§10 Čech cohomology for étale sheaves

Let $\mathcal{U} = \{U_i \longrightarrow X\}$ be an étale covering, P a presheaf on X_{et}. We can form Čech complex in usual way:

$$C^r(\mathcal{U}, P) = \prod_{(i_0, \dots, i_r)} P(U_{i_0} \times_X U_{i_1} \times \dots \times U_{i_r})$$

differential given by

$$(d^r s)_{i_0, \dots, i_{r+1}} = \sum_{j=0}^{r+1} (-1)^j s_{i_0 \dots i_{j-1} i_{j+1} \dots i_{r+1}}$$

Complex, form homology, etc.

If $\mathcal{F}$ is a sheaf, then $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

Say $\mathcal{V}$ refines $\mathcal{U}$ if every $V_j \to X$ factors thru some $U_i \to X$. Get $C^\bullet(\mathcal{U}, P) \to C^\bullet(\mathcal{V}, P)$

which induces well-defined map $\check{H}^\bullet(\mathcal{U}, P) \to \check{H}^\bullet(\mathcal{V}, P)$

Def. $\check{H}^r(X, P) := \varinjlim_{\mathcal{U}} \check{H}^r(\mathcal{U}, P)$

Properties: $\check{H}^0(X, F) = F(X)$ for any sheaf F/X (coverings)

$\check{H}^r(X, I) = 0$ for injectives (show $C^\bullet(\mathcal{U}, F) = \mathrm{Hom}(Z^\bullet, I)$ for some exact $Z^\bullet$ presheaves)

These imply $H = \check{H}$ whenever SBS of sheaves $\Rightarrow$ LES in $\check{C}$ cohom. This follows if for SBS induces

$$0 \to \varinjlim C^\bullet(\mathcal{U}, F') \to \varinjlim C^\bullet(\mathcal{U}, F) \to \varinjlim C^\bullet(\mathcal{U}, F'') \to 0$$

Thm: This holds if every fin. subset of X is contained in an open affine & X is qc.

Pf. Key difficulty: right exactness. Have local surj, i.e.

for each $s \in F''(U_{i_0, \dots, i_n})$ have cover $V_j \to U_{i_0 \dots i_n}$

s.t. $s|_{V_j}$ lifts to $F(V_j)$, but we specifically need V_j of the form $V_{i_0} \times_X V \ldots \times_X V_{i_n} \to V_{i_m} (\to U_{i_m}) \to X$.

CA lemma: Df A ring, $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, strict henselizations $A_1, \ldots, A_r$ of $A_{\mathfrak{p}_1}, \ldots, A_{\mathfrak{p}_r}$, then $A' = A_1 \otimes_A \cdots \otimes_A A_r$ have

ff-etale $A' \to B$ has sxn $B \to A'$.

Lastly: $H^1(X_{et}, \mathcal{F}) = \check{H}^1(X_{et}, \mathcal{F})$ always

§ 11 : Principal homog. spaces

If $\mathcal{G}$ is a sheaf of gps on X_{et}, then given a covering $\mathcal{U}$.

can define 1-cocycles for $\mathcal{U}$ in the usual way: g_{ij} on U_{ij},

$$g_{ij} \cdot g_{jk} = g_{ik} \text{ on overlap } U_{ijk}.$$

Cohomologous if $g'_{ij} = (h_i|U_{ij}) \cdot g_{ij} \, (h_j|U_{ij})^{-1}$ for some (h_i)

$\check{H}^1(X_{et}, \mathcal{G})$ is not always a gp, just ptd set. Get exact seq.

$$1 \longrightarrow \mathcal{G}'(X) \to \mathcal{G}(X) \to \mathcal{G}''(X) \to \check{H}^1(X, \mathcal{G}') \to \check{H}^1(X, \mathcal{G}) \to \check{H}^1(X, \mathcal{G}'')$$

PHS. Let $\mathcal{G}/X_{et}$ sh of gps, S a right $\mathcal{G}$-shf.

We say S is a PHS for $\mathcal{G}$ if

(a) $\exists$ an étale covering $(U_i \to X)$ s.t. $S(U_i) \neq \emptyset$

(b) If $U \to X$ is any étale map and $s \in S(U)$, then $g \longmapsto sg$ is an iso $\mathcal{G}|_U \to S|_U$.

Such a covering is called a splitting of $\mathcal{S}$.

Given a PHS, get 1-cocycle for $\mathcal{G}$ by choosing $s_i \in \mathcal{S}(U_i)$,
then $\exists!\ g_{ij} \in \mathcal{G}(U_{ij})$ s.t. $s_i \circ g_{ij} = s_j$ on U_{ij}.

