Integers that can be written as the sum of two rational cubes

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Contents

Abstract

We are interested in finding for which positive integers D we have rational solutions for the equation $x^3+y^3=D$. The aim of the paper is to compute the value of the L-function $L(E_D,1)$, for $E_D: x^3+y^3=D$. For the case of p prime $p\equiv 1 \mod 9$, two formulas have been computed by Rodriguez-Villegas and Zagier in [14]. We compute several formulas using automorphic methods.

1 Introduction

In the current paper we are interested in finding an algorithm to decide which positive integers can be written as the sum of two rational cubes:

$$x^3+y^3=D,\ x,y\in\mathbb{Q} \tag{1}$$

After making the equation homogeneous, we get the equation $x^3+y^3=Dz^3$ that has a rational point at $\infty=[1,-1,0]$. Moreover, after a change of coordinates $X=12D\dfrac{z}{x+y}$, $Y=36D\dfrac{x-y}{x+y}$ the equation becomes

$$E_D: Y^2=X^3-432D^2$$

which defines an elliptic curve over $\mathbb{Q}$.

We will assume D is cube free and $D\neq 1,2$ (trivial cases) throughout the paper. It is known that $E_D(\mathbb{Q})$ has trivial torsion for $D\neq 1,2$ (see [16]). Thus, (1) has a solution iff $E_D(\mathbb{Q})$ has positive rank. From the BSD conjecture, this is equivalent to the vanishing of $L(E_D,1)$.

Without assuming BSD, from the work of Coates-Wiles [2], or more generally Gross-Zagier [6] and Kolyvagin [10], when $L(E_D,1)\neq 0$, we have $\operatorname{rank} E_D(\mathbb{Q})=0$, thus no rational solutions in (1). We define an invariant S_D of E_D as follows:

$$S_D=\frac{L(E_D,1)}{\Omega_{D,\infty}R_{E_D}},$$

where:

- $\Omega_{D,\infty}$is the real period
- R_{E_D} is the regulator

The definition is made such that in the case of $L(E_D, s) \neq 0$ we expect to get from the full BSD conjecture:

$$S_D = \#Ш(E_D) \prod_{p|6D} c_p, \tag{2}$$

where $\#Ш$ is the order of the Tate-Shafarevich group and c_p are the the Tamagawa numbers corresponding to the elliptic curve E_D. From work of Cassels [1], using the Cassels-Tate pairing, we have that when Ш is finite, its order is going to be a square. Thus we expect that S_D to be an *integer square* up to the Tamagawa numbers.

For the case of prime numbers, Sylvester conjectured that the answer is affirmative in the case of $p \equiv 4, 7, 8 \mod 9$. In the cases of $p \equiv 2, 3, 5 \mod 9$ we have $L(E_p, 1) \neq 0$ and p is not the sum of two cubes. This follows either from a 3-descent argument (Sylvester, Lucas and Pepin) or from the theorem of Coates-Wiles [2].

In [14], Rodriguez-Villegas and Zagier computed formulas for $L(E_p, 1)$ in the case of primes $p \equiv 1 \mod 9$. In this case it is predicted by BSD that the rank of $E_D(\mathbb{Q})$ is either 0 or 2. They compute two formulas for S_D. In the current paper, we are extending their results to all integers D. If we let $K = \mathbb{Q}[\sqrt{-3}]$, we have:

Theorem 1.1. *For all integers D, S_D is an integer and we have the formula:*

$$S_D = \mathrm{Tr}_{H_{3D}/K} \left(D^{1/3} \frac{\Theta_K(D\omega)}{\Theta_K(\omega)} \right), \tag{3}$$

where:

- *H_{3D} is the ring class field associated to the order $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$,*
- *$\omega = \frac{-1+\sqrt{-3}}{2}$ is a third root of unity, and*
- *$\Theta_K(z) = \sum_{a,b\in\mathbb{Z}} e^{2\pi i z(a^2+b^2-ab)}$ is the theta function associated to the number field K.*

