Higher Direct Images:

$\pi: Y \to X$ morphism. $\mathcal{F}/Y_{et}$, $\pi_* \mathcal{F}/X_{et}$

$$\Gamma(U, \pi_* \mathcal{F}) = \Gamma(U_Y, \mathcal{F}) \qquad \pi_* \text{ left exact}$$

$R^r \pi_*$ higher direct images.

Facts:
- $R^r \pi_* \mathcal{F}$ is sheaf assoc w/ presheaf $U \mapsto H^r(U_Y, \mathcal{F})$
- $(R^r \pi_* \mathcal{F})_{\bar{x}}$ $\bar{x} \to X$ is $\varinjlim_{(U,u)} H^r(U_Y, \mathcal{F})$ ← étale nbhds

Ex: X connected, normal $\quad g: \eta \to X$ generic pt

$$(R^r g_* \mathcal{F})_{\bar{x}} = H^r(\operatorname{Spec} k_{\bar{x}}, \mathcal{F}) \qquad k_{\bar{x}} = \operatorname{Frac}(\mathcal{O}_{X,\bar{x}})$$

$\eta = \operatorname{Spec} K \quad G = \operatorname{Gal}(K^{sep}/K) \quad M = M_{\mathcal{F}} = \varinjlim \mathcal{F}(k')$

$G \curvearrowright M$ trivial $\Rightarrow \mathcal{F}$ constant.

$$\bar{\eta} = \operatorname{Spec} K^{sep} \longrightarrow X$$
$$\searrow \operatorname{Spec} K \nearrow$$
$$\operatorname{Spec} K = \eta, \qquad K^{sep} \supset$$

$\mathcal{O}_{X,\bar{\eta}} = K^{sep}$ (nmlization of X in L/K finite étale over X on some nonempty open)

Thus $$(R^r g_* \mathcal{F})_{\bar{\eta}} = H^r(\operatorname{Spec} k^{sep}, \mathcal{F}) = H^r(\operatorname{Gal}(k^{sep}/k^{sep}), M) = \begin{cases} M & r = 0 \\ 0 & \text{o.w.} \end{cases}$$

In general $k_{\bar{x}}$ is union of $L/K \subset K^{sep}$ s.t. nmlization of X in L unram at some pt $/x$.

$$(R^r g_* F)_{\bar{x}} = H^r(\underbrace{\operatorname{Gal}(k_{sep}/k_{\bar{x}})}_{= H}, M) \quad \therefore (g_* \mathcal{F})_{\bar{x}} = M^H.$$

Thus $g_* \mathcal{F}$ constant if $\mathcal{F}$ constant.

$X = \operatorname{Spec} A$ Dedekind $\quad \tilde{A} \subset k^{sep}$ int closure. $\quad k_{\bar{x}} = (k^{sep})^{I(\tilde{p})}$

Thm (Leray S.S.) $H^r(X_{et}, R^s \pi_* \mathcal{F}) \Rightarrow H^{r+s}(Y_{et}, F)$.

CJ talk: $H^1(X_{et}, \mathbb{G}_m) = H^1(X_{zar}, \mathcal{O}_X^\times) = \operatorname{Pic} X$. Want for $\forall r$, can use Kummer seq to get $H^1(X_{et}, \mu_n)$ n coprime to char.

Weil Divisor Exact Sequence:

Recall from 256 A:

When A integral $$0 \to A^\times \to K^\times \xrightarrow{\operatorname{ord}_p} \bigoplus_{ht(p)=1} \mathbb{Z} \to 0$$ exact iff A UFD.

X com. normal

$$\Rightarrow \quad 0 \to \mathcal{O}_X^\times \to K^\times \to \operatorname{Div} \to 0 \quad \text{on } X_{zar}$$

left exact, exact when X regular.

Note: X irred has η generic, $\Gamma(U, g_* K^\times) = K^\times$ $U \subset X$ nonempty

Z prime div, z generic pt. $\Gamma(U, \bigoplus_z i_z \mathbb{Z}) = \operatorname{Div}(U)$ $i_z : z \hookrightarrow X$.

$$0 \to \mathcal{O}_X^\times \to g_* K^\times \to \bigoplus_z i_{z*} \mathbb{Z} \to 0,$$

Étale top:

$$0 \to \mathbb{G}_m \to g_* \mathbb{G}_{m,K} \to \bigoplus i_{z*} \mathbb{Z} \to 0$$

exact w/ same assumptions

Cohom of $\mathbb{G}_m$ on curve:

Some NT inputs:

k is quasi-alg closed if $\forall$ nonconstant homogeneous $f(T_1, \dots, T_n) \in k[T_1, \dots, T_n]$ deg $d<n$ has nontrivial 0 in k^n.

