critical point of the form

$$(x, 0) \in \mu^{-1}(0) \times \mathfrak{g}.$$

If a critical point has a non-trivial holonomy $g = \exp(-2\pi\eta)$ for $\eta \in \mathfrak{g}$, then it is gauge equivalent to a critical point of the form

$$(\exp(-\theta\eta)x, \eta)$$

for $\theta \in [0, 2\pi]$ and $x \in (\mu^{-1}(0))^g$.

Let $(x, \eta) \in$ Crit($\mathcal{L}$), the Hessian operator of $\mathcal{L}$ at (x, η)

$$\mathcal{Q}_{(x,\eta)} : W^{1,2}(S^1, x^*TX \times \mathfrak{g}) \longrightarrow L^2(S^1, x^*TX \times \mathfrak{g})$$

is defined by the second Fréchet derivative

$$\langle (v_1, \xi_1), \mathcal{Q}_{(x,\eta)}(v_2, \xi_2)\rangle = D^2\mathcal{L}_{(x,\eta)}((v_1, \xi_1), (v_2, \xi_2))$$

for $(v_1, \xi_1), (v_2, \xi_2) \in W^{1,2}(S^1, x^*TX \times \mathfrak{g})$.

Proposition 3.9. *At the critical point of the form $(x, \eta) = (\exp(-\theta\eta)x_0, \eta)$ for $\eta \in \mathfrak{g}$ and $x_0 \in (\mu^{-1}(0))^g$ with $g = \exp(-2\pi\eta)$, the Hessian operator is given by*

$$\mathcal{Q}_{(x,\eta)}(v, \xi) \mapsto \Big(J(L_{-\tilde{\eta}}v + \tilde{\xi}_x), d\mu_x(v)\Big).$$

Here $L_{-\tilde{\eta}}v$ is the Lie derivative of v along the vector field $-\tilde{\eta}$. In particular, if $x_0 \in \mu^{-1}(0)$, the Hessian operator

$$\mathcal{Q}_{(x_0,0)}(v, \xi) = \left(J(\frac{\partial v}{\partial\theta} + \tilde{\xi}_{x_0}), d\mu_{x_0}(v)\right).$$

Proof. For $(v_1, \xi_1), (v_2, \xi_2) \in T_{(x,\eta)}\mathcal{C} = \Gamma_{C^\infty}(S^1, x^*TX \times \mathfrak{g})$, denote by $\bar{v}_1$ the parallel transport of v_1 along a path $\exp_x(sv_2)$.

$$\begin{aligned} & D^2\mathcal{L}_{(x,\eta)}((v_1, \xi_1), (v_2, \xi_2)) \\ = & \frac{d}{ds}\Big|_{s=0} \big((D\mathcal{L})_{\exp_x(sv_2),\eta+s\xi_2}(\bar{v}_1, \xi_1)\big) \\ = & \int_{S^1} \frac{d}{ds}\Big|_{s=0} \left(\omega_{\exp_x(sv_2)}(\frac{\partial(\exp_x(sv_2))}{\partial\theta} + (\widetilde{\eta + s\xi_2})_{\exp_x(sv_2)}, \bar{v}_1) + \langle\mu(\exp_x(sv_2)), \xi_1\rangle\right) d\theta. \end{aligned}$$

At the critical point $(x_0, 0)$ for $x_0 \in \mu^{-1}(0)$, we have $(v_1, \xi_1), (v_2, \xi_2) \in C^\infty(S^1, T_{x_0}X \times \mathfrak{g})$, and

$$\begin{aligned} & D^2\mathcal{L}_{(x_0,0)}((v_1, \xi_1), (v_2, \xi_2)) \\ = & \int_{S^1} (\omega(\frac{\partial v_2}{\partial\theta} + \tilde{\xi}_2, v_1) + \langle d\mu_x(v_2), \xi_1\rangle) d\theta \\ = & \langle (v_1, \xi_1), (J(\frac{\partial v_2}{\partial\theta} + \tilde{\xi}_2), d\mu_{x_0}(v)\rangle. \end{aligned}$$

So the Hessian operator

$$\mathcal{Q}_{(x_0,0)}(v,\xi) = \left(J(\frac{\partial v}{\partial \theta} + \tilde{\xi}_{x_0}), d\mu_{x_0}(v)\right).$$

When the critical point (x,η) has non-vanishing $\dot{x}(\theta)$, we can continue the calculation as follows

$$\begin{aligned} & D^2\mathcal{L}_{(x,\eta)}((v_1,\xi_1),(v_2,\xi_2)) \\ = & \int_{S^1} \Big(\omega(\nabla_{\dot{x}(\theta)} v_2, v_1) d\theta + \omega_x(\nabla_{v_2}\tilde{\eta} + \tilde{\xi}_2, v_1) d\theta + \langle d\mu_x(v_2), \xi_1\rangle\Big) \\ = & \int_{S^1} \Big(\omega\big(\nabla_{\dot{x}(\theta)} v_2 + \tilde{\xi}_2 + \nabla_{v_2}\tilde{\eta}, v_1\big) + \langle d\mu_x(v_2), \xi_1\rangle\Big)\, d\theta. \end{aligned}$$

At the critical point $(x,\eta) = (\exp(-\theta\eta)x_0, \eta)$, the vector field $\dot{x}(\theta)$ along the loop $x = \exp(-\theta\eta)x_0$ agrees with $-\tilde{\eta}$, then

$$\nabla_{\dot{x}(\theta)} v_2 + \nabla_{v_2}\tilde{\eta} = L_{-\tilde{\eta}} v_2$$

as vector fields along the loop $\exp(-\theta\eta)x_0$. Hence,

$$D^2\mathcal{L}_{(x,\eta)}((v_1,\xi_1),(v_2,\xi_2)) = \langle (v_1,\xi_1), (J(L_{-\tilde{\eta}}v_2 + \tilde{\xi}_2), d\mu_{x_0}(v))\rangle.$$

The Hessian operator at this critical point as given by $(v,\xi) \mapsto \Big(J(L_{-\tilde{\eta}}v + \tilde{\xi}_x), d\mu_x(v)\Big)$.

□

Denote by $\mathcal{C}^{\#}_{1,2}$ the submanifold of $\mathcal{C}_{1,2}$ consisting of elements with finite stabilisers under the gauge group $\mathcal{G}_{2,2}$. Then

$$\mathcal{B}^{\#}_{1,2} = \mathcal{C}^{\#}_{1,2}/\mathcal{G}_{2,2}$$

is a smooth Banach orbifold. Let $(x,\eta) \in \mathcal{C}^{\#}_{1,2}$ and let

$$\mathcal{G}_{(x,\eta)} = \{g \in \mathcal{G}_{2,2} | g\cdot(x,\eta) = (x,\eta)\}$$

be the stabiliser group of (x,η), a finite group in $\mathcal{G}_{2,2}$. Then the tangent space at $\gamma = [x,\eta] \in \mathcal{B}^{\#}_{1,2}$ in orbifold sense is a $\mathcal{G}_{(x,\eta)}$-quotient of the Banach space

$$\{(v,\xi) \in T_{(x,\eta)}\mathcal{C}_{1,2} | d\mu_x(Jv) + \frac{\partial \xi}{\partial \theta} + [\eta,\xi] = 0\}.$$

The action function $\mathcal{L}$ descends locally to a circle-valued function on the Banach orbifold $\mathcal{B}^{\#}_{1,2}$. The L^2-gradient vector field $\nabla\mathcal{L}$ defines an orbifold L^2-gradient vector field on $\mathcal{B}^{\#}_{1,2}$. As 0 is a regular value of the moment map μ, the critical point set

$$\text{Crit} = \text{Crit}(\mathcal{L})/\mathcal{G}_{2,2} \subset \mathcal{B}^{\#}_{1,2}$$

is a smooth orbifold, diffeomorphic to the inertia orbifold of the symplectic reduction

$$\mathcal{X}_0 = [\mu^{-1}(0)/G].$$

Each component (called a twisted sector) is a finite dimensional suborbifold of $\mathcal{B}^{\#}_{1,2}$. The next proposition implies that t the functional $\mathcal{L}$ satisfies the Morse-Bott property.

Proposition 3.10. *Assume that* 0 *is a regular value of the moment map* μ. *Let* (x, η) *be a critical point of Crit*$(\mathcal{L})$ *which is gauge equivalent to* $(\exp(-\theta\eta)x_0, \eta)$ *for* $x_0 \in \mu^{-1}(0)$ *and* $\eta \in \mathfrak{g}$ *such that* x_0 *is a fixed point of* $g = \exp(-2\pi\eta)$ *, then the kernel of Hessian operator Hess*$(\mathcal{L})_{(x,\eta)}$ *of* $\mathcal{L}$ *at* (x, η)

$$\mathcal{Q}_{(x,\eta)} : W^{1,2}(S^1, x^*TX \times \mathfrak{g}) \longrightarrow L^2(S^1, x^*TX \times \mathfrak{g}),$$

modulo the image of infinitesimal action of $W^{2,2}$*-gauge group action at* (x, η)*, is isomorphic to*

$$T_{x_0}(\mu^{-1}(0))^g / T_{x_0}(C(g) \cdot x_0),$$

where $C(g) \cdot x_0$ *is the orbit space of the centralizer of* g *in* G.

Proof. The Hessian operator $\mathcal{Q}_{(x,\eta)}$ of $\mathcal{L}$ at (x, η)

$$\mathcal{Q}_{(x,\eta)} : W^{1,2}(S^1, x^*TX \times \mathfrak{g}) \longrightarrow L^2(S^1, x^*TX \times \mathfrak{g}),$$

is given by $(v, \xi) \mapsto \Big(J(L_{-\tilde{\eta}}v + \tilde{\xi}_x), d\mu_x(v)\Big)$. By a direct calculation, we have for any $\zeta \in L^2(S^1, \mathfrak{g})$,

$$(L_{\tilde{\eta}}\tilde{\zeta})(x(\theta)) = \widetilde{\dot{\zeta}(\theta)}_{x(\theta)} + \widetilde{[\eta, \zeta(\theta)]}_{x(\theta)}. \tag{3.11}$$

where $\dot{\zeta}(\theta) = \dfrac{d\zeta}{d\theta}$, and $\widetilde{[\eta, \zeta(\theta)]}_{x(\theta)}$ is the infinitesimal action of $[\eta, \zeta(\theta)] \in \mathfrak{g}$ at $x(\theta)$. So

$$\mathcal{Q}_{(x,\eta)}(\tilde{\zeta}_x, \dot{\zeta} + [\eta, \zeta(\theta)]) = 0,$$

which means, the tangent space of $W^{2,2}$-gauge orbit at (x, η),

$$T_{(x,\eta)}(\mathcal{G}_{2,2} \cdot (x, \eta)) = \{(\tilde{\zeta}_x, \dot{\zeta} + [\eta, \zeta(\theta)]) | \zeta \in L^2(S^1, \mathfrak{g})\}$$

is contained the kernel of $\mathcal{Q}_{(x,\eta)}$. This implies that the Hessian operator $\mathcal{Q}_{(x,\eta)}$ is well-defined on the quotient space of $W^{1,2}(S^1, x^*TX \times \mathfrak{g})$ by the the tangent space of $W^{2,2}$-gauge orbit at (x, η). Let $(v, \xi) \in W^{1,2}(S^1, x^*TX \times \mathfrak{g})$ be in the kernel of $\mathcal{Q}_{(x,\eta)}$. Then we have the following two equations for (v, ξ)

(1) $-L_{\tilde{\eta}}v + \tilde{\xi} = 0.$

(2) $d\mu_{x(\theta)}(v) = 0.$

We can take the following gauge fixing condition

$$d\mu_{x(\theta)} \circ J(v) = 0,$$

which is equivalent to the condition that $v(\theta) \in T_{x(\theta)}X$ is orthogonal to the infinitesimal action of G at $x(\theta)$ with respect to the Riemannian metric on X defined by ω and J. Note that second equation implies that $v(\theta) \in T_{x(\theta)}(\mu^{-1}(0))$. So $v(\theta)$ is orthogonal to the infinitesimal action of G at $x(\theta)$. With this gauge fixing condition, the first equation becomes

$$L_{\tilde{\eta}}v = 0, \qquad \widetilde{\xi(\theta)}_{x(\theta)} = 0,$$

as the metric on X is invariant under the action of G. The G-action on $\mu^{-1}(0)$ is locally free means $\xi = 0$. Then the equation $L_{\tilde{\eta}}v = 0$ says that v is invariant under the flow Φ_t^η generated by $-\tilde{\eta}$. Therefore v is determined by the value of $v(0) \in T_{x_0}(\mu^{-1}(0))$. As the flow Φ_t^η is given by the action of $\exp(-t\eta)$ on X, we get

$$v(0) = v(2\pi) = (\Phi_{2\pi}^\eta)_* v(0).$$

This implies that $v(0) \in T_{x_0}(\mu^{-1}(0))^g$. Notice that $v(0) \in T_{x_0}\mu^{-1}(0)$ is orthogonal to the infinitesimal action of G at $x(0)$. Therefore $v(0) \in T_{x_0}(\mu^{-1}(0))^g$ is orthogonal to the infinitesimal action of $C(g)$ at $x(0)$ as $(\mu^{-1}(0))^g$ is only invariant under the action of $C(g)$. This completes the proof of the proposition. □