A second result makes S_D more easily computable. We also hope to extend this result to show that S_D is an integer square up to Tamagawa numbers:

Theorem 1.2. *In the case of $D = \prod_{p_i \equiv 1 \mod 3} p_i^{e_i}$, S_D is an integer and we have:*

$$S_D = \left| Tr_{H_{\mathcal{O}}/H_0} \frac{\theta_1(z_0)}{\theta_0(z_0)} D^{-1/3} \right|^2 \tag{4}$$

where:

- *$\theta_1(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i (n+1/D-1/6)^2 z}$ a 1/2-weight modular form*
- *$z_0 = \frac{-b+\sqrt{-3}}{2}$ a CM-point, with $b^2 \equiv -3 \mod 4D^2$,*

- *$H_{\mathcal{O}}$ is the ray class field of modulus $3D$ and H_0 is an intermediate field $K \subset H_0 \subset H_{\mathcal{O}}$ that is the fixed field of a certain Galois subgroup $G_0 \cong \mathrm{Cl}(\mathcal{O}_{3D})$.*

Conjecture 1.1. *We conjecture that the term $I_0 = \mathrm{Tr}_{H_{\mathcal{O}}/H_{3D}} \frac{\theta_1(z_0)}{\theta_0(z_0/D^2)} D^{1/3}$ is an integer and I_0^2 is the order of the Tate-Shafaravich group.*

Using similar methods, we obtain a general formula for all integers N, for which Theorem 1.2 is a particular case.

Theorem 1.3. *Using the same notation as in Theorem 1.2, we have for all integers D:*

$$S_D = \sum_{r=0}^{D-1} \left| Tr_{H_{\mathcal{O}}/H_{3D}} \frac{\theta_r(Dz_0)}{\theta_0(z_0/D)} D^{-1/3} \right|^2, \tag{5}$$

where:

- *$\theta_r(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i (n+r/D-1/6)^2 z}$ a 1/2-weight modular form*
- *$z_0 = \frac{-b+\sqrt{-3}}{2}$ a CM-point*

Conjecture 1.2. *We conjecture that all terms $I_r = \left| Tr_{H_{\mathcal{O}}/H_{3D}} D^{-1/3} \frac{\theta_r(Dz_0)}{\theta_0(z_0/D)} \right|^2$ are equal for all r such that $(r, D) = 1$. This is indeed the case when D is a product of primes that split.*

Further results not included in this draft:

In Appendix A we provide a second proof of Theorem 1.1 based on an idea of Xinyi Yuan. In Appendix B we compute a different formula for S_D inspired by the Rallis inner product. This expresses S_N in the following way:

Theorem 1.4. *For D a product of primes $p \equiv 1 \mod 4$, we have*

$$L(E_D, s) = c_D \sum_{\mathcal{A}\in Cl(\mathcal{O}_{3D})} E_0(s, g_{\mathcal{A}}, \Phi)\overline{\chi_D(\mathcal{A})}(-1)^{\frac{\mathrm{Nm}\,\mathcal{A}-1}{2}},$$

where $g_{\mathcal{A}}$ is the embedding of the generator of the ideal $\mathcal{A} = (a + b\sqrt{-3})$ into $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ and $E_0(s, g_z, \Phi) = \sum_{m\equiv 0 \mod D, (n,D)=1} \frac{1}{(mz+n)|mz+n|^s}$ is a sum of Eisenstein series defined by Hecke in [7].

2 Background.

Let $K = \mathbb{Q}[\sqrt{-3}]$. Note that K is a PID and has the ring of integers $\mathcal{O}_K = \mathbb{Z}[\omega]$, where $\omega = \frac{-1+\sqrt{-3}}{2}$ is a fixed root of unity. We will denote K_v the localization of K at the place v. We will denote by $K_p := \prod_{v|p} K_v \cong \mathbb{Q}_p[\sqrt{-3}]$.

