Milne Etale Cohomology Seminar: §1.15-1.17

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Weil pairing example of Poincare duality

Before starting the new material, we go over an example from the curve cohomology chapter. Recall Poincare duality:

Theorem 0.0.1. *Let U be a connected regular curve over an algebraically closed field. For a locally constant sheaf $\mathcal{F}$ on U and integer r, there is a canonical perfect pairing of finite groups*

$$H^r_c(U, \mathcal{F}) \times H^{2-r}(U, \check{\mathcal{F}}(1)) \to H^2_c(U, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}.$$

Example 0.0.2. Let X be a complete connected smooth curve over an algebraiclly closed field. Then we can take etale cohomology instead of compactly supported cohomology in the pairing. Set $\mathcal{F}$ to be the constant sheaf $\underline{\mathbb{Z}/n\mathbb{Z}}$. We tensor the first factor of the pairing and the target with μ_n to get a canonical pairing

$$H^1(X, \mu_n) \times H^1(X, \mu_n) \to H^2(X, \mu_n \otimes \mu_n) \simeq \mu_n,$$

noting also that $\check{\mathbb{Z}/n\mathbb{Z}}(1)\mathscr{H}\text{om}(\mathbb{Z}/n\mathbb{Z}, \mu_n) \simeq \mu_n$. Recall that we may canonically $H^1(X, \mu_n) \simeq \text{Jac}(X)[n]$, which follows from the Kummer sequence and the identification $H^r(X_{et}, \mathbb{G}_m) \simeq \text{Pic}(X)$. This turns out to be exactly the same as the Weil pairing. This is the canonical pairing $\text{Jac}(X)[n] \times \text{Jac}(X)^\vee[n] \to \mu_n$ defined by Cartier duality. Jacobians of a smooth curve are always canonically principally polarizable, so this yields a corresponding perfect alternating pairing $\text{Jac}(X)[n] \times \text{Jac}(X)[n] \to \mu_n$. Tracing the maps given in Poincare duality to verify that these are actually the same seems hard.

15 Cohomological dimension

The cohomological dimension of X is the least integer c such that $H^r(X_{et}, \mathcal{F}) = 0$ for all $r > c$ and torsion sheaves $\mathcal{F}$.

The cohomological dimension of a field Spec K is the same as the cohomological dimension of $\text{Gal}(K^{\text{sep}}/K)$ (from group cohomology). Hence $\text{cd}(K) = 0$ if K is algebraically closed.

Finite fields have cohomological dimension 1. Note that $H^2(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), \mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \neq 0$, but this doesn't count since $\mathbb{Z}$ is not torsion. $\mathbb{R}$ has infinite cohomological dimension since the cohomology of a cyclic group is periodic.

As one expects from intuition from manifolds:

Theorem 15.0.1. *If X is a variety over an algebraically closed field, then $\mathrm{cd}(X) \leq 2\dim(X)$, or $\dim(X)$ if X is affine. Moreover $H^{2\dim X}(X, \mathbb{Z}/n\mathbb{Z})$ when X is proper.*

Proof. We'll only prove the first statement.

We can assume that $\mathcal{F}$ is of the form $g_*\mathcal{F}$, where $g : \eta \to X$ is the generic point, since the map $\mathcal{F} \to g_*g^*\mathcal{F}$ via adjunction is an isomorphism at the generic point, hence has kernel and cokernel supported on a proper closed subscheme and we can apply induction to conclude that it suffices to prove the statement for $g_*g^*\mathcal{F}$ instead.

Fact: If A is a Henselian local ring containing a field k such that the residue field of K is separable over k, then A contains a field mapping isomorphically onto K. As a corollary, the Henselization A of $\mathcal{O}_{X,Z}$ has field of fractions F containing $k(Z)$ with tr. deg. $\dim X - \dim Z$.

Then the main technical lemma we need is:

Lemma 15.0.2. *The sheaf $R^s g_*\mathcal{F}$ has support in dimension $\leq m - s$.*

Proof. Let Z be closed irr. subvar. of X with generic point z and corresponding choice of geometric point $\overline{z}$. Then the strictly local ring at $\overline{z}$ is the maximal unramified extension of $\mathcal{O}^h_{X,Z}$, so its residue field is $k(Z)^{\mathrm{sep}}$ and its field of fractions is an algebraic extension of $k(X)$. Hence it has transcendence degree $\dim X - \dim Z$ over $k(Z)^{\mathrm{sep}}$. Since

$$(R^s g_*\mathcal{F})_{\overline{z}} = H^s(K_{\overline{z}}, \mathcal{F})$$

this vanishes for $s > m - \dim Z$, since one can show that $\mathrm{cd}(K) \leq \mathrm{cd}(k) + d$ for a field extension K/k of transcendence degree d, and $k(Z)^{\mathrm{sep}}$ has $\mathrm{cd} = 0$. Therefore $R^s g_*\mathcal{F}$ is nontrivial only if $\dim Z \leq m - s$, i.e. its support is in dimension at most $m - s$. ∎

By inductive hypothesis we conclude that $H^r(X, R^s g_*\mathcal{F}) = 0$ for $s \neq 0$, $r > 2(m - s)$. Moreover, we also know that $H^{r+s}(\eta, \mathcal{F}) = 0$ whenever $r + s > m$, since the generic point has transcendence degree m over the algebraically closed base field. The Leray spectral sequence says

$$E_2^{rs} = H^r(X, R^s g_*\mathcal{F}) \implies H^{r+s}(\eta, \mathcal{F}).$$

The fact that $H^r(X, R^s g_*\mathcal{F}) = 0$ for $s \neq 0$, $r > 2(m - s)$ is enough information to tell us $E_2^{r0} = E_\infty^{r0}$, hence $H^r(X, g_*\mathcal{F}) = 0$. ∎

16 Purity/Gysin sequence

Let X be a variety in char. not divisible by n. Define $\Lambda(r)$ to be the sheaf such that $\Gamma(U, \Lambda(r)) = \mu_n(\Gamma(U, \mathcal{O}_U))^{\otimes r}$ for all etale affine $U \to X$, ($\Lambda(0) = \Lambda = \mathbb{Z}/n\mathbb{Z}$) and let $\mathcal{F}(r) = \mathcal{F} \otimes \Lambda(r)$. We say a smooth pair (Z, X) is smooth X with smooth subvar Z.