Remark 3.11. Proposition 3.10 can be also proved using the gauge fixing condition for $(v, \xi) \in W^{1,2}(S^1, x^*TX \times \mathfrak{g})$ given by

$$d\mu_x(Jv) + \frac{\partial \xi}{\partial \theta} + [\eta, \xi] = 0, \tag{3.12}$$

that is, (v, ξ) is L^2-orthogonal to the tangent space of the gauge orbit at the critical point (x, η). Denote by $\mathcal{T}_{(x,\eta)}$ the L^2-completion of the subspace of the space of smooth section of $x^*TX \times \mathfrak{g}$ satisfying the gauge fixing condition (3.12). Then the Hessian operator

$$\mathcal{Q}_{(x,\eta)} : \mathcal{T}_{(x,\eta)} \longrightarrow \mathcal{T}_{(x,\eta)}$$

is a closed, essentially self-adjoint, Fredholm operator with discrete real spectrum of finite multiplicity. The domain of $\mathcal{Q}_{(x,\eta)}$ is the $W^{1,2}$-completion of $\mathcal{T}_{(x,\eta)}$, which is the subspace of $W^{1,2}(S^1, x^*TX \times \mathfrak{g})$ with the gauge fixing condition (3.12). This is due to the fact that $\mathcal{Q}_{(x,\eta)}$ on $\mathcal{T}_{(x,\eta)}$ is equivalent to a first order elliptic differential operator (called the extended Hessian operator). This operator on $L^2(S^1, \mathfrak{g} \times x^*TX \times \mathfrak{g})$ is obtained by combining the infinitesimal action of gauge transformations and the gauging fixing into the Hessian opeator, sending (ζ, v, ξ) to $(d\mu_x(Jv) + \frac{\partial \xi}{\partial \theta} + [\eta, \xi], (-\tilde{\zeta}_x, \frac{d\zeta}{d\theta} + [\eta, \zeta]) + \mathcal{Q}_{(x,\eta)}(v, \xi))$.

In the next proposition, we establish the inequality for $\tilde{\mathcal{L}}$ which is important in analyzing gradient flow lines near any critical point.

Proposition 3.12. *For any x in a critical manifold Crit$(\tilde{\mathcal{L}}) \subset \tilde{\mathcal{C}}_{1,2}$, there exist a constant δ and a small $W^{1,2}$ ϵ-ball neighborhood $B_\epsilon(x)$ of x in $\tilde{\mathcal{C}}_{1,2}$ such that*

$$\|\nabla \tilde{\mathcal{L}}(y)\|_{L^2}^2 \geq \delta |\tilde{\mathcal{L}}(y) - \tilde{\mathcal{L}}(x)|$$

for any $y \in B_\epsilon(x)$. Here ϵ and δ are independent of x (as $\mu^{-1}(0)$ is compact).

We remark that though the above inequality is written in a small ϵ-ball of a critical point of $\tilde{\mathcal{L}}$ on $\tilde{\mathcal{C}}_{1,2}$, in fact the inequality still holds in a sufficiently small ϵ-ball of a critical point of $\mathcal{L}$ on $\mathcal{C}_{1,2}$. This is due to the local diffeomorphism between $\tilde{\mathcal{C}}_{1,2}$ and $\mathcal{C}_{1,2}$. That is, the difference function $\mathcal{L}(y) - \mathcal{L}(x)$ makes sense for $y \in B_\epsilon(x)$ when ϵ is small.

Proof. By the gauge invariance, we only need to verify the inequality at critical points of the form

$$(\exp(-\theta\eta)x_0, \eta)$$

for $x_0 \in \mu^{-1}(0)$ and $\eta \in \mathfrak{g}$ such that $\exp(2\pi\eta) \in G_{x_0}$ (a finite group). Assume that the holonomy $Hol(\eta) = \exp(2\pi\eta)$ is trivial, then $(\exp(-\theta\eta)x_0, \eta)$ is gauge equivalent to $(x_0, 0)$. A small neighborhood of $(x_0, 0)$ in $\tilde{\mathcal{C}}_{1,2}$ can be identified with a small ball in

$$T_{(x_0,0)}\mathcal{C}_{1,2} = W^{1,2}(S^1, T_{x_0}X \times \mathfrak{g})$$

centred at the origin with radius ϵ for a sufficiently small ϵ. Let $(u, \xi) \in W^{1,2}(S^1, T_{x_0}X \times \mathfrak{g})$ satisfying $\|(u, \xi)\|_{W^{1,2}} < \epsilon$. The corresponding point in $\tilde{\mathcal{C}}_{1,2}$ is

$$(\tilde{u}(\theta), \xi(\theta)) = (\exp_{x_0}(u(\theta)), \xi(\theta)).$$

With respect to the canonical metric defined by $\omega(\cdot, J(\cdot))$, we have the following orthogonal decomposition

$$T_{x_0}X \cong T_{x_0}\mu^{-1}(0) \oplus \nu_{x_0}$$

where $\nu_{x_0} = \{J(\tilde{\zeta}_{x_0}) | \zeta \in \mathfrak{g}\}$. This decomposition provides a local coordinate of X at x_0, denoted by (u_0, u_μ). Under this coordinate, vector fields will be parallel transported to the origin along the geodesic rays, and then be treated as vectors in $T_{x_0}X$. In particular, $\tilde{\xi}_{\tilde{u}}(\theta) \in T_{\tilde{u}}X$ will be considered as a tangent vector in $T_{x_0}X$.

Write $\dfrac{du}{d\theta} = (\dot{u}_0, \dot{u}_\mu)$. With a choice of gauge transformation, we can assume that

$$\langle \dot{u}_0(\theta), \tilde{\xi}_{x_0}(\theta) \rangle = 0$$

for any $\theta \in S^1$ by gauging away the component of $u_0(\theta)$ along the infinitesimal action of the gauge group. Now we calculate

$$\|\nabla\tilde{\mathcal{L}}(\tilde{u}, \xi)\|_{L^2}^2 = \int_{S^1} \left(\left\| J\left(\frac{du}{d\theta} + \tilde{\xi}_{\tilde{u}}\right) \right\|^2 + \|\mu(\tilde{u})\|^2 \right) d\theta$$

as follows. Note that

$$\begin{aligned}
&\int_{S^1} \left\| J\left(\frac{du}{d\theta} + \tilde{\xi}_{\tilde{u}}\right) \right\|^2 d\theta \\
= &\int_{S^1} \|(\dot{u}_0 + \tilde{\xi}_{x_0}) + \dot{u}_\mu + (\tilde{\xi}_{\tilde{u}} - \tilde{\xi}_{x_0})\|^2 d\theta \\
\geq &\int_{S^1} \left(\frac{1}{2}\|(\dot{u}_0 + \tilde{\xi}_{x_0})\|^2 + \frac{1}{2}\|\dot{u}_\mu\|^2 - \|\tilde{\xi}_{\tilde{u}} - \tilde{\xi}_{x_0}\|^2 \right) d\theta \\
= &\int_{S^1} \left(\frac{1}{2}\|(\dot{u}_0\|^2 + \frac{1}{2}\|\tilde{\xi}_{x_0})\|^2 + \frac{1}{2}\|\dot{u}_\mu\|^2 - \|\tilde{\xi}_{\tilde{u}} - \tilde{\xi}_{x_0}\|^2 \right) d\theta \\
\geq &\frac{1}{2}\|\dot{u}\|_{L^2}^2 + (\frac{1}{2} - \epsilon C) \int_{S^1} \|\tilde{\xi}_{x_0}\|^2 d\theta
\end{aligned}$$

for some constant $C > 0$, and

$$\|\tilde{\xi}_{x_0}\|^2 = \omega(\tilde{\xi}_{x_0}, J_{x_0}\tilde{\xi}_{x_0}) > C_0|\xi|^2$$

for some constant $C_0 > 0$ due to the locally free action of G on $\mu^{-1}(0)$. Hence, for a sufficiently small ϵ, we obtain

$$\|\nabla\tilde{\mathcal{L}}(\tilde{u}, \xi)\|_{L^2}^2 \geq \frac{1}{2}\|\dot{u}\|_{L^2}^2 + \|\mu(\tilde{u})\|_{L^2}^2 + (\frac{1}{2} - \epsilon C)C_0\|\xi\|_{L^2}^2.$$

On the other hand, let $(\tilde{u}(\theta, t), \tilde{\xi}) = (\exp_{x_0}(tu(\theta)), t\xi)$ for $t \in [0,1]$ be a path connecting $(\tilde{u}, \xi)$ and $(x_0, 0)$, then

$$\tilde{\mathcal{L}}(u, \xi) - \tilde{\mathcal{L}}(x_0, 0) = -\int_{S^1\times[0,1]} \tilde{u}^*\omega + \int_{S^1} \langle \mu(\tilde{u}), \xi\rangle d\theta.$$

A direct calculation gives

$$|\int_{S^1\times[0,1]} \tilde{u}^*\omega| = |\frac{1}{2}\int_{S^1} \omega_{x_0}(u, \dot{u})| d\theta \leq C_1(\|u_0\|^2_{L^2} + \|\dot{u}\|^2_{L^2})$$

for some constants C_1. Assume that $\int_{S^1} u_0 d\theta = 0 \in T_{x_0}\mu^{-1}(0)$, the Wirtinger's inequality implies that there exists a constant C_2 depending only ω_{x_0} and $J(x_0)$ such that

$$\|u_0\|^2_{L^2} \leq C_2\|\dot{u}_0\|^2_{L^2}.$$

Therefore, we have

$$|\tilde{\mathcal{L}}(\tilde{u}, \xi) - \tilde{\mathcal{L}}(x_0, 0)| \leq C_3\|u_0\|^2_{L^2} + \frac{1}{2}(\|\mu(\tilde{u})\|^2_{L^2} + \|\xi\|^2_{L^2}\|)$$

for some constant $C_3 > 0$. For a properly chosen $\delta > 0$ and sufficiently small ϵ, we have

$$\|\nabla\tilde{\mathcal{L}}(\tilde{u}, \xi)\|^2_{L^2} \geq \delta|\tilde{\mathcal{L}}(\tilde{u}, \xi) - \tilde{\mathcal{L}}(x_0, 0)|.$$

If $\int_{S^1} u_0 d\theta \neq 0$, we can replace x_0 and $x_0' = x_0 + \int_{S^1} u_0 d\theta$. Then we have

$$\tilde{\mathcal{L}}(x_0, 0) = \tilde{\mathcal{L}}(x_0', 0).$$

The above calculation applied to $(x_0', 0)$ implies

$$\left\|\nabla\tilde{\mathcal{L}}(\tilde{u}, \xi)\right\|^2_{L^2} \geq \delta|\tilde{\mathcal{L}}(\tilde{u}, \xi) - \tilde{\mathcal{L}}(x_0', 0)|.$$

So the inequality has been proved for any critical point which is gauge equivalent to $(x_0, 0)$. The above argument can be adapted for a critical point gauge equivalent to $(\exp(-\theta\eta)x_0, \eta)$ by lifting technique: since $Hol(\eta)$ is of finite order, say k, then we consider a k covering $S^1 \to S^1$, then the pull-back of connection has a trivial holonomy. Then we can use the above argument to get the required estimate. □

At the end of this subsection, we discuss the energy of the gradient flow line. Let $\gamma = (\tilde{x}, \tilde{\eta}) : [a, b] \to \mathcal{C}_{1,p}$ be a path connecting (x_1, η_1) and (x_2, η_2). Let $(\tilde{x}_1, \tilde{\eta}_1)$ be a path γ_1 connecting the based point to (x_1, η_1); then we set $(\tilde{x}_2, \tilde{\eta}_2)$ be the path $\gamma_2 = \gamma_1\sharp\gamma$. As in Remark 3.3, we treat $(\tilde{x}_1, \tilde{\eta}_1)$ and $(\tilde{x}_2, \tilde{\eta}_2)$ as elements in $\tilde{\mathcal{C}}$.