2.1 The L-function

Our goal is to compute several formulas for the special value of the L-function $L(E_D, 1)$ of the elliptic curve $E_D : x^3 + y^3 = Dz^3$. The elliptic curve E_D has complex multiplication (CM) by $\mathcal{O}_K$. Then $L(E_D, s)$ is the L-function of a Hecke character that is computed explicitly in Ireland and Rosen [8]. We have:

$$L(E_D, s) = L(s, \overline{\chi_D}\varphi),$$

where χ_D and φ are classical Hecke characters such that $\varphi\chi_D$ is the Hecke character corresponding to the elliptic curve E_D. The Hecke character φ is the Hecke character corresponding to E_1 and χ_D is the Hecke character corresponding to the cubic twist. More precisely, the Hecke characters are defined to be:

- $\varphi : I(3) \to K^\times$ is defined on the ideals prime to 3 by $\varphi(\mathcal{A}) = \alpha$, where α is the unique generator of the ideal $\mathcal{A}$ such that $\alpha \equiv 1 \mod 3$.
- $\chi_D : I(3D) \to \{1, \omega, \omega^2\}$ is the cubic character defined below in Section 2.3; it is defined on the space $I(3D)$ of all fractional ideals of $\mathcal{O}_K$ prime to $3D$. Moreover, it is well-defined over $\mathrm{Cl}(\mathcal{O}_{3D})$ the ring class group corresponding to the order $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$.

The L-function can be expanded:

$$L(E_D, s) = \sum_{\mathcal{A} \in I(3D)} \frac{\overline{\chi_D}(\mathcal{A})\varphi(\mathcal{A})}{(\mathrm{Nm}\,\mathcal{A})^s} = \sum_{\alpha \in \mathcal{O}_K, \alpha \equiv 1 (\mathrm{mod}\ 3)} \frac{\overline{\chi_D(\alpha)}\alpha}{\mathbf{N}\alpha^s}.$$

2.2 Ring class fields.

Recall that an order $\mathcal{O}$ of K is a subring of $\mathcal{O}_K$ that is a finitely generated $\mathbb{Z}$-module and such that $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = K$. As K is a quadratic number field, each order is of the form $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ and we call $f = [\mathcal{O}_K : \mathcal{O}]$ the conductor of $\mathcal{O}$. We can also write $\mathcal{O}$ using a $\mathbb{Z}$-basis in the form $\mathcal{O} = [1, f\omega]_{\mathbb{Z}}$.

We define the class group $\mathrm{Cl}(\mathcal{O})$ of the order $\mathcal{O}$ of conductor f is defined to be:

$$\mathrm{Cl}(\mathcal{O}) := I_{\mathcal{O}}(f)/P_{\mathcal{O}}(f),$$

where $I_{\mathcal{O}}(f)$ is the set of fractional $\mathcal{O}$-ideals prime to the conductor f, and $P_{\mathcal{O}}(f)$ the subgroup of $I_{\mathcal{O}}(f)$ of principal fractional $\mathcal{O}$-ideals.

We define the ring class field to be the abelian extension $H_{\mathcal{O}}$ of K corresponding to the Galois group $\mathrm{Cl}(\mathcal{O})$ from class field theory, meaning:

$$\mathrm{Gal}(H_{\mathcal{O}}/K) \cong \mathrm{Cl}(\mathcal{O}).$$

We denote by $I(N)$ the group of fractional ideals in K prime to N. We denote the subgroup $P_{\mathbb{Z},N} = \{(\alpha)\colon\ \alpha \in K$ such that $\alpha \equiv a \mod N$ for some integer a such that $gcd(a, N) = 1\}$. Furthermore, let $\mathcal{O}_N := \mathbb{Z} + N\mathcal{O}_K$ be the order of K of conductor N. Then we can define the ring class field of the order $\mathcal{O}_N$ to be

$$\mathrm{Cl}(\mathcal{O}_N) := I(N)/P_{\mathbb{Z},N}$$

Note that K has class number one and thus by the Strong Approximation theorem we have:

$$\mathbb{A}_K^\times = K^\times \mathbb{C}^\times \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times.$$