Theorem 16.0.1. *For any smooth pair of k-vars (Z, X) of codim c and locally constant sheaf of Λ-modules on X, there are canonical isomorphisms*

$$H^{r-2c}(Z, \mathcal{F}(-c)) \to H^r_Z(X, \mathcal{F})$$

for all $r \geq 0$.

Corollary 16.0.2. *There are isomorphisms $H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F})$ for $0 \leq r < 2c - 1$ and an exact sequence (Gysin sequence)*

$$0 \longrightarrow H^{2c-1}(X, \mathcal{F}) \longrightarrow H^{2c-1}(U, \mathcal{F}) \longrightarrow H^{r-2c})Z, \mathcal{F}(-c)) \longrightarrow \dots$$

Proof. By the exact sequence of the pair $(X, X \backslash Z)$. ■

Example 16.0.3. We have $H^1(\mathbb{A}^1, \mathbb{G}_m) = \mathrm{Pic}(\mathbb{A}^1) = 0$. Then the Kummer sequence shows that $H^r(\mathbb{A}^1, \mu_n) = 0$ for $r > 0$, too, and we can use Kunneth to get more generally $H^r(\mathbb{A}^1, \mu_n) = 0$ for $r > 0$. Considering the pair $(\mathbb{P}^n, \mathbb{P}^{n-1})$, the corollary tells us (with $\mathcal{F} = \mu_n = \Lambda$) that $H^0(\mathbb{P}^m, \Lambda) \simeq H^0(\mathbb{A}^m, \Lambda) \simeq \Lambda$ and $H^1(\mathbb{P}^m, \Lambda) \hookrightarrow H^1(\mathbb{A}^m, \Lambda) = 0$. The theorem directly tells us that $H^{r-2}(\mathbb{P}^{m-1}, \Lambda(-1)) \simeq H^r(\mathbb{P}^m, \Lambda)$. We can use induction to show that this implies $H^r(\mathbb{P}^m, \Lambda) = \Lambda(-r/2)$.

We can more generally use the Gysin sequence to analyze the $\mathbb{Z}/n\mathbb{Z}$-cohomology of smooth complete intersections.

The theorem also has the followling form. For a sheaf $\mathcal{F}$ on X, we define $\mathcal{F}^!$ to be the largest subsheaf of $\mathcal{F}$ with support on Z, which is also given by $\mathcal{F}^! = \ker(\mathcal{F} \to j_* j^* \mathcal{F})$ for the open immersion $j : X \backslash Z \to X$. We let $i^! \mathcal{F}$ to be $i^* \mathcal{F}^!$ for the inclusion $i : Z \to X$. $i^!$ is a right adjoint to i_*, hence it is left exact and preserves injectives.

Theorem 16.0.4. (Cohomological purity.) *With hypotheses as above, $R^{2c} i^! \mathcal{F} \simeq (i^* \mathcal{F})(-c)$ and all other degrees vanish.*

This implies the first version using the Grothendieck spectral sequence for the composition of functors $\Gamma(Z, -), i^!$, which is

$$E_2^{r,s} = H^r(Z, R^s i^! \mathcal{F}) \implies H^{r+s}_Z(X, \mathcal{F}).$$

$H^r(Z, R^s i^! \mathcal{F})$ vanishes by the purity theorem unless $2 = 2c$, so we conclude that the spectral sequence is already degenerate at the second page and that $H^r_Z(X, \mathcal{F}) \simeq H^{r-2c}(Z, R^{2c} i^! \mathcal{F}) \simeq H^{r-2c}(Z, \mathcal{F}(-c))$.

To prove cohomological purity, one shows that the problem is étale local, and then that any smooth pair of codimension c is locally isomorphic to $(\mathbb{A}^{m-c}, \mathbb{A}^m)$. Then one proceeds by induction.

17 Proper base change

A sheaf on X_{et} is called constructible if it has finite stalks and it is locally constant on a nonempty open subset of every closed irreducible subscheme Z.

Constructibility is preserved by (higher) pushforwards and it remains true that $(R^r\pi_*\mathcal{F})_{\overline{s}} = H^r(X_{\overline{s}}, \mathcal{F}|X_{\overline{s}})$ for geometric points of S. Consequently, $H^r(X_{et}, \mathcal{F})$ is finite for all r if X is proper over a separably closed field k, since constructible sheaves on $\operatorname{Spec} k$ are equivalent to finite abelian groups.

Theorem 17.0.1. *Let $\pi : X \to S$ be proper, and let $X \times_S T \to T$ for some morphism $f : T \to S$. For any torsion sheaf on X, there is a canonical isomorphism*

$$f^*(R^r\pi_*\mathcal{F}) \to R^r\pi'_*(f'^*\mathcal{F}).$$