Lemma 3.13. *Suppose that $\gamma = \gamma(t) : [a, b] \to \mathcal{C}_{1,p}$ is a gradient flowline of $\mathcal{L}$. Then the following quantities are equal:*

(1) $\tilde{\mathcal{L}}(\tilde{x}_1, \tilde{\eta}_1) - \tilde{\mathcal{L}}(\tilde{x}_2, \tilde{\eta}_2)$
(2) $-\int_{S^1\times[a,b]} \tilde{x}^*\omega + \int_{S^1} (\langle\mu(x_1), \eta_1\rangle - \langle\mu(x_2), \eta_2\rangle) d\theta$
(3) *the Yang-Mills-Higgs energy $E(\tilde{x}, \tilde{\eta})$;*
(4) $\int_a^b \left\|\frac{\partial\gamma(t)}{\partial t}\right\|^2_{L^2} dt.$

Proof. Recall that a path $\gamma = (x, \eta)$ is a gradient flow line of $\nabla\mathcal{L}$ if it satisfies the equations

$$\frac{\partial}{\partial t}(x(t), \eta(t)) = -\left(J(\frac{\partial x}{\partial \theta} + \tilde{\eta}_x), \mu(x)\right).$$

Since $(\tilde{x}, \tilde{\eta})$ solves the symplectic vortex equation,

$$E(\tilde{x}, \tilde{\eta}) = -\int_{S^1\times[a,b]} \tilde{x}^*\omega + d\langle\mu(\tilde{x}), \tilde{\eta}\rangle.$$

This implies that (1)=(2)=(3). Now we show that (1)=(4).

$$\tilde{\mathcal{L}}(\tilde{x}_2, \tilde{\eta}_2) - \tilde{\mathcal{L}}(\tilde{x}_1, \tilde{\eta}_1) = \int_a^b \frac{d}{dt}\mathcal{L}(\tilde{x}(t), \tilde{\eta}(t))dt = \int_a^b \langle\nabla\mathcal{L}, \frac{d\gamma}{dt}\rangle dt = \int_a^b \left\|\frac{\partial\gamma(t)}{\partial t}\right\|_{L^2}^2 dt.$$

□

3.2. **Asymptotic behaviour of finite energy symplectic vortices on a cylinder.**

In this subsection, we establish the existence of a limit point for any gradient flow line

$$\gamma : [0, \infty) \to \mathcal{C}_{1,p}$$

with finite energy $E(\gamma)$. Then by Lemma 3.13, we have

$$\int_0^\infty \left\|\frac{\partial\gamma(t)}{\partial t}\right\|_{L^2}^2 dt = \int_0^\infty \|\nabla\mathcal{L}(\gamma(t))\|_{L^2}^2 dt = E(\gamma) < \infty. \tag{3.13}$$

Theorem 3.14. *Let $\gamma : [0, \infty) \to \mathcal{C}_{1,p}$ be a gradient flow line of $\mathcal{L}$ with finite energy. Then there exists a unique critical point $(x_\infty, \eta_\infty) \in Crit(\mathcal{L})$ and constants $\delta, C > 0$ such that the L^2-distance*

$$dist_{L^2}\big(\gamma(T), (x_\infty, \eta_\infty)\big) \leq Ce^{-\delta T}$$

for any sufficient large T. Here the constant δ is the half of the constant δ in Proposition 3.12.

Proof. **Step 1.** For any sequence $\{\gamma(t_i) | \lim_{i\to\infty} t_i = \infty\}$, we show that there exists a convergent subsequence, still denoted by $\{\gamma(t_i)\}$, such that up to gauge transformations in $\mathcal{G}_{2,p}$, the sequence $\{\gamma(t_i)\}$ converges to a critical point y_∞ of $\mathcal{L}$ in the C^∞-topology.

Let $(u_i, A_i) = \gamma(t)$ be the symplectic vortex on $S^1 \times [-1, 1]$ in temporal gauge, obtained from $\gamma : [t_i - 1, t_i + 1] \to \mathcal{C}_{1,p}$. Then we have

$$\lim_{i\to\infty} E(u_i, A_i) = 0,$$

where the energy $E(u_i, A_i)$ agrees with the Yang-Mills-Higgs energy

$$E(u_i, A_i) = \int_{S^1\times[-1,1]} \frac{1}{2}\left(|d_{A_i}u_i|^2 + |F_{A_i}|^2 + |\mu(u_i)|^2\right) d\theta dt.$$

Applying the standard regularity result and Uhlenbeck compactness, see Theorem 3.2 in [13], there exists a sequence of $W^{2,p}$-gauge transformations g_i of $P_{S^1} \times [-1, 1]$ such that the sequence

$$g_i \cdot (u_i, A_i)$$

has a C^∞-convergent subsequence. Let (u_∞, A_∞) be the limit, then (u_∞, A_∞) satisfies the following equations

$$F_{A_\infty} = 0, \qquad d_{A_\infty}u_\infty = 0, \qquad \mu(u_\infty) = 0. \tag{3.14}$$

We can find a smooth gauge transformation h of $P_{S^1} \times [-1,1]$ such that $h \cdot (u_\infty, A_\infty)$ is in temporal gauge. So we can write

$$h \cdot (u_\infty, A_\infty) = (x_\infty(\theta, t), \eta_\infty(\theta, t) d\theta)$$

as a path in $\mathcal{C}_{1,p}$. Then the equations (3.14) become

$$\begin{cases} \dfrac{\partial \eta_\infty(\theta,t)}{\partial t} = 0, \dfrac{\partial x_\infty(\theta,t)}{\partial t} = 0 \\ \dfrac{\partial x_\infty}{\partial \theta} + \tilde{\eta}_\infty(x_\infty) = 0, \mu(x_\infty) = 0. \end{cases}$$

These equations imply that $(x_\infty, \eta_\infty) = h \cdot (u_\infty, A_\infty) \in \mathrm{Crit}(\mathcal{L})$ and

$$\lim_{i\to\infty} (hg_i) \cdot (u_i, A_i) = (x_\infty, \eta_\infty)$$

in the C^∞-topology. Hence, up to gauge transformations in $\mathcal{G}_{2,p}$, the subsequence $\{\gamma(t_i)\}$ converges to a critical point (x_∞, η_∞) of $\mathcal{L}$ in the C^∞-topology. We denote it by y_∞.

Step 2. Set $\gamma^i = g_i \gamma$. We claim that there exists t_i such that $\gamma^i(t) \in B_\epsilon(y_\infty)$ for $t \geq t_i$. Here ϵ and the δ below ara the same constants given in Proposition 3.12.

If not, for each i there exists $s_i > t_i$ such that the path $\gamma^i(t), t_i \leq t \leq s_i$ locates in $B_\epsilon(y_\infty)$ and $\gamma^i(s_i) \in \partial B_\epsilon(y_\infty)$. Applying Proposition 3.12, we have

$$\frac{d\big(\mathcal{L}(\gamma(t)) - \mathcal{L}(y_\infty)\big)^{1/2}}{dt} = -\frac{\|\nabla(\mathcal{L}(\gamma(t))\|^2_{L^2}}{2\big(\mathcal{L}(\gamma(t)) - \mathcal{L}(y_\infty)\big)^{1/2}} \leq -\frac{\delta^{1/2}}{2} \|\nabla(\mathcal{L}(\gamma(t) - \mathcal{L}(y_\infty))\|_{L^2}.$$

Therefore,

$$\begin{aligned} \mathrm{dist}_{L^2}(\gamma^i(t_i), \gamma^i(s_i)) &\leq \int_{t_i}^{s_i} \|\frac{\partial \gamma^i(t)}{\partial t}\|_{L^2} dt = \int_{t_i}^{s_i} \|\nabla(\mathcal{L}(\gamma^i(t) - \mathcal{L}(y_\infty))\|_{L^2} dt \\ &\leq -2\delta^{-1/2} \int_{t_i}^{s_i} \frac{d}{dt}((\mathcal{L}(\gamma^i(t) - \mathcal{L}(y_\infty))^{1/2}) dt \\ &\leq 2\delta^{-1/2}((\mathcal{L}(\gamma^i(t_i) - \mathcal{L}(y_\infty))^{1/2} - (\mathcal{L}(\gamma^i(s_i) - \mathcal{L}(y_\infty))^{1/2}). \end{aligned}$$

As $i \to \infty$, this goes to 0. On the other hand, by Step 1, there exists h_i such that $h_i\gamma^i(s_i)$ uniform converges to some critical point y'_∞. Since $\gamma^i(s_i) \in B_\epsilon(y_\infty)$ and $h_i\gamma^i(s_i)$ uniformly converges to y'_∞, h_i is uniformly bounded at least in $C^{1,\alpha}$ for some $\alpha > 0$. This means that there exists a subsequence of h_i that converges. We may relabel the sequence and assume that h_i converges to h. We conclude that $\gamma^i(s_i)$ converges to $h^{-1}y'_\infty$. Therefore,

$$\mathrm{dist}_{L^2}(\gamma^i(t_i), \gamma^i(s_i)) \to \mathrm{dist}_{L^2}(y_\infty, h^{-1}y'_\infty) = 0.$$

This implies that $y_\infty = h^{-1}y'_\infty$. However, $\gamma^i(s_i)$ is on the boundary of the ball $B_\epsilon(y_\infty)$, this is impossible. The contradiction implies that $\gamma^i(t) \in B_\epsilon(y_\infty)$ for $t \geq t_i$.

Step 3: From Step 2, suppose that $\gamma^i(t)$ locates in $B_\epsilon(y_\infty)$ when t sufficiently large. Reset y_∞ to be $g_i^{-1}y_\infty$. Then we may assume that $\gamma(t)$ locates in $B_\epsilon(y_\infty)$ when t large. Now we show that

$$\mathrm{dist}_{L^2}(\gamma(t), y_\infty) \leq Ce^{-\delta t}$$

for t large.

We can assume that for $t > T_0$, $\gamma(t) \in B_\epsilon(y_\infty)$ so that the crucial inequality in Proposition 3.12 can be applied to get

$$\frac{d\big(\mathcal{L}(\gamma(t)) - \mathcal{L}(y_\infty)\big)^{1/2}}{dt} = -\frac{\|\nabla(\mathcal{L}(\gamma(t))\|^2_{L^2}}{2\big(\mathcal{L}(\gamma(t)) - \mathcal{L}(y_\infty)\big)^{1/2}} \leq -\frac{\delta}{2}\big(\mathcal{L}(\gamma(t)) - \mathcal{L}(y_\infty)\big)^{1/2}.$$

Hence, for any $t > T_0$, we have

$$\big(\mathcal{L}(\gamma(t)) - \mathcal{L}(y_\infty)\big)^{1/2} \leq e^{-\frac{\delta}{2}(t-T_0)}\big(\mathcal{L}(\gamma(T_0)) - \mathcal{L}(y_\infty)\big)^{1/2}.$$

That is, for $t > T_0$

$$\mathrm{dist}_{L^2}\big(\gamma(t), y_\infty\big) \leq 2\delta^{-1/2}e^{-\frac{\delta}{2}(t-T_0)}\big(\mathcal{L}(\gamma(T_0)) - \mathcal{L}(y_\infty)\big)^{1/2}.$$

With $C = 2\delta^{-1/2}e^{\frac{\delta}{2}T_0}\big(\mathcal{L}(\gamma(T_0)) - \mathcal{L}(y_\infty)\big)^{1/2}$, we get the exponential decay estimate for $\mathrm{dist}_{L^2}\big(\gamma(t), y_\infty\big)$.

□

Remark 3.15. By a similar calculation as in the above proof, one can establish the following exponential decay for the Yang-Mills-Higgs energy of a finite energy gradient flow line $\gamma : [0,\infty) \to \mathcal{C}_{1,p}$, that is, there exist constants $\delta, C > 0$ such that

$$\int_T^\infty \|\nabla\mathcal{L}(\gamma(t))\|^2_{L^2} dt \leq Ce^{-\delta T}$$

for a sufficiently large T. Moreover, let y_∞ be the limit of $\gamma(t)$ at infinity, by a gauge transformation, we may assume that $y_\infty \in$ Crit, then for any $k \in \mathbb{N}$, there exist $C, \delta > 0$ such that

$$|\nabla^k\gamma(t)| \leq Ce^{-\delta t} \tag{3.15}$$

for t sufficiently large. To get the above point-wise estimate, we apply the elliptic regularity to the symplectic vortex $\gamma|_{[T-2,T+2]\times S^1}$ for a sufficiently large T to get a C^k-estimate

$$\|g\cdot\gamma\|_{C^k} \leq C.$$

for some constant $C > 0$ and any $k \in \mathbb{N}$. Write $\gamma = (\alpha, u)$, then the curvature F_α and $\mu(u)$ are gauge invariant and hence bounded. Then (3.15) follows from applying the standard elliptic estimates to the gradient flow equations. We also remark that the decay rate δ can be chosen such that δ is smaller than the minimum absolute value of non-zero eigenvalues of the Hessian operator of $\mathcal{L}$ at y_∞.