We would like to describe $Cl(\mathcal{O}_N)$ adelically. We do this below:

Lemma 2.1. *For N a positive integer, we can think of the ring class group adelically as:*

$$\mathrm{Cl}(\mathcal{O}_N) \cong U(N) \backslash \mathbb{A}_{K,f}^\times / K^\times,$$

where $U(N) = \prod_p (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$.

Proof: From the Strong approximation theorem, as K is a PID, we have:

$$\mathbb{A}_K^\times \cong K^\times \mathbb{C}^\times \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times.$$

Taking the quotient by $K^\times \mathbb{C}^\times$, we get:

$$\mathbb{A}_{K,f}^\times / K^\times \cong \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times / \left(K^\times \cap \prod_v \mathcal{O}_{K_v}^\times \right) \cong \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times / \langle -\omega \rangle,$$

where $\langle -\omega \rangle$ is the group of sixth roots of unity.

Furthermore, note that $U(N) \cong \prod_{v \nmid N} \mathcal{O}_{K_v}^\times \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$. Moreover note that $\langle -\omega \rangle U(N) = U(N)$. Thus we have:

$$\mathbb{A}_{K,f}^\times / K^\times U(N) \cong \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times / \langle -\omega \rangle U(N) \cong \prod_{v|N} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times \cong \prod_{p|N} \prod_{v|p} \mathcal{O}_{K_v}^\times / (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$$

Finally, we need to show an isomorphism between $\mathrm{Cl}(\mathcal{O}_N) = I(N)/P_{\mathbb{Z}}(N)$ and $\prod_{v|N} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$. We construct the map:

$$I(N) \to \prod_{v|N} \mathcal{O}_{K_v}^\times \to \prod_{v|p} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$$

Let $(k_0) \in I(N)$ be an ideal. Then we can map $k_0 \to (k_0)_{v|N}$. After taking the projection map, we want to look at the kernel of the composition $I(N) \to \prod_{v|p} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$. This consists of ideals $(k_0) \in I(N)$ such that $k_0 \equiv a_p \mod N\mathbb{Z}_p[\omega]$, where $a_p \in \mathbb{Z}$ and $(a_p, p) = 1$.

By the Chinese remainder theorem, we can find $a \in \mathbb{Z}$ such that $a \equiv a_p \mod N$ for all $p|N$. Then we have $k_0 \equiv a \mod N\mathbb{Z}_p[\omega]$ for all $a \in \mathbb{Z}$. Thus $(k_0) \in P_{\mathbb{Z}}(N)$ and $P_{\mathbb{Z}}(N)$ is the kernel of the above map. Thus we get:

$$I(N)/P_{\mathbb{Z}}(N) \cong \prod_{v|p} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times,$$

which proves our claim.

Another easy result that we will use is the following straight forward application of the Chinese remainder theorem. This map will be important in our proof:

Lemma 2.2. *For any* $(l_{1,v})_{v|N} \in \prod_{v|N} \mathcal{O}_{K_v}^{\times}$, *we can find* $k_1 \in \mathcal{O}_K$ *such that for all* $v|N$ *we have:*

$$l_{1,v} \equiv k_1 \mod N\mathcal{O}_{K_v},$$

Proof: For any $v|N$ we can find $a_{1,v} \in \mathcal{O}_K$ such that $l_{1,v} \equiv a_{1,v} \mod N\mathcal{O}_{K_v}$. We will pick for $N = \prod_{v|N} \mathfrak{p}_v^{e_v}$, where $\mathfrak{p}_v$ is the prime corresponding to the place v:

$$k_1 = \sum_{v|N} a_{1,v} m_v \frac{N}{\mathfrak{p}_v^{e_v}},$$

where $m_v \in \mathcal{O}_K$, $m_v \frac{N}{\mathfrak{p}_v^{e_v}} \equiv 1 \mod \mathfrak{p}_v^{e_v}$. We can find such an inverse since $\mathcal{O}_K$ is a PID, thus $\mathcal{O}_K/N\mathcal{O}_K \cong \prod_{v|N} \mathcal{O}_K/\mathfrak{p}_v^{e_v}\mathcal{O}_K$.