Ex: (a) alg closed field

(b) function field of dim 1 / alg closed field

(c) $K = \operatorname{Frac}(R)$, R Henselian DVR w/ alg closed res field if $\hat{K}/K$ sep

↳ $R = \mathcal{O}_{X,x}^h$ X finite type / field and K char 0.

Prop: k quasi-alg closed, $G = Gal(k^{sep}/k)$

(a) Brauer group of k is 0, $H^2(G, (k^{sep})^\times) = 0$ ($= 0 \; \forall r > 0$)

(b) $H^r(G, M) = 0 \quad \forall r > 1$, torsion G-mod

(c) $H^r(G, M) = 0 \quad \forall r > 2$, G-mod

Pf: (a) come back if have time (b) Lot of CFT (c) omit.

Thm: Connected, nonsingular curve $X / k = \bar{k}$

$$H^r(X_{et}, \mathbb{G}_m) = \begin{cases} \Gamma(X, \mathcal{O}_X^\times), & r = 0 \\ Pic(X) & r = 1 \\ 0 & r > 1 \end{cases}$$

Lemma: $H^r(X_{et}, g_* \mathbb{G}_{m,\eta})$, $H^r(X_{et}, Div_X) = 0 \quad \forall r > 0$.

Lemma + Weil Divisor Exact $\Rightarrow$ Thm.

Pf of Lemma: x closed pt of X, i_{x*} exact, $H^r(X_{et}, i_{x*} \mathcal{F}) = H^r(x_{et}, F) = 0$

$H^r(X_{et}, Div_X) = 0 \quad \forall r > 0$.

Consider $R^r g_* \mathbb{G}_{m,\eta}$

$$(R^r g_* \mathbb{G}_{m,\eta})_{\bar{y}} = \begin{cases} 0 & y = \eta \quad r > 0 \\ H^r(Spec\, k_{\bar{y}}, \mathbb{G}_m) & \end{cases}$$

$K_{\bar{x}}$ is $Frac(\mathcal{O}_{X,\bar{x}})$, thus quasi-alg closed. So $H^r(Spec\, k_{\bar{x}}, \mathbb{G}_m) = H^r(Gal(k_{sep}/k_{\bar{x}}), (k^{sep})^\times) = 0 \quad \forall r > 0$.

Leray $\Rightarrow H^r(X_{et}, g_* \mathbb{G}_{m,\eta}) = H^r(\eta, \mathbb{G}_m) = H^r(G, (k^{sep})^\times)$ $\quad G = Gal(k^{sep}/k)$

$H^r(G, (k^{sep})^\times) = 0$ $\quad$ $r = 1$ Hilbert 90; $r > 1$ above. $\quad$ (b) of ex

□

See Milne for other cases.

Pf of Prop:

(a) WTS $\forall$ central div alg $/k$ deg 1. $[D:k]=n^2$ $e_1, \ldots, e_{n^2}$ basis

$\alpha = \sum a_i e_i \in D$

$\exists f(X_1, \ldots, X_{n^2})$ hom poly deg n s.t.

$f(a_1, \ldots, a_{n^2})$ is reduced norm.

But also $Nm_{\mathbb{Q}[\alpha]/\mathbb{Q}}(\alpha)^r$ $r = \frac{n}{\mathbb{Q}[\alpha]:\mathbb{Q}}$

(b) idea: k quasi-alg closed $\Rightarrow$ so is finite ext.

(a) + infl-restriction exact seq $\Rightarrow H^r(Gal(L/k), L^\times) = 0$ $r = 1, 2$

Tate's Thm (on Tate cohom): $H^r(Gal(L/k), L^\times) = 0$ $\forall r > 0$.

inverse limit:

$H^r(G, (k^{sep})^\times) = 0$ $\forall r > 0$.

Kummer seq

$$0 \to \mu_n \to (k^{sep})^\times \xrightarrow{n} (k^{sep})^\times \to 0$$

$H^r(G, \mu_n) = 0$ $\forall r > 1$, n coprime to char k.

$p \neq$ char k. $\exists K/k$ finite Gal containing μ_p.

$$H^r(G, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{res} H^r(H, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{cores} H^r(G, \mathbb{Z}/p\mathbb{Z}) \qquad H = Gal(k^{sep}/K)$$

comp is $[K:k]$ mult, isom. $H^r(H, \mathbb{Z}/p\mathbb{Z}) = H^r(H, \mu_p) = 0$, $H^r(G, \mathbb{Z}/p\mathbb{Z}) = 0$. $r > 1$.

Artin Schreier $\Rightarrow$ $p =$ char k

For a torsion G-mod, prime by prime works. See Milne for details.

$\square$