4. L^2-MODULI SPACE OF SYMPLECTIC VORTICES ON A CYLINDRICAL RIEMANN SURFACE

In this section, we consider the symplectic vortices of finite energy on a Riemann surface Σ with cylindrical end. For simplicity, Σ is assumed to have just one end, isometrically diffeomorphic to a half cylinder $S^1 \times [0,\infty)$. Let K be a compact set of Σ such that $\Sigma\backslash K$ is isometrically diffeomorphic to $S^1 \times (1,\infty)$ with the flat metric $d\theta^2 + dt^2$. Let P be a principal G-bundle over Σ with a fixed trivialization over the cylindrical end $S^1 \times [0,\infty)$. Let $\mathcal{N}_\Sigma(X, P)$ be the moduli space of symplectic vortices with finite energy associated to P and a closed Hamiltonian G-manifold (X, ω) with an ω-compatible G-invariant almost complex structure J. It is the space of gauge equivalence classes of

$$(A, u) \in \mathcal{A}(P) \times C^\infty_G(P, X)$$

satisfying the symplectic vortex equations (2.2) and with the property that the Yang-Mills-Higgs energy (Cf. (2.5)) is finite. For any $p > 2$, by the elliptic regularity, $\mathcal{N}_\Sigma(X, P)$ is the moduli space of the symplectic vortex equations (2.2) for

$$(A, u) \in \widetilde{\mathcal{B}}_{W^{1,p}_{\rm loc}} = \mathcal{A}_{W^{1,2}_{\rm loc}}(P) \times W^{1,p}_{{\rm loc},G}(P, X),$$

the space of $W^{1,p}_{\rm loc}$-connections on P and $W^{1,p}_{\rm loc}$-equivariant maps from P to X such that

$$E(A, u) = \int_\Sigma \frac{1}{2}(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2)\nu_\Sigma < \infty$$

modulo the the action of the group $\mathcal{G}_{W^{2,p}_{loc}}(P)$ of all $W^{2,p}_{\rm loc}$ gauge transformations on P.

Given a finite energy symplectic vertex (u, A), restricted to the cylindrical end $S^1 \times [0, \infty)$, (u, A) is gauge equivalent to a gradient flow line of $\mathcal{L}$ with finite energy. Then by Theorem 3.14, we know that there is a unique asymptotic limit $(A_\infty, u_\infty) \in \text{Crit}(\mathcal{L})$. Modulo a gauge transformation in $W^{2,p}(S^1, G)$, (A_∞, u_∞) is gauge equivalent to

$$(\eta, \exp(-\theta\eta) \cdot x_\infty) \in \text{Crit}(\mathcal{L})$$

where $\eta \in \mathfrak{g}$ is treated as a connection $d + \eta d\theta$ on the principal G-bundle over S^1 with respect to a fixed trivialization, $x_\infty \in \big(\mu^{-1}(0)\big)^g$ for $g = \exp(2\pi\eta) \in G$ of finite order m. Therefore, there is an asymptotic limit map

$$\partial_\infty : \mathcal{N}_\Sigma(X, P) \longrightarrow \text{Crit}, \tag{4.1}$$

where Crit is the critical point set modulo the gauge transformations as Section 3. We remark that Crit is diffeomorphic to the inertia orbifold $I\mathcal{X}_0$ associated to the reduced symplectic orbifold $\mathcal{X}_0 = [\mu^{-1}(0)/G]$, as we assume that 0 is a regular value of the moment map μ.

We first prove the continuity for the asymptotic limit map (4.1).

Proposition 4.1. *Let Σ be a Riemann surface Σ with one cylindrical end, P be a principal G-bundle over Σ and $\mathcal{N}_\Sigma(X, P)$ be the moduli space of symplectic vortices with finite energy associated to P and a closed Hamiltonian manifold (X, ω). Then the asymptotic limit of symplectic vortices in $\mathcal{N}_\Sigma(X, P)$ define a continuous map*

$$\partial_\infty : \mathcal{N}_\Sigma(X, P) \longrightarrow \textit{Crit} \cong I\mathcal{X}_0.$$

Proof. Let $[(u, A)] \in \mathcal{N}_\Sigma(X, P)$ and $\partial_\infty([(u, A)]) = [x_0] \in (\mu^{-1}(0))^g/C(g)$ for an element $g = \exp(2\pi\eta_0)$ of finite order. Fix an open neighborhood V of $[x_0]$ in $(\mu^{-1}(0))^g/C(g)$. We need to find an open neighborhood $U \subset \mathcal{N}_\Sigma(X, P)$ of $[(u, A)]$ such that

$$\partial_\infty(U) \subset V.$$

Let $\tilde{\mathcal{V}}$ be a $\mathcal{G}_{2,p}(S^1)$-invariant open neighborhood of $(\exp(-\theta\eta_0)x_0, \eta_0)$ in $\mathcal{C}_{1,p}$ such that $\tilde{\mathcal{V}} \cap \mathrm{Crit}(\mathcal{L})$ is mapped to a subset of V under the identification

$$\mathrm{Crit}(\mathcal{L})/\mathcal{G}_{2,p}(S^1) \cong I\mathcal{X}_0.$$

Denote by $\tilde{\mathcal{U}}$ the solutions $(u, A) \in \widetilde{\mathcal{B}}_{W^{1,p}_{\mathrm{loc}}(\Sigma)}$ to the symplectic vortex equations with finite energy such that for sufficiently large T, the restriction of (u, A) to $S^1 \times [T, \infty)$ is gauge equivalent to an element in $\tilde{\mathcal{V}}$. Then $U = \tilde{\mathcal{U}}/\mathcal{G}_{W^{2,p}(\Sigma)}$ is an open neighborhood of $[(u, A)]$ in $\mathcal{N}_\Sigma(X, P)$ and $\partial_\infty(U) \subset V$. □

For any $[(u, A)] \in \mathcal{N}_\Sigma(X, P)$, there is a canonical degree 2 equivariant homology class in $H_2^G(X, \mathbb{Z})$ described as follows. Assume that $\partial_\infty([(u, A)]) \in \mathcal{X}_0^{(g)}$, the twisted sector of $\mathcal{X}_0$ defined by the conjugacy class of g in G. So $\partial_\infty([(u, A)])$ can be represented by $(\exp(-\theta\eta) \cdot x_0)$ for $x_0 \in \left(\mu^{-1}(0)\right)^g$, and $g = \exp(2\pi\eta)$ is of order m. We can identify the cylinder $S^1 \times [0, \infty)$ with a unit disc $\mathbb{D}^* = \mathbb{D} - \{0\}$ in $\mathbb{C}$ using the biholomorphic coordinate change $(i\theta, t) \mapsto e^{-(t+i\theta)}$. Then the cylindrical surface Σ become a punctured Riemann surface. Denote this punctured Riemann surface by Σ^*. Let P^* be the associated principal G-bundle over Σ^*. Note that P^* has a fixed trivialization near the puncture. As the connection A_∞ has a non-trivial holonomy g of order m, we can construct an orbifold principal G-bundle P_{orbi} over the orbifold Riemann surface Σ_{orbi} such that A can be extended to a connection on P_{orbi}. The orbifold Riemann surface Σ_{orbi} is obtained by gluing Σ^* and the orbifold disc $\mathbb{D}/\mathbb{Z}_m$, so Σ_{orbi} is the closure of Σ^* with one orbifold singular point of order m at the puncture. Note that over the punctured disc $\mathbb{D}^*$, there is a Z_m-equivariant map

$$\varphi : \mathbb{D}^* \times G \longrightarrow \mathbb{D}^* \times G$$

sending (z, h) to $(z^m, \exp(-2\pi\eta)h)$. Over $\mathbb{D}^*$, the map φ identifies the $\mathbb{Z}_m$-equivariant principal G-bundle on the left hand side $\mathbb{D} \times G \longrightarrow \mathbb{D}$ where $\mathbb{Z}_m$-action on $\mathbb{D} \times G$ is given by

$$e^{2\pi i/m}(z, h) = (e^{2\pi i/m} z, \exp(2\pi\eta)h)$$

with the trivial G-bundle on the right hand side. The orbifold principal G-bundle P_{orbi} is obtained by gluing P^* over Σ^* with the $\mathbb{Z}_m$-equivariant principal G-bundle by the gluing map φ. The trivial connection with the constant section x_0 define a symplectic vortex on the $\mathbb{Z}_m$-equivariant G-bundle over $\mathbb{D}$. We can extend the symplectic vortex (A, u) on (Σ, P, X) to a pair $(\tilde{A}, \tilde{u})$ on $(\Sigma_{orbi}, P_{orbi}, X)$ where $\tilde{A}$ is a connection on P_{orbi} and $\tilde{u} : P_{orbi} \to X$ is a smooth G-equivariant map. Hence, with the classifying map $P_{orbi} \to EG$, $\tilde{u}$ gives rise to a degree 2 equivariant homology class in $H_2(X_G, \mathbb{Z})$. For simplicity, we still denote this class by $[u_G]$ which is called the **homology class** of (A, u). Then by a direct calculation, the energy of (A, u) is

$$E(A, u) = \langle [\omega - \mu], [u_G] \rangle.$$

Fix an equivariant homology class $B \in H_2(X_G, \mathbb{Z})$ such that $\langle [\omega - \mu], B \rangle > 0$. Let $\mathcal{N}_\Sigma(X, P, B)$ be the moduli space of symplectic vortices on Σ associated to (P, X) with the homology class B. Then $\mathcal{N}_\Sigma(X,P,B) \subset \mathcal{N}_\Sigma(X,P)$ and there is a continuous asymptotic limit map

$$\partial_\infty : \mathcal{N}_\Sigma(X,P,B) \longrightarrow \text{Crit} \cong I\mathcal{X}_0.$$

4.1. **Fredholm theory for L^2-moduli space of symplectic vortices.** To understand the moduli space $\mathcal{N}_\Sigma(X,P,B)$, we need to introduce the weighted Sobolev space for the fiber of the asymptotic limit map

$$\partial_\infty : \mathcal{N}_\Sigma(X,P,B) \longrightarrow \text{Crit} \cong I\mathcal{X}_0.$$

Any symplectic vortex $[(u,A)] \in \mathcal{N}_\Sigma(X,P,B)$ decays exponentially to its asymptotic limit at a rate $\delta > 0$ for some δ as Theorem 3.14. Note that Crit is compact, so we can choose a constant δ such that restricted to the cylindrical end $[S^1 \times [0,\infty)$, $[(u,A)] \in \partial_\infty^{-1}(y_\infty)$ decays exponentially to its limit at the rate δ for any $y_\infty \in$ Crit. We fix such a δ throughout this section.

Fix a smooth function $\beta : \Sigma \to [0,\infty)$ such that the follow conditions hold:

(1) On $S^1 \times [1,\infty)$, $\beta(\theta,t) = t$ is the coordinate function on the cylinder.
(2) $\beta = 0$ on $\Sigma \setminus \{S^1 \times [0,\infty)\}$.
(3) $\beta|_{S^1\times[0,1]}$ is an increasing function.

The weighted $W^{k,p}$-norm on a compact support section ξ of an Euclidean vector bundle V over Σ with a covariant derivative ∇ is defined by

$$\|\xi\|_{W^{k,p}_\delta} = \left(\int_\Sigma e^{\delta\beta}\big(|\xi|^p + |\nabla(\xi)|^p + \cdots + |\nabla^p(\xi)|^p\big) d\nu_\Sigma \right)^{1/p}.$$

We denote $W^{k,p}_\delta(\Sigma, E)$ the completion of all compact support sections of E with respect to the weighted $W^{k,p}$-norm, which is also called the Banach space of $W^{k,p}_\delta$-sections of E. When $k=0$, we simply denote by $L^p_\delta(\Sigma,E)$ the $W^{0,p}$-sections of E.