2.2.1 Characterization of ideals in ring class fields

Recall that a primitive ideal is an ideal not divisible by any integral ideal. It is easy to prove:

Lemma 2.3. *Any primitive ideal of* $\mathcal{O}_K$ *can be be written in the form* $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]$ *as a* $\mathbb{Z}$*-module, where* b *is an integer (determined modulo* $2a$*) such that* $b^2 \equiv -3 \mod 4a$ *and* $\operatorname{Nm} \mathcal{A} = a$. *This implies that for* $\mathcal{A} = (\alpha)$, *we have* $\|\alpha\| = a$.

Conversely, given an integer satisfying the above congruence and $\mathcal{A}$ *defined as above, we get that* $\mathcal{A}$ *is an ideal in* $\mathcal{O}_K$ *of norm* a.

2.3 The cubic character

In the following we will define the cubic character χ_D and check that it is well defined on the class group $\mathrm{Cl}(\mathcal{O}_{3D})$. Let $\omega = \frac{-1+\sqrt{-3}}{2}$ be a fixed cube root of unity. Then we can define the cubic residue character following Ireland and Rosen [8].

Definition 2.1. *For* $\alpha \in \mathbb{Z}[\omega]$ *such that* α *is prime to 3, we define a cubic residue character* $\chi_\alpha : I(3\alpha) \to \{1, \omega, \omega^2\}$ *on the fractional ideals of* K *prime to* 3α. *For every prime ideal* $\mathfrak{p}$ *of* $\mathbb{Z}[\omega]$, *the character is defined to be:*

$$\chi_\alpha(\mathfrak{p}) = \omega^j,$$

for $j \in \{0, 1, 2\}$ *such that* ω^j *is the unique third root of unity for which:*

$$\alpha^{(\operatorname{Nm}\mathfrak{p}-1)/3} \equiv \omega^j \mod \mathfrak{p}, \text{ for } \operatorname{Nm}\mathfrak{p} \neq 3.$$

It is further defined multiplicatively on the fractional ideals of $I(3\alpha)$.

Notation: We will also denote $\chi_D(\cdot) =: \left(\frac{D}{\cdot}\right)_3$.

First let us check that this definition makes sense. Since K is a PID, any prime ideal $\mathfrak{p}$ has a generator of the form $\pi = a + b\omega \in \mathbb{Z}_p[\omega]$. Then the norm $\mathbf{N}\mathfrak{p} = a^2 - ab + b^2$ which is congruent to $0, 1 \mod 3$. Then, if $\mathfrak{p}$ is prime to 3, we must have $\mathbf{N}\mathfrak{p} \equiv 1 \mod 3$, implying that 3 divides $\mathbf{N}\mathfrak{p} - 1$.

Furthermore, the group $(\mathbb{Z}[\omega]/\mathfrak{p}\mathbb{Z}[\omega])^\times$ has $\operatorname{Nm}\mathfrak{p}-1$ elements, thus we have $\alpha^{\operatorname{Nm}\mathfrak{p}-1} \equiv 1 \mod \mathfrak{p}$. Then since $\operatorname{Nm}\mathfrak{p}-1$ is divisible by 3, we can factor out:

$$\mathfrak{p}|(\alpha^{(\operatorname{Nm}\mathfrak{p}-1)/3}-1)(\alpha^{(\operatorname{Nm}\mathfrak{p}-1)/3}-\omega)(\alpha^{(\operatorname{Nm}\mathfrak{p}-1)/3}-\omega^2)$$

Finally since $K=\mathbb{Q}[\sqrt{-3}]$ is an UFD, $\mathfrak{p}$ divides exactly one of these terms, say $(\alpha^{(\operatorname{Nm}\mathfrak{p}-1)/3}-\omega^i)$. Thus we can take $\chi_\alpha(\mathfrak{p})=\omega^i$ and it is well-defined.