$\Rightarrow (g_{ij})_{U_{ij}}$ is a 1-cocycle.

Prop: This defines a bijection from iso. classes of PHS split by $\mathcal{U}$
to $\check{H}(\mathcal{U}, \mathcal{G})$

Pf: Mostly routine

Ex: Let $\mathcal{G}$ be const. sh. on X_{et} defined by a fin. gp. G

If $Y \to X$ is a Galois covering w/ gp. G, we can

let $\mathcal{S}(U) = \{$ sections $U \to Y \times_X U$ of $Y \times_X U \to U \}$. Then

G acts rt. on $Y \times_X U$

$\mathcal{S}$ is a PHS for X; $\{Y \to X\}$ is a trivializing cover.

Conversely, all PHS for $\mathcal{G}$ arise in this way, inducing

$$H^1(X_{et}, \mathcal{G}) \simeq \mathrm{Hom}_{cont}(\pi_1(X, \bar{x}), G)$$

Thm: The ~~set~~ iso classes of locally free (Zariski) sheaves of rank n
on X $(= \mathrm{Lu}(X))$ is in bij. w/ $\check{H}^1(X_{Zar}, GL_n)$

$\check{H}^1(X_{et}, GL_n)$

$\check{H}^1(X_{fl}, \mathbb{G}_m)$

flat site (flat finite type maps)

Pf. $L_n(X) = \check{H}^1(X_{Zar}, \mathbb{G}_m)$ is standard; need to show that the $\check{H}^1$ all agree. Need to show:

(a) all locally free sheaves on flat / étale top. come from Zar;

(b) If M coherent $\mathcal{O}_{X_{Zar}}$, then $M^{fl/et}$ locally free $\Rightarrow$ M locally free

(c) $M \simeq N \in L_n(X_{Zar})$ iff $M^{fl/et} \simeq N^{fl/et}$.

Suffices to work in affine case.

(b) Sketch:

CA: Projective A-modules $\Leftrightarrow$ locally free A-mods.

If $A \to B$ ff, then

always if M fg

$$B \otimes_A \mathrm{Hom}_A(M, N) = \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N)$$

"proj"

$\Rightarrow$ exactness of $\mathrm{Hom}_B(B \otimes_A M, -)$

$\Rightarrow$ exactness of $B \otimes_A \mathrm{Hom}_A(M, -)$

ff $\Rightarrow$ exactness of $\mathrm{Hom}_A(M, -)$

Can now work affine locally to show that if M^{fl} locally free $\Rightarrow$ M locally free

Cor: $\mathrm{Pic}(X) = H^1(X_{et}, \mathbb{G}_m)$

Spectral Sequences:

Groth. SS: Let $A \xrightarrow{F} B \xrightarrow{G} C$ left exc.

and assume $(R^r G)(FI) = 0$ for $r > 0$ ($\Leftarrow$ F sends injectives to inj. $\Leftarrow$ F has a left adj)

Then there is a SS

(composition of der. functors)

$$E_2^{rs} = (R^r G)(R^s F)(A)$$

$$\Longrightarrow R^{r+s}(FG)(A)$$

(derived functor of composition)

For us:

$$Sh(X_{et}) \xrightarrow{i} PreSh(X_{et}) \xrightarrow{\check{H}^0(X_{et}, -)} AB$$

i has derived functor

$$\underset{\text{presheaf}}{\mathcal{H}^r(\mathcal{F})} \overset{\text{sheaf}}{:} U \mapsto H^r(U, \mathcal{F}|_U)$$

Prop: Sheafification of $\mathcal{H}^r(\mathcal{F})$ is 0 for $r>0$

$$a(\mathcal{H}^r(\mathcal{F})) = a(H^r(i\overset{\text{inj. res of } \mathcal{F}}{I^\bullet}))$$

$$= H^r(ai I^\bullet) \quad \text{since } a \text{ is exact}$$

$$= H^r(I^\bullet) = 0$$

$\gg$ Given $s \in H^r(X, \mathcal{F})$ $r>0$,

$\exists$ étale covering $(U_i \to X)$ s.t. s restricts

to 0 in $H^r(U_i, \mathcal{F})$

Map on 1:

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{G}(X) \to H^1(X,\mathcal{F}) \to 0$$

$$t \longmapsto s$$

Locally $s=0$, so $t|_{U_i}$ lifts to $\tilde{t}_i \in \mathcal{I}(U_i)$.

Thus at $s_{ij} = \tilde{t}_j|_{U_{ij}} - \tilde{t}_i|_{U_{ij}}$ in $\mathcal{F}(U_{ij})$