Let $\mathcal{X}_0^{(g)}$ be a twisted sector corresponding to the locally free action of $C(g)$ on $(\mu^{-1}(0))^g$ for $g = \exp(2\pi\eta) \in G$ with finite order. Let $\mathcal{N}_\Sigma(X,P,B;(g))$ be the subset of $\mathcal{N}_\Sigma(X,P,B)$ consisting of symplectic vortices $[(A,u)]$ such that

$$\partial_\infty([A,u]) \in \mathcal{X}_0^{(g)} \subset \text{Crit}.$$

Note that the critical manifold corresponding to $\mathcal{X}_0^{(g)}$ is diffeomorphic to the orbifold defined by the action of constant $C(g)$-valued gauge transformations on the space

$$\{(d+\eta d\theta, \exp(-\theta\eta)x) | x \in (\mu^{-1}(0))^g\} \cong (\mu^{-1}(0))^g.$$

Define $\widetilde{\mathcal{B}}_\delta(P,B,(g))$ be the subspace of $\widetilde{\mathcal{B}}_{W^{1,p}_{loc}}$ consisting of elements (A,u) with the following property:

(1) on $S^1\times[1,\infty)$, $A - d - \eta d\theta \in W^{1,p}_\delta(S^1\times[1,\infty), \Lambda^1\otimes ad\,P)$,
(2) there exist $x \in (\mu^{-1}(0))^g$ and a sufficiently large T such that $u|_{S^1\times[T,\infty)} = \exp_{u_x}(v)$. Here $u_x(\theta,t) = \exp(-\theta\eta)x$ and $v \in W^{1,p}_\delta(S^1\times[T,\infty), u_x^*TX)$.
(3) the canonical class of u in $H_2^G(X,\mathbb{Z})$ is B.

Then $\widetilde{\mathcal{B}}_\delta(P, B, (g))$ is a Banach manifold whose tangent space at (A, u) is given by

$$T_{(A,u)}\widetilde{\mathcal{B}}_\delta(P, B, (g)) = W^{1,p}_\delta(\Sigma, \Lambda^1 \otimes ad\ P \oplus u^*T^{\text{vert}}Y) \oplus (\mu^{-1}(0))^g$$

with $Y = P \times_G X$. There is an obvious smooth submersion (a locally trivial fibration)

$$\widetilde{\mathcal{B}}_\delta(P, B, (g)) \longrightarrow (\mu^{-1}(0))^g \tag{4.2}$$

defined by the asymptotic limit of (A, u) over the end. The fiber over $x \in (\mu^{-1}(0))^g$ consists of those (A, u) satisfying the above three conditions for a fixed x in (2). This is the product of an affine Banach manifold modelled on $W^{1,p}_\delta(\Sigma, \Lambda^1 \otimes ad\ P)$ and a Banach manifold of $W^{1,p}$-sections of P over Σ with prescribed asymptotics over $S^1 \times [T, \infty)$. Then the local triviality of the submersion (4.2) can be obtained by the parallel transport along geodesics in $(\mu^{-1}(0))^g$ with respect to the Levi-Civita connection on TX.

Let $\widetilde{\mathcal{G}}_\delta$ be the gauge group of $W^{2,p}_{loc}$ gauge transformations that converge in $W^{2,p}_\delta$ to constant $C(g)$-valued gauge transformations over the cylindrical end $S^1 \times [1, \infty)$. There is a subgroup $\mathcal{G}_\delta$ of $\widetilde{\mathcal{G}}_\delta$ consisting of those gauge transformations that converge in $W^{2,p}_\delta$ to the trivial gauge transformation. Then the Lie algebra of $\mathcal{G}_\delta$ is $W^{2,p}_\delta(\Sigma, ad\ P)$. There is an exact sequence of group homomorphisms

$$1 \to \mathcal{G}_\delta \longrightarrow \widetilde{\mathcal{G}}_\delta \longrightarrow C(g) \to 1$$

where $C(g)$ is thought as the constant gauge transformations.

The symplectic vortex equation (2.2) on Σ defines a smooth $\widetilde{\mathcal{G}}_\delta$-invariant section $\widetilde{S}$ of the $\widetilde{\mathcal{G}}_\delta$-equivariant Banach bundle $\widetilde{\mathcal{E}}_\delta(P, B, (g))$ over $\widetilde{\mathcal{B}}_\delta(P, B, (g))$ whose fiber at (A, u) is given by

$$L^p_\delta(\Sigma, \Lambda^{0,1} \otimes u^*T^{\text{vert}}Y \oplus ad\ P).$$

Then the moduli space $\mathcal{N}_\Sigma(X, P, B; (g))$ can be identified with the zero set $\widetilde{S}^{-1}(0)$ modulo the gauge group $\widetilde{\mathcal{G}}_\delta$.

Note that $\mathcal{G}_\delta$ acts freely on $\widetilde{\mathcal{B}}_\delta(P, B, (g))$ and does not change the asymptotic behaviour of (A, u) over the end. The fibration (4.2) defines a smooth $C(g)$-equivariant fibration

$$\widetilde{\mathcal{B}}_\delta(P, B, (g))/\mathcal{G}_\delta \longrightarrow (\mu^{-1}(0))^g. \tag{4.3}$$

The $\widetilde{\mathcal{G}}_\delta$-invariant section S defines a section of Banach bundle $\widetilde{\mathcal{E}}_\delta(P, B, (g))/\mathcal{G}_\delta$ over the Banach manifold $\widetilde{\mathcal{B}}_\delta(P, B, (g))/\mathcal{G}_\delta$. As in the case of moduli space of symplectic vortices over a closed Riemann surface, the deformation complex associated to a symplectic vortex $[A, u]$ in the fiber of (4.3) is given by

$$W^{2,p}_\delta(\Sigma, ad\ P) \xrightarrow{L_{A,u}} W^{1,p}_\delta(\Sigma, \Lambda^1 \otimes ad\ P \oplus u^*T^{\text{vert}}Y) \xrightarrow{\mathcal{D}_{A,u}} L^p_\delta(\Sigma, \Lambda^{0,1} \otimes u^*T^{\text{vert}}Y \oplus ad\ P),$$

which is elliptic in the sense that the cohomology groups are finite dimensional. The proof of this statement using the Atiyah-Singer-Singer boudnary condition is quite standard nowadays so we omit it here. This ensures that

$$\big(\widetilde{\mathcal{B}}_\delta(P, B, (g))/\mathcal{G}_\delta, \widetilde{\mathcal{E}}_\delta(P, B, (g))/\mathcal{G}_\delta, \widetilde{S}\big)$$

is a family of Fredholm systems over $(\mu^{-1}(0))^g$.

Note that the twisted sector $\mathcal{X}_0^{(g)}$ is defined by the locally free $C(g)$-action on $(\mu^{-1}(0))^g$. From the $C(g)$-equivariant fibration (4.3), we obtain a Banach orbifold structure on

$$\widetilde{\mathcal{B}}_\delta(P, B, (g))/\widetilde{\mathcal{G}}_\delta,$$

denoted by $\mathcal{B}_\delta(P, B, (g))$. Denote $\mathcal{E}_\delta(P, B, (g)) = \widetilde{\mathcal{E}}_\delta(P, B, (g))/\widetilde{\mathcal{G}}_\delta$. Then $\mathcal{E}_\delta(P, B, (g))$ is a Banach orbifold bundler over $\mathcal{B}_\delta(P, B, (g))$ and the $\widetilde{\mathcal{G}}_\delta$-invariant section $\widetilde{S}$ defines a section of $\mathcal{E}_\delta(P, B, (g))$. Therefore, we obtain an orbifold Fredholm system

$$\big(\mathcal{B}_\delta(P, B, (g)), \mathcal{E}_\delta(P, B, (g)), S\big)$$

for the moduli space $\mathcal{N}_\Sigma(X, P, B; (g))$. It is a family of orbifold Fredholm systems over $\mathcal{X}_0^{(g)}$ realising the asymptotic limit map

$$\partial_\infty : \mathcal{N}_\Sigma(X, P, B; (g)) \longrightarrow \mathcal{X}_0^{(g)}$$

in the sense that the fiber $\partial_\infty^{-1}(x, (g))$ can be identified with the zero set of the section S restricted to the fiber of the orbifold fibration

$$\mathcal{B}_\delta(P, B, (g)) \longrightarrow \mathcal{X}_0^{(g)}$$

over the point $(x, (g)) \in \mathcal{X}_0^{(g)}$.

To calculate the expected dimension of components in $\mathcal{N}_\Sigma(X, P, B, (g))$, we introduce a degree shift as in [11]. We first define the degree shift of an element g in G of order m acting linearly on $\mathbb{C}^n$. Let the complex eigenvalues of g on $\mathbb{C}^n$ be

$$e^{2\pi i m_1/m}, e^{2\pi i m_2/m}, \cdots, e^{2\pi i m_n/m}$$

for an n-tuple of integers $(m_1, m_2, \cdots, m_n)$ with $0 \leq m_j < m$ for $j = 1, 2, \cdots, n$. Then the degree shift of an element g on $\mathbb{C}^n$, denoted by $\iota(g, \mathbb{C}^n)$, is given by

$$\iota(g, \mathbb{C}^n) = \sum_{j=1}^{n} \frac{m_j}{m}.$$

From the definition, we have

$$\iota(g, \mathbb{C}^n) + \iota(g^{-1}, \mathbb{C}^n) = n.$$

For the orbifold $\mathcal{X}_0 = [\mu^{-1}(0)/G]$, if $g \in G$ has a non-empty fixed point set $(\mu^{-1}(0))^g$, then the Chen-Ruan degree shift of g on $\mathcal{X}_0$, denoted by $\iota_{CR}(g, \mathcal{X}_0)$ at $x \in (\mu^{-1}(0))^g$, is defined to be

$$\iota_{CR}(g, \mathcal{X}_0) = \iota(g, T_{[x]}(\mathcal{X}_0)).$$

For a twisted sector $\mathcal{X}_0^{(g)}$ of $\mathcal{X}_0$, the corresponding degree shift as in [11] is defined to

$$\iota_{CR}(\mathcal{X}_0^{(g)}, \mathcal{X}_0) = \iota_{CR}(g, \mathcal{X}_0)$$

for any g such that $\mathcal{X}_0^{(g)}$ is diffeomorphic to the orbifold defined by the action of $C(g)$ on the fixed point manifold $(\mu^{-1}(0))^g$.

Theorem 4.2. *Let $\mathcal{N}_\Sigma(X, P, B; (g))$ be the subset of $\mathcal{N}_\Sigma(X, P, B)$ consisting of symplectic vortices $[(A, u)]$ such that*

$$\partial_\infty(A, u) \in \mathcal{X}_0^{(g)}$$

Then $\mathcal{N}_\Sigma(X, P, B; (g))$ admits a Fredhom system with its virtual dimension given by

$$2\langle c_1^G(TX), B\rangle + 2(n - \dim G)(1 - g_\Sigma) - 2\iota_{CR}(\mathcal{X}_0^{(g)}, \mathcal{X}_0)$$

where g_Σ is the genus of the Riemann surface Σ.

Proof. With the Fredholm set-up and the orbifold model discussed above, we know that (A, u) has a unique extension to an orbifold symplectic vortex $(\tilde{A}, \tilde{u})$ on the principal G-bundle P_{orbi} over $\Sigma_{orib} = (|\Sigma_{orbi}|, (p, m))$, an orbifold Riemann surface with one orbifold point p of order m (the order of g). So by the excision property, we only need to calculate the index of the linearisation operator for the orbifold symplectic vortex $(\tilde{A}, \tilde{u})$ modulo based gauge transformations. Note that the remaining gauge transformations consist of constant ones taking values in $C(g)$, the centralizer of g in G, as we require that $\tilde{u}(p) \in \big(\mu^{-1}(0)\big)^g$.