Following Ireland and Rosen, it is natural to look at the primary elements of K:

Definition 2.2. *For a prime ideal $\mathfrak{p}$ of K we call π* ***primary*** *if π generates $\mathfrak{p}$ a prime ideal and $\pi \equiv 2 \mod 3$.*

Lemma 2.4. *For any ideal $\mathcal{A}$ prime to 3, we can find a generator $\alpha \in \mathbb{Z}[\omega]$ such that $\alpha \equiv 2 \mod 3$.*

Proof: Since K is a PID, we can find a generator $\alpha_0 = a+b\omega$ be a generator of $\mathcal{A}$. Then note that $\pm\alpha_0, \pm\alpha_0\omega, \pm\alpha_0\omega^2$ also generate the ideal $\mathcal{A}$ and exactly one of them is $\equiv 2 \mod 3$.

Remark 2.1. Note that from the definition of χ_{π_1} we have $\chi_{\pi_1}(\pi_2)=\chi_{-\pi_1}(\pi_2)$, as $\pi_1^{(\operatorname{Nm}\pi_2-1)/3} = (-\pi_1)^{(\operatorname{Nm}\pi_2-1)/3}$ when $\operatorname{Nm}\pi_2$ is odd and $\pi_1^{(\operatorname{Nm}2-1)/3} \equiv (-\pi_1)^{(\operatorname{Nm}2-1)/3} \equiv 1 \mod 2$ when $\pi_2=2$. Moreover $\chi_{\pi_1}(\pi_2)=\chi_{\pi_1}(-\pi_2)$, as $\chi_{\pi_1}(-1)=1$. Then we actually have for any choices of $\pm$:

$$\chi_{\pm\pi_1}(\pm\pi_2)=\chi_{\pm\pi_2}(\pm\pi_1)$$

Theorem 2.1. ***(Cubic reciprocity law)****. For $\pi_1,\pi_2 \equiv 2 \mod 3$ primary generators of primes $\mathfrak{p}_1,\mathfrak{p}_2$, $N\pi_1 \neq N\pi_2$ and $N\pi_1, N\pi_2 \neq 3$, then:*

$$\left(\frac{\pi_1}{\pi_2}\right)_3 = \left(\frac{\pi_2}{\pi_1}\right)_3$$

Corollary 2.1. *For $\pi_i,\pi_i' \equiv 2 \mod 3$, we have*

$$\chi_{\pm\pi_1\ldots\pi_n}(\pm\pi_1'\ldots\pi_n')=\chi_{\pm\pi_1'\ldots\pi_n'}(\pm\pi_1\ldots\pi_n)$$

Proof: We will first show that $\chi_{\pi_1\ldots\pi_n}(\pi_i')=\chi_{\pi_i}(\pi_1\ldots\pi_n)$. By definition, we have:

$$\chi_{\pi_1\ldots\pi_n}(\pi_i') \equiv (\pi_1\ldots\pi_n)^{(\operatorname{Nm}\pi_i'-1)/3} \mod \pi_i'$$

Thus, we have:

$$\chi_{\pi_1\ldots\pi_n}(\prod_{i=1}^m \pi_i') = \prod_{i=1}^m \chi_{\pi_1\ldots\pi_n}(\pi_i') = \prod_{i=1}^m\prod_{j=1}^n \chi_{\pi_j}(\pi_i')$$

Using the cubic reciprocity, we have $\chi_{\pi_j}(\pi_i')=\chi_{\pi_i'}(\pi_j)$, thus we get $\prod_{i=1}^m\prod_{j=1}^n \chi_{\pi_j}(\pi_i') = \prod_{i=1}^m\prod_{j=1}^n \chi_{\pi_i'}(\pi_j)$, which furthermore implies:

$$\chi_{\pi_1\ldots\pi_n}(\prod_{i=1}^m \pi_i') = \chi_{\pi_1'\ldots\pi_m'}(\prod_{j=1}^n \pi_i).$$

Note that we can always write the elements of $\mathbb{Z}[\omega]$ that are congruent to $\pm 1 \mod 3$ as a product of primary elements up to sign. Using the above corollary for α and D, we get:

Corollary 2.2. *If $\alpha \equiv \pm 1 \mod 3$ and D an integer prime to 3, then we have:*

$$\chi_D(\alpha) = \chi_\alpha(D)$$

Proof: Since $\alpha, D \equiv \pm 1 \mod 3$, we can write each of them in the form $\alpha = \pm \prod_{i=1}^n \pi_i$ and $D = \pm \prod_{j=1}^m \pi'_j$.

Then using the previous Corollary and Remark 2.1, we have $\chi_{\pm \prod\limits_{i=1}^n \pi_i}(\pm \prod\limits_{j=1}^m \pi_j) = \chi_{\pm \prod\limits_{j=1}^m \pi_j}(\pm \prod\limits_{i=1}^n \pi_i)$.

Lemma 2.5. *Let α be prime to 3 and $\mathfrak{p}$ a prime ideal prime to 3. Then the cubic residue can also be rewritten as:*

$$\chi_\alpha(\mathfrak{p}) \equiv \left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\pi - 1)/3} \mod \pi$$

Proof: We have by definition $\chi_\alpha(\mathfrak{p}) \equiv \alpha^{(\mathrm{Nm}\,\pi-1)/3} \equiv \omega^i \mod \mathfrak{p}$. Taking the complex conjugate we have $\overline{\alpha}^{(\mathrm{Nm}\,\pi-1)/3} \equiv \omega^{2i} \mod \mathfrak{p}$. Then by taking the ratio we get:

$$\left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\pi-1)/3} \equiv \frac{\omega^{2i}}{\omega^i} \mod \mathfrak{p}$$

Thus we have $\chi_\alpha(\mathfrak{p}) \equiv \alpha^{(\mathrm{Nm}\,\pi-1)/3} \equiv \omega^i \equiv \left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\pi-1)/3} \mod \mathfrak{p}$ which finishes the proof of the lemma.

Corollary 2.3. *Let $D = \prod\limits_{i=1}^m \mathfrak{p}_i$. For $\alpha \in \mathcal{P}_{\mathbb{Z},3D}$, we have $\chi_D(\alpha) = 1$. Thus χ_D is well defined on $\mathrm{Cl}(\mathcal{O}_{3D})$.*

Proof: Recall from the previous Lemma that if $\alpha \equiv \pm 1 \mod 3$, then we have:

$$\chi_\alpha(\mathfrak{p}) \equiv \left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} \mod \mathfrak{p}$$

Let $\mathfrak{p}|D$. Since $\alpha \in \mathcal{P}_{\mathbb{Z},3D}$, we have $\alpha \equiv a \mod 3D$ for some $a \in \mathbb{Z}$ and $(a, 3D) = 1$. Thus $\alpha \equiv a \mod \mathfrak{p}$, which also $\overline{\alpha} \equiv \overline{a} \mod \mathfrak{p}$, which implies:

$$\chi_\alpha(\mathfrak{p}) \equiv \left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} \equiv \left(\frac{\overline{a}}{a}\right)^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} \equiv 1 \mod \mathfrak{p}$$

Thus we get $\chi_\alpha(\mathfrak{p}) = 1$ for all $\mathfrak{p}|D$. Thus we have $\chi_\alpha(D) = 1$. Moreover, using Corollary 2.2, we have $\chi_D(\alpha) = \prod\limits_{i=1}^m \chi_{\mathfrak{p}_i}(\alpha) = \prod\limits_{i=1}^m \chi_\alpha(\mathfrak{p}_i) = 1$.