The underlying Fredholm operator is the a compact perturbation of the direct sum of the operator $(-d_A^*, *d_A)$

$$\Omega^1_\delta(\Sigma, ad\ P) \to \Omega^0_\delta(\Sigma, ad\ P) \oplus \Omega^0_\delta(\Sigma, ad\ P)$$

in the original cylindrical model, with its index given by $-\dim G(1 - 2g_\Sigma)$, and the Cauchy-Riemann operator $\bar{\partial}_{\tilde{A},\tilde{u}}$ on the orbifold Σ_{orbi} with values in the complex vector bundle $\tilde{u}^*T^{\mathrm{vert}}Y$. Hence, the virtual dimension is given by

$$\mathbf{Index}\bar{\partial}_{\tilde{A},\tilde{u}} - \dim G(1 - 2g_\Sigma) - \dim C(g). \tag{4.4}$$

By the orbifold index theorem, we have

$$\mathbf{Index}\bar{\partial}_{\tilde{A},\tilde{u}} = 2\langle c_1(u^*T^{vert}Y), [|\Sigma_{orbi}|]\rangle + 2n(1 - g_\Sigma) - 2\iota_{CR}(g, T_{\tilde{u}(\tilde{p})}X). \tag{4.5}$$

By the definition of $c_1^G(u^*T^{vert}Y)$ and $[u_G]$, we have

$$\langle c_1(u^*T^{\mathrm{vert}}Y), [|\Sigma_{orbi}|]\rangle = \langle c_1^G(TX), B\rangle.$$

To calculate the degree shift for the g-action on $T_{\tilde{u}(\tilde{p})}X$, we apply the following decomposition

$$T_{\tilde{u}(\tilde{p})}X \cong \mathfrak{g} \oplus \mathfrak{g}^* \oplus T_{[x_\infty]}\mathcal{X}_0.$$

Here the actions of g on $\mathfrak{g}$ and $\mathfrak{g}^*$ are adjoint to each other and the zero eigenspace of the g-action on $\mathfrak{g}$ is the Lie algebra of $C(g)$. By the definition of degree shift, this implies that

$$\begin{aligned} & 2\iota_{CR}(g, T_{\tilde{u}(\tilde{p})}X) \\ = \ & 2\iota_{CR}(g, \mathcal{X}_0) + 2\dim_{\mathbb{C}} G/C(g) \\ = \ & 2\iota_{CR}(g, \mathcal{X}_0) + \dim_{\mathbb{R}} G/C(g). \end{aligned} \tag{4.6}$$

Put these formula (4.4), (4.5) and (4.6) together, we get the virtual dimension as claimed in the theorem. □

4.2. **L^2-moduli space of symplectic vortices on punctured Riemann surface.**

Let $C = (\Sigma, p_1, \ldots, p_k)$ be a Riemann surface with k marked points. We assume that C is stable, i.e, $2 - 2g(\Sigma) - k < 0$ where g_Σ is the genus of the Riemann surface Σ. It is well known that there is a canonical hyperbolic metric on the punctured Riemann surface $\Sigma \setminus \{p_1, \ldots, p_k\}$. This hyperbolic metric provides a disjoint union of horodiscs centred at each punctured point. We may deform the metric on the disc such that the metric becomes cylindrical. For simplicity, we use the same notation Σ for this Riemann surface with k cylindrical ends. Denote the metric and the corresponding volume form by ρ_Σ and ν_Σ respectively.

Let P be a principal G-bundle over Σ. Let $\mathcal{N}_\Sigma(X, P)$ be the moduli space of symplectic vortices with finite energy on Σ associated to P and a $2n$-dimensional Hamiltonian G-space (X, ω). Then $\mathcal{N}_\Sigma(X, P)$ is the space of gauge equivalence classes of solutions to the symplectic vortex equations (2.2) for

$$(A, u) \in \widetilde{\mathcal{B}}_{W^{1,p}_{\rm loc}(\Sigma)} = \mathcal{A}_{W^{1,p}_{\rm loc}(\Sigma)} \times W^{1,p}_{{\rm loc},G}(P, X)$$

such that

$$E(A, u) = \int_\Sigma \frac{1}{2}(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2)\nu_\Sigma < \infty.$$

Then the asymptotic limit map

$$\partial_\infty : \mathcal{N}_\Sigma(X, P) \longrightarrow \big(\text{Crit}\big)^k$$

is continuous. Let δ be a positive real number which is smaller than the minimum absolute value of eigenvalues of the Hessian operators of $\mathcal{L}$ along the compact critical manifold Crit, then $[u, A] \in \mathcal{N}_\Sigma(X, P)$ decays exponentially to its asymptotic limit along each end. Moreover, the energy function on $\mathcal{N}_\Sigma(X, P)$ takes values in a discrete set

$$\{\langle [\omega - \mu], B\rangle | B \in H_2^G(X, \mathbb{Z})\}.$$

Fix an equivariant homology class $B \in H_2^G(X, \mathbb{Z})$ such that $\langle [\omega - \mu], B\rangle > 0$. Let $\mathcal{N}_\Sigma(X, P, B)$ be the moduli space of symplectic vortices on Σ associated to (P, X) with the homology class B. We remark that the homology class of (u, A) is defined by the associated orbifold model as in the previous section for one cylindrical end case.

Then the Fredholm analysis for the one cylindrical end case in the previous section can be adapted to establish the following theorem.

Theorem 4.3. *Let $\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1,\cdots,k})$ be the subset of $\mathcal{N}_\Sigma(X, P, B)$ consisting of symplectic vortices $[(A, u)]$ such that*

$$\partial_\infty(A, u) \in \big(\mathcal{X}_0^{(g_1)} \times \cdots \times \mathcal{X}_0^{(g_k)}\big) \subset (\textit{Crit})^k$$

Then $\mathcal{N}_\Sigma(X, P, B; \{g_i\}_{i=1,\cdots,k})$ admits an orbifold Fredhom system with its virtual dimension given by

$$2\langle c_1^G(TX), B\rangle + 2(n - \dim G)(1 - g_\Sigma) - 2\sum_{i=1}^k \iota_{CR}(\mathcal{X}_0^{(g_i)}, \mathcal{X}_0)$$

where g_Σ is the genus of the Riemann surface Σ.

5. Compactness of L^2-moduli space of symplectic vortices

In this section, we establish a compactness result for the underlying topological space of the moduli space $\mathcal{N}_\Sigma(X, P, B)$ of symplectic vortices on a Riemann surface Σ with k cylindrical ends. We assume that $k > 0$. By reversing the orientation on S^1 if necessarily, we can assume that all these ends are modelled on $S^1 \times [0, \infty)$.

Given an orbifold topological space $\mathcal{N}$, the underlying topological space (also called the coarse space of $\mathcal{N}$) will be denoted by $|\mathcal{N}|$. We will provide a compactification of the coarse moduli space $|\mathcal{N}_\Sigma(X, P, B)|$ by adding certain limiting data consisting of bubbling off J-holomorphic spheres in (X, ω, J) in the interior of Σ and chains of symplectic vortices on cylinders. When X is Kähler, the compactness theorem for the L^2-moduli spaces of symplectic vortices on a Riemann surface with cylindrical end have been studied in [40]. The compactified space is a stratified topological space over a partially order finite index set, whose topology is inductively induced by the convergence properties for sequences of symplectic vortices on Σ.

To describe the limiting data for a sequence of symplectic vortices on Σ, we first introduce an index set for the topological type of the domain. Let g be the topological genus of Σ and $B \in H_2^G(X, \mathbb{Z})$ such that $\langle [\omega - \mu], B\rangle > 0$. Recall that a tree is a connected graph without any closed cycle of edges.

Definition 5.1. A web of stable weighted trees of the type $(\Sigma; B)$ is a finite disjoint union of trees

$$\Gamma = \Gamma_0 \sqcup \Gamma_1 \sqcup \cdots \sqcup \Gamma_k$$

consisting of a principal tree Γ_0 with ordered k tails and a collection of chains (ordered sequences) of trees

$$\Gamma_i = \bigsqcup_{j=1}^{m_i} T_i(j)$$

for each tail $i = 1, \cdots, k$, together with the following additional conditions.

(1) The principal tree Γ_0 has a distinguished vertex (called the principal vertex) with a weight (g, B_0) and ordered k tails labelled by $\{1, 2, \cdots, k\}$. Here g_0 is a non-negative integer and $B_0 \in H_2^G(X, \mathbb{Z})$ satisfying the following positivity condition

$$\langle [\omega - \mu], B_0\rangle \geq 0.$$

(2) For the i-th tail in Γ_0, there is a chain of trees of length m_i

$$\Gamma_i = T_i(1) \sqcup T_i(2) \sqcup \cdots \sqcup T_i(m_i)$$

such that each $T_i(j)$ has a distinguished vertex (called a branch vertex) with a weight given by a class $B_{i,j} \in H_2^G(X, \mathbb{Z})$ such that

$$\langle [\omega - \mu], B_{i,j}\rangle \geq 0.$$

If $B_{i,j} = 0$, the tree $T_i(j)$ is non-trivial in the sense that the branch vertex is not the only vertex.

(3) Any undistinguished vertex v in Γ has its weight given by a class $B_v \in H_2(X, \mathbb{Z})$, such that

$$\langle [\omega], B_v \rangle \geq 0.$$

If $B_v = 0$, the number of edges at v is at least 3, two of which connect to vertices of non-zero weights.

(4) Under the natural homomorphism $H_2(X, \mathbb{Z}) \to H_2^G(X, \mathbb{Z})$,

$$B_0 + \sum_{i=1}^{k} \sum_{j=1}^{m_i} B_{i,j} + \sum_{i=0}^{k} \sum_{v \in V(\Gamma_i)} B_v = B \tag{5.1}$$

holds in $H_2^G(X, \mathbb{Z})$. Here $V(\Gamma_i)$ is the set of undistinguished vertices in Γ_i.

Two webs of stable weighted trees are called equivalent if there is a weight preserving isomorphism between them. Denote by $\mathcal{S}_{\Sigma;B}$ be the set of equivalence classes of webs of stable weighted trees of the type $(\Sigma; B)$.

Given two element $[\Gamma]$ and $[\Gamma']$ in $\mathcal{S}_{\Sigma;B}$, we say $[\Gamma] \prec [\Gamma']$ if any representative Γ' in $[\Gamma']$ can be obtained, up to equivalence, from any representative Γ in $[\Gamma]$ by performing finitely many steps of the following three operations.

(1) Contracting an edge connecting two undistinguished vertices, say v_1 and v_2, in Γ to obtain a web with a replaced undistinguished vertex of a combined weight $B_{v_1} + B_{v_2}$.
(2) Identifying two branch vertices of adjacent trees in a chain Γ_i of length m_i to get a chain of trees of length $m_i - 1$ with a weight given by the sum of the two assigned weights.
(3) Identifying the principal vertex in Γ with a first branch vertex in a chain (say Γ_i), such that the new principal vertex is endowed with a new weight $B_0 + B_{i,1}$ and the i-th chain becomes $T_i(2) \sqcup \cdots \sqcup T_i(m_i)$.

Lemma 5.2. $(\mathcal{S}_{\Sigma;B}, \prec)$ *is a partially ordered finite set.*

Proof. It is easy to see that the order $\prec$ is a partial order. By the condition (5.1), we see that there are only finitely many collections of

$$\{(B_0, B_{i,j}, B_v) | \langle [\omega - \mu], B_{i,j} \rangle > 0, \langle [\omega - \mu], B_v \rangle > 0\}.$$

The stability conditions for branch vertices or undistinguished vertices with zero weight implies that there are only finitely many possibilities. This ensures that $\mathcal{S}_{\Sigma;B}$ is a finite set. □

Given an element $\Gamma = \sqcup_{i=0}^k \Gamma_i$ in $\mathcal{S}_{\Sigma;B}$, we can associate a bubbled Riemann surface of genus g and k cylindrical ends, and a collection of chains of bubbled cylinders as follows. Associated to Γ_0, we assign a bubbled Riemann surface Σ_0 which is the nodal Riemann surface obtained by attaching trees of $\mathbb{CP}^1$'s to Σ. Associated to an i-th chain of trees $\Gamma_i = \sqcup_{j=1}^{m_i} T_i(j)$ we assign a chain of bubbled cylinders

$$C_i = \{C_i(1), \cdots, C_i(m_i)\}$$

where each $C_i(j)$ is a nodal cylinder with trees of $\mathbb{CP}^1$'s attached according to the tree.

Now we construct a moduli space of stable symplectic vortices with the domain curve being the bubbled Riemann surface Σ_0 or one of the bubbled cylinders in $\{C_i(j)|i = 1, \cdots, k; j = 1, \cdots, m_i\}$ as follows.

Let $\tilde{\Gamma}_0$ be the new weighted graph obtained by severing all edges in Γ_0 which are attached to the principal vertex. Assume that Γ_0 has l_0 trees attached to the principal vertex. Then $\tilde{\Gamma}_0$ consists a single vertex (the principal vertex) with l_0 half-edges and k ordered tails. The remaining part of Γ_0, denoted by $\hat{\Gamma}_0$, becomes a disjoint union of l_0 trees, each of which has a half-edge attached one particular vertex (the adjacent vertex to the principal vertex). Equivalently,

$$\Gamma_0 = \big(\tilde{\Gamma}_0 \sqcup \hat{\Gamma}_0\big)/\sim$$

where the equivalence relation is given by the identification of l_0-tuple half-edges in $\tilde{\Gamma}_0$ with the l_0-tuple half-edges in $\hat{\Gamma}_0$.

Denote by $\mathcal{N}_{\tilde{\Gamma}_0}$ by the moduli space of symplectic vortices of homology class B_0 over Σ with l_0 marked points and k cylindrical ends. Then there is a continuous map

$$\widetilde{ev}_0 : \mathcal{N}_{\tilde{\Gamma}_0} \longrightarrow X^{l_0}$$

given by the evaluations at the l_0 marked points. Moreover, there is a continuous asymptotic limit map along each of the k cylindrical ends

$$\partial_0 : \mathcal{N}_{\tilde{\Gamma}_0} \longrightarrow (\mathrm{Crit})^k.$$

Associated to $\hat{\Gamma}_0$, as a disjoint union of l_0 trees, there is a moduli space of the Gromov-Witten moduli space of unparametrized stable pseudo-holomorphic spheres with l_0-marked points and the weighted dual graph given by $\hat{\Gamma}_0$, see Chapter 5 in [31]. We denote this moduli space by $\mathcal{M}^{GW}_{\hat{\Gamma}_0}$. Then there is a continuous map

$$\widehat{ev}_0 : \mathcal{M}^{GW}_{\hat{\Gamma}_0} \longrightarrow X^{l_0}$$

given by the evaluations at the l_0 marked points. The moduli space of bubbled symplectic vortices of type Γ_0, denoted by $\mathcal{N}_{\Gamma_0}$, is defined to be the orbifold topological space *generated* by the fiber product

$$\mathcal{N}_{\tilde{\Gamma}_0} \times_{X^{l_0}} \mathcal{M}^{GW}_{\hat{\Gamma}_0}$$

with respect to the maps $\widetilde{ev}_0$ and $\widehat{ev}_0$. Then the coarse moduli space $|\mathcal{N}_{\Gamma_0}|$ inherits a continuous asymptotic limit map

$$\partial_{\Gamma_0} : |\mathcal{N}_{\Gamma_0}| \longrightarrow (\mathrm{Crit})^k.$$

Remark 5.3. We remark that there is an ambiguity here with regarding the orbifold structure on $\mathcal{N}_{\Gamma_0}$. A proper way to make this precise is to employ the language of proper étale groupoids to describe the spaces of objects and arrows on $\mathcal{N}_{\tilde{\Gamma}_0} \times_{X^{l_0}} \mathcal{M}^{GW}_{\hat{\Gamma}_0}$, and then add further arrows to include all equivalences relations to get an orbifold structure on $\mathcal{N}_{\Gamma_0}$. As we are dealing with the compactification of the coarse moduli space, there is no ambiguity for the coarse space $|\mathcal{N}_{\Gamma_0}|$. We will return to this issue when we discuss weak Freholm systems for these moduli spaces in [9] and [10].

Similarly, for the i-th chain of trees $\Gamma_i = \bigsqcup_{j=1}^{m_i} T_i(j)$, we define a chain of moduli spaces of stable symplectic vortices of type Γ_i as follows. Associated to the tree $T_i(j)$, we excise the branch vertex away to get a graph consisting of a single vertex with $l_{i,j}$ half-edges and $l_{i,j}$ trees with one half-edge for each tree. Let $\tilde{T}_i(j)$ and $\hat{T}_i(j)$ be these two graphs respectively.

Denote by $\mathcal{N}_{\tilde{T}_i(j)}$ be the moduli space of symplectic vortices of homology class $B_{i,j}$ over the cylinder $C_i(j) \cong S^1 \times \mathbb{R}$ with $l_{i,j}$-marked points. Then there are a continuous evaluation map

$$\widetilde{ev}_{\tilde{T}_i(j)} : \mathcal{N}_{\tilde{T}_i(j)} \longrightarrow X^{l_{i,j}}$$

and continuous asymptotic value maps

$$\partial^{\pm}_{\tilde{T}_i(j)} : \mathcal{N}_{\tilde{T}_i(j)} \longrightarrow \text{Crit}$$

associated to the two ends at $\pm\infty$ respectively. Denote by $\mathcal{M}^{GW}_{\hat{T}_i(j)}$ the Gromov-Witten moduli space of unparametrized stable pseudo-holomorphic spheres with $l_{i,j}$-marked points and the weighted dual graph given by $\hat{T}_i(j)$. Note that $\mathcal{M}^{GW}_{\hat{T}_i(j)}$ is equipped with a continuous evaluation map

$$\widetilde{ev}_{\hat{T}_i(j)} : \mathcal{M}^{GW}_{\hat{T}_i(j)} \longrightarrow X^{l_{i,j}}.$$

Then forming the fiber product

$$\mathcal{N}_{\tilde{T}_i(j)} \times_{X^{l_{i,j}}} \mathcal{M}^{GW}_{\hat{T}_i(j)},$$

we get the moduli space $\widehat{\mathcal{N}}_{T_i(j)}$ of stable symplectic vortices of type $T_i(j)$. In particular, $|\widehat{\mathcal{N}}_{T_i(j)}|$ inherits continuous asymptotic limit maps

$$\hat{\partial}^{\pm}_{T_i(j)} : |\widehat{\mathcal{N}}_{T_i(j)}| \longrightarrow \text{Crit} \tag{5.2}$$

along the two ends. Note that the group of rotations and translations $S^1 \times \mathbb{R}$ on the cylinder induces a free action of $S^1 \times \mathbb{R}$ on the moduli space $\widehat{\mathcal{N}}_{T_i(j)}$ which preserves the asymptotic limit maps $\hat{\partial}^{\pm}_{T_i(j)}$. We quotient the moduli space $\widehat{\mathcal{N}}_{T_i(j)}$ by the group $\mathbb{R} \times S^1$, and denote the resulting moduli space by

$$\mathcal{N}_{T_i(j)} = \widehat{\mathcal{N}}_{T_i(j)} / (\mathbb{R} \times S^1).$$

The induced asymptotic limit maps on the coarse moduli space is denoted by

$$\partial^{\pm}_{T_i(j)} : |\mathcal{N}_{T_i(j)}| \longrightarrow \text{Crit}.$$

By taking the consecutive fiber products with respect to maps $\partial^{+}_{T_i(j)}$ and $\partial^{-}_{T_i(j+1)}$ for $j = 1, \cdots, m_i$, we get the coarse moduli spaces of chains of stable symplectic vortices of type Γ_i, that is,

$$|\mathcal{N}_{\Gamma_i}| = |\mathcal{N}_{T_i(1)}| \times_{\text{Crit}} |\mathcal{N}_{T_i(2)}| \times_{\text{Crit}} \cdots \times_{\text{Crit}} |\mathcal{N}_{T_i(m_i)}|.$$

Then there are two asymptotic limit maps given by $\partial^{-}_{T_i(1)}$ and $\partial^{+}_{T_i(m_i)}$, simply denoted by ∂^{-}_i and ∂^{+}_i,

$$\partial^{\pm}_i : |\mathcal{N}_{\Gamma_i}| \longrightarrow \text{Crit}.$$

Definition 5.4. Given $\Gamma = \Gamma_0 \sqcup \Gamma_1 \sqcup \cdots \sqcup \Gamma_k$ a web of stable weighted trees in $\mathcal{S}_{\Sigma;B}$, the coarse moduli space of stable symplectic vortices of type Γ, denoted by $|\mathcal{N}_\Gamma|$, is defined to be the fiber product

$$|\mathcal{N}_\Gamma| = |\mathcal{N}_{\Gamma_0}| \times_{(\text{Crit})^k} \prod_{i=1}^{k} |\mathcal{N}_{\Gamma_i}|,$$

where $\prod_{i=1}^{k} |\mathcal{N}_{\Gamma_i}| = |\mathcal{N}_{\Gamma_1}| \times |\mathcal{N}_{\Gamma_2}| \times \cdots \times |\mathcal{N}_{\Gamma_k}|$, and the fiber product is defined via the maps $\partial_{\Gamma_0} : |\mathcal{N}_{\Gamma_0}| \to (\text{Crit})^k$ and

$$\prod_{i=1}^{k} \partial_i^- : \prod_{i=1}^{k} |\mathcal{N}_{\Gamma_i}| \to (\text{Crit})^k.$$

There exists a continuous map

$$\partial_\Gamma : |\mathcal{N}_\Gamma| \longrightarrow (\text{Crit})^k$$

given by $\prod_{i=1}^{k} \partial_i^+$.

For any k-tuple $((g_1), \cdots, (g_k))$ conjugacy classes in G such that each representative g_i in (g_i) has a non-empty fixed point set in $\mu^{-1}(0)$, then we define

$$|\mathcal{N}_\Gamma((g_1), \cdots, (g_k))| = \partial_\Gamma^{-1} \left(|\mathcal{X}_0^{(g_1)}| \times \cdots \times |\mathcal{X}_0^{(g_k)}| \right).$$

Now we can state the compactness theorem for the coarse L^2-moduli space $|\mathcal{N}_\Sigma(X, P, B)|$ of symplectic vortices on Σ.

Theorem 5.5. *Let Σ be a Riemann surface of genus g with k-cylindrical ends. The coarse L^2-moduli space $|\mathcal{N}_\Sigma(X, P, B)|$ can be compactified to a stratified topological space*

$$|\overline{\mathcal{N}}_\Sigma(X, P, B)| = \bigsqcup_{\Gamma \in \mathcal{S}_{\Sigma;B}} |\mathcal{N}_\Gamma|$$

indexed by the set $\mathcal{S}_{\Sigma;B}$ of webs of stable weighted trees, such that the top stratum is $|\mathcal{N}_\Sigma(X, P, B)|$. Moreover, the coarse moduli space

$$|\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1,\cdots,k})|$$

of the moduli space $\mathcal{N}_\Sigma(X, P, B; \{(g_i)\}_{i=1,\cdots,k})$ in Theorem 4.3 can be compactified to a stratified topological space

$$|\overline{\mathcal{N}}_\Sigma(X, P, B; \{(g_i)\}_{i=1,\cdots,k})| = \bigsqcup_{\Gamma \in \mathcal{S}_{\Sigma;B}} |\mathcal{N}_\Gamma((g_1), \cdots, (g_k))|.$$

Proof. For simplicity, we assume that the Riemann surface Σ has only one outgoing cylindrical end, that is, diffeomorphic to $S^1 \times [0, \infty)$. The proof for the general case is essentially the same. Under this assumption, any web of stable weighted trees of the type $(\Sigma; B)$ has only one chain of trees denoted by $\{T(1), T(2), \cdots, T(m)\}$.

Given any sequence $[A_i, u_i] \in \mathcal{N}_\Sigma(X, P, B)$, we shall show that there exists a subsequence with a limiting datum in $\mathcal{N}_\Gamma$ for some $\Gamma \in \mathcal{S}_{\Sigma;B}$. The strategy to prove this claim is quite standard now, for example see [17], [14] and [34].

Note that the energy function on this sequence

$$E(A_i, u_i) = \int_\Sigma \frac{1}{2}(|d_{A_i} u_i|^2 + |F_{A_i}|^2 + |\mu \circ u_i|^2)\nu_\Sigma$$

is constant given by $\langle [\omega - \mu], B\rangle$. For any non-constant pseudo-holomorphic map from a closed Riemann surface, the energy is bounded from below by a positive number

$$\min\{\langle [\omega], \beta\rangle | \beta \in H_2(X, \mathbb{Z}), \langle [\omega], \beta\rangle > 0\},$$

which is greater than the minimal energy of non-trivial symplectic vortices on Σ associated to (P, X, ω)

$$\hbar = \min\{\langle [\omega - \mu], \beta\rangle | \beta \in H_2^G(X, \mathbb{Z}), \langle [\omega - \mu], \beta\rangle > 0\},$$

Step 1. (Convergence for the sequence with bounded derivative) Without loss of generality, we suppose that $\{(A_i, u_i)\}$ is a sequence of symplectic vortices in $\mathcal{N}_\Sigma(X, P, B)$ with a uniform bound

$$\|d_{A_i} u_i\|_{L^\infty} < C$$

for a constant C. Then there exists a sequence of gauge transformations $\{g_i\}$ such that $\{g_i \cdot (A_i, u_i)\}$ has a C^∞ convergent subsequence.

This claims follows from Theorem 3.2 in [13].

Step 2. (Bubbling phenomenon at interior points) Assume that the sequence $\|d_{A_i} u_i\|_{L^\infty}$ is unbounded over a compact set in Σ, then the rescaling technics in the proof of Theorem 3.4 in [13] can be applied here to get the standard pseudo-holomorphic sphere at the point in Σ where a sphere is attached to Σ.

Hence, combining Steps 1-2, we know that there may exist a subset of finite points, say $\{q_1, \cdots, q_{l_0}\}$, of Σ such that for any compact set $Z \subset \Sigma' = \Sigma - \{q_1, \cdots, q_{l_0}\}$, there exists a subsequence of (A_i, u_i) and gauge transformation g_i such that $g_i(A_i, u_i)$ uniformly converge in Z. As Z exhausts Σ', we get a symplectic vortex (A_∞, u_∞) on Σ'. By the removable singularity theorem, this symplectic vortex (A_∞, u_∞) can be defined on Σ.

Moreover, at each point q_j, we get a bubble tree of holomorphic sphere attached to q_i. As in the Gromov-Witten theory, there is certain energy bounded from below lost when the bubbling phenomenon happens at interior points. This gives rise to a principal tree Γ_0 in a web of stable weighted trees in $\mathcal{S}_{\Sigma;B}$.