Remark 2.2. For any fractional ideal $\mathcal{A}$ of K, when we write $\chi_D(\mathcal{A})$ we will mean:

$$\chi_D(\mathcal{A}) := \chi_D(\alpha),$$

where α is the unique generator of $\mathcal{A}$ such that $\alpha \equiv 1 \mod 3$.

2.3.1 Relating χ_D to the Galois conjugates of $D^{1/3}$.

There is another way to look at the cubic character using the Galois conjugates of $D^{1/3}$. We have the following lemma:

Lemma 2.6. *Let D be an integer prime to 3. Then for a prime ideal $\mathfrak{p}$ of K prime to $3D$, we have:*

$$D^{1/3}\chi_D(\mathfrak{p}) = (D^{1/3})^{\sigma_{\mathfrak{p}}},$$

where $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(\mathbb{C}/K)$ is the Galois action corresponding to the ideal $\mathfrak{p}$ in the Artin correspondence.

Proof: It is enough to prove the claim for $\sigma_i \in \mathrm{Gal}(F/K)$, where $L = K[D^{1/3}, D^{1/3}\omega, D^{1/3}\omega^2]$. Let $\sigma_{\mathfrak{p}} = \left(\frac{L/K}{\mathfrak{p}}\right)$ the Frobenius element corresponds to $\mathfrak{p}$ the prime ideal of $\mathcal{O}_K$. Then using the definition of the Frobenius element for $D^{1/3} \in L$, we get:

$$(D^{1/3})^{\sigma_{\mathfrak{p}}} \equiv (D^{1/3})^{\mathrm{Nm}\,\mathfrak{p}} \mod \mathfrak{p}\mathcal{O}_L$$

Furthermore, note that $(D^{1/3})^{\mathrm{Nm}\,\mathfrak{p}} = D^{1/3}D^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} \equiv D^{1/3}\chi_D(\mathfrak{p}) \mod \mathfrak{p}\mathcal{O}_L$.Since the Galois conjugates of $D^{1/3}$ are the roots of x^3-D, the Galois conjugate $(D^{1/3})^{\sigma_{\mathfrak{p}}} \in \{D^{1/3}, D^{1/3}\omega, D^{1/3}\omega^2\}$ and from the congruences above we get:

$$(D^{1/3})^{\sigma_{\mathfrak{p}}} = D^{1/3}\chi_D(\mathfrak{p})$$

Corollary 2.4. *Let D be an integer prime to 3 and $\mathcal{A}$ an ideal of K prime to $3D$. Moreover, let $\sigma_A \in \mathrm{Gal}(K^{ab}/K)$ be the Galois action corresponding to the ideal $\mathcal{A}$ through the Artin map. Then for the cubic character χ_D, we have:*

$$(D^{1/3})^{\sigma_{\mathcal{A}}} = D^{1/3}\chi_D(\mathcal{A}). \tag{6}$$

Proof: Let $\mathcal{A} = \prod_j \mathfrak{p}_j^{f_j}$ the prime decomposition of $\mathcal{A}$ in K. Note that $\chi_D(\mathfrak{p}_i) \in K$, thus it is preserved by the Galois action. Applying the above Lemma we get:

$$((D^{1/3})^{\sigma_{\mathfrak{p}_i}})^{\sigma_{\mathfrak{p}_j}} = (D^{1/3}\chi_D(\mathfrak{p}_i))^{\sigma_{\mathfrak{p}_i}} = D^{1/3}\chi_D(\mathfrak{p}_j)\chi_D(\mathfrak{p}_j)$$

Using this step repeatedly, we get $(D^{1/3})^{\sigma_{\mathcal{