Step 3. (Bubbling phenomenon at the infinite end) Assuming that for a sufficiently large T, the sequence $\{(A_i, u_i)\}$ converges to (A_∞, u_∞) on $\Sigma - (S^1 \times [T, \infty))$, where (A_∞, u_∞) is of the type Γ_0, a principal tree in Definition 5.1. Now we study the sequence over the cylindrical end. We may further assume that the Yang-Mills-Higgs energy

$$\int_{S^1 \times [T, \infty)} \frac{1}{2}(|d_{A_i} u_i|^2 + |F_{A_i}|^2 + |\mu \circ u_i|^2)$$

is greater than the minimum energy $\hbar$ defined as above. Otherwise, the limit of the sequence is in $\mathcal{N}_{\Gamma_0}$.

We replace the sequence $\{(A_i, u_i)|_{(S^1 \times [T, \infty)}\}$ by their translations to the left by $\{\delta_i\}$ such that the Yang-Mills-Higgs energy of the translate for (A_i, u_i) over $[T - \delta_i, 0]$ is $\hbar/4$. Then $\delta_i \to \infty$

as $i \to \infty$. Applying the above standard convergence theorem to the translated sequence with or without the bounded derivative condition on $\|d_{A_i} u_i\|_{L^\infty}$, there exists a subsequence which converges to a bubbled symplectic vertex (A'_∞, u'_∞) on any compact subset of $S^1 \times \mathbb{R}$. This gives rise to a stable symplectic vortex of type $\Gamma(1)$, where $\Gamma(1)$ is a tree with a branch vertex as in Definition 5.1.

Step 5. (No energy loss in between) Now we show that there is no energy loss on the connecting neck between (A'_∞, u'_∞) and (A_∞, u_∞). Equivalently, associated to the subsequence (still denoted by $\{(A_i, u_i)\}$, for each i, there exist

$$N_i < N_i + K_i < N'_i - K_i < N'_i$$

such that N_i, K_i and $N'_i - N_i - 2K_i \to \infty$ as $i \to \infty$, and under the temporal gauge,

(1) the sequence (A_i, u_i) on $S^1 \times [N_i, N_i + K_i]$ coneverges to (A_∞, u_∞) on any compact set after translation;
(2) (A_i, u_i) on $S^1 \times [N'_i - K_i, N_i]$ coneverges to (A'_∞, u'_∞) on any compact set after translation.

We shall show that the Yang-Mills-Higgs energy of (A_i, u_i) on $S^1 \times [N_i + K_i, N'_i - K_i]$ tends to 0 as $i \to \infty$.

Let y_∞ and $y'_{-\infty}$ be the limit of (A_∞, u_∞) as $t \to \infty$ and (A'_∞, u'_∞) as $t \to -\infty$ respectively. Let $\bar{y}'_{-\infty}$ be the pair obtained from $y'_{-\infty}$ by reversing the orientation of S^1. Suppose that

$$y_\infty = (a, \alpha), \quad \bar{y}'_{-\infty} = (b, \beta).$$

Then $(A_i(t), u_i(t)), t \in [N_i, N_i + K_i]$ is arbitrary close to y_∞ and $(A_i(t'), u_i(t')), t' \in [N'_i - K_i, N'_i]$ is arbitrary close to $\bar{y}'_{-\infty}$ as $i \to \infty$.

We claim that $\widetilde{\mathcal{L}}(y_\infty) = \widetilde{\mathcal{L}}(\bar{y}'_{-\infty})$. Otherwise, the difference would be larger than $\hbar$. However the Yang-Mills-Higgs energy of (A_i, u_i) on $[N_i, N'_i]$ is less than $\hbar/2$. This is impossible.

Now we explain the Yang-Mills-Higgs energy of (A_i, u_i) at $[N_i + t, N'_i - t]$ decays exponentially with respect to t. We normalize the band by translation such that $[N_i + K_i, N'_i - K_i]$ becomes $[-d, d]$ where $d = \frac{N'_i - N_i}{2} - K_i$.

Denote the Yang-Mills-Higgs energy of $y_i = (A_i, u_i)$ on $S^1 \times [-t, t]$ by

$$E_i(t) = \int_{S^1 \times [-t,t]} \frac{1}{2}(|d_{A_i} u_i|^2 + |F_{A_i}|^2 + |\mu \circ u_i|^2) dvol$$

for $0 \le t \le d$. Then

$$\frac{dE_i(t)}{dt} = \|\nabla \widetilde{\mathcal{L}}_{y_i(t)}\|^2 + \|\nabla \widetilde{\mathcal{L}}_{y_i(-t)}\|^2$$

Replace $\widetilde{\mathcal{L}}$ by $\widetilde{\mathcal{L}} - \widetilde{\mathcal{L}}(y_\infty)$, then by the crucial inequality (Proposition 3.12), we obtain the following differential inequality

$$\frac{dE_i(t)}{dt} \ge \delta(|\tilde{\mathcal{L}}(y_i(t))| + |\tilde{\mathcal{L}}(y_i(-t))|) \ge \delta E_i(t). \tag{5.3}$$

Here we use the fact that

$$E_i(t) = |\widetilde{\mathcal{L}}(y_i(t)) - \widetilde{\mathcal{L}}(y_i(-t))|.$$

Then the differential inequality (5.3) gives rise to

$$e^{-\delta t}E_i(t) \leq e^{-\delta d}E_i(d),$$

for any $t < d$. Apply to our case, this implies

$$E(A_i, u_i)|_{[N_i+K_i, N_i'-K_i]} \leq e^{-\delta K_i}E(A_i, u_i)|_{[N_i, N_i']}. \tag{5.4}$$

As $i \to \infty$, the Yang-Mills-Higgs energy goes to 0.

This ensures that

$$\partial_{\Gamma_0}([A_\infty, u_\infty]) = \partial^-_{\Gamma_1}([A'_\infty, u'_\infty]) \in \text{Crit}.$$

If the sum of Yang-Mills-Higgs energies of (A_∞, u_∞) and (A'_∞, u'_∞) agrees with $\langle[\omega - \mu], B\rangle$, the limit of the sequence is in $\mathcal{N}_\Gamma$ for $\Gamma = \Gamma_0 \sqcup \Gamma_1$.

Step 6. (Energy loss at the $+\infty$ end in the limit) If the sum of Yang-Mills-Higgs energies of (A_∞, u_∞) and (A'_∞, u'_∞) is less than $\langle[\omega - \mu], B\rangle$, then

$$\nu = \langle[\omega - \mu], B\rangle - E(A_\infty, u_\infty) - E(A'_\infty, u'_\infty) \geq \hbar.$$

In this case, we loss some energy at the $+\infty$ end in the limit, we repeat Steps 3-4 to get a limit in $\mathcal{N}_\Gamma$ with a chain of trees of length $m \geq 2$. This same process will stop after a finitely many steps due to the fact that each tree in the chain carries at least $\hbar$ energy. Put all these limiting data together, we get

$$\bigsqcup_{\Gamma \in \mathcal{S}_{\Sigma;B}} |\mathcal{N}_\Gamma|,$$

with each $\mathcal{N}_\Gamma$ admitting an orbifold Fredholm system. The topology on this disjoint union can be defined in a similar way as for Gromov-Witten moduli spaces. The compactness and Hausdorff properties of this topology can be established in the same way as in [21], [28] and [35]. This completes the compactification of $|\mathcal{N}_\Sigma(X, P, B)|$.

The compactification of $|\mathcal{N}_{\Sigma_{\mathbf{p}}}(X, P, B; \{(g_i)\}_{i=1,\cdots,k})|$ can be obtained in the similar manner.

□

6. OUTLOOK

In this paper, we mainly discuss the L^2-moduli space of symplectic vortices on a Riemann surface with cylindrical end. The analysis can be generalised to the case of a family of Riemann surfaces with cylindrical end. Then we get a moduli space of L^2-symplectic vortices fibered over Deligne-Mumford moduli spaces. In particular, for a Riemann surface

$$\Sigma_{g,k} = (\Sigma, (z_1, \cdots, z_k), \mathrm{j})$$

of genus g and with k-marked points, when $2 - 2g - k < 0$, we can consider $\Sigma_{g,k}$ as a Riemann surface of genus g and with n-punctures. By the uniformization theorem, for each complex structure on $\Sigma_{g,k}$, there is a unique complete hyperbolic metric on the corresponding punctured surface. This defines a canonical horodisc structure at each puncture, see [7]. This horodisc structure at each puncture is also called a hyperbolic cusp. Using the canonical horodisc structure at each point defined by the complete hyperbolic metric on the punctured $\Sigma_{g,k}$, we can identify the moduli space $\mathcal{M}_{g,k}$ with the moduli space of hyperbolic metrics with a canonical horodisc structure at each punctured disc. Each horodisc can be equipped with a canonical cylindrical metric on the punctured disc. In particular, we get a smooth universal family of Riemann surface with k cylindrical ends over the moduli space $\mathcal{M}_{g,k}$. Then the analysis in this paper on the L^2-moduli space of symplectic vortices can be carried over to get a continuous family of Fredholm system defined by the symplectic vortex equations. The corresponding L^2-moduli spaces of symplectic vortices without and with prescribed asymptotic data will denoted by

$$\mathcal{N}_{g,k}(X,P,B) \qquad \text{and} \qquad \mathcal{N}_{g,k}(X,P,B;\{(g_i)\}_{i=1,\cdots,k})$$

respectively. Then we have the similar compactness result for this L^2-muduli space where the index set $\mathcal{S}_{\Sigma;B}$ is replaced by $\mathcal{S}_{g,k;B}$ where the principal vertex of each web is replaced a dual graph as in the Gromov-Witten moduli space with weights in $H_2^G(X,\mathbb{Z})$ at each vertex, each vertex carries bubbling trees (with weights in $H_2(X,\mathbb{Z})$) and each tail is assigned a chain of trees.

In the subsequence paper, we shall also establish a weak orbifold Fredholm system and a gluing principle for the compactified moduli space $\overline{\mathcal{N}}_{g,k}(X,P,B;\{(g_i)\}_{i=1,\cdots,k})$ so that the virtual neighborhood technique developed in [8] can be applied to define a Gromov-Witten type invariant from these moduli spaces. We will show that the compactified moduli space $\overline{\mathcal{N}}_{g,k}(X,P,B;\{g_i\}_{i=1,\cdots,k})$ admits an oriented orbifold virtual system and the virtual integration

$$\int^{vir}_{\overline{\mathcal{N}}_{g,k}(X,P,B;\{(g_i)\}_{i=1,\cdots,k})} : H^*(I\mathcal{X}_0,\mathbb{R})^k \to \mathbb{R}$$

is well-defined. Here $I\mathcal{X}_0$ is the inertial orbifold of the symplectic reduction $\mathcal{X}_0 = \mu^{-1}(0)/G$. The Gromov-Witten type invariant is defined to be

$$\langle \alpha_1,\cdots,\alpha_k\rangle^{\ell HGW}_{g,k,B} = \int^{vir}_{\overline{\mathcal{N}}_{g,k}(X,P,B;\{(g_i)\}_{i=1,\cdots,k})} \partial^*_\infty(\pi_1^*\alpha_1\wedge\cdots\pi_k^*\wedge\alpha_k)$$

for any k-tuple of cohomology classes

$$(\alpha_1,\cdots,\alpha_k)\in H^*(\mathcal{X}_0^{(g_1)},\mathbb{R})\times\cdots\times H^*(\mathcal{X}_0^{(g_k)},\mathbb{R}).$$

Here $\pi_i : \mathrm{Crit}^k \to \mathrm{Crit} = I\mathcal{X}_0$ denotes the projection to the i-the component. We emphasize that this is an invariants on $H^*_{CR}(\mathcal{X}_0)$ rather than on $H^*_G(X)$. It is different from usual HGW invariants. We call the invariant L^2-Hamiltonian GW invariants (abbreviated as ℓHGW). In particular, when $(g,k)=(0,\geq 3)$, the above invariant can be assembled to get a new (big) quantum product $*^{HR}$ on $H^*(I\mathcal{X}_0,\mathbb{R})$. Here HR stands for Hamiltonian reduction. In a separate paper ([10]), we will introduce an augmented symplectic vortex equation to define an equivariant version of this invariant on $H^*_G(X)$ when G is abelian. This enables us to define a quantum product $*_G$ on $H^*_G(X)$. In a sequel work, we will investigate its relation to $*^{HR}$, in particular, we combine symplectic vortex equation with the augmented one to define the quantum Kirwan map Q_κ and show that Q_κ is a ring morphism with respect to $*_G$ and $*^{HR}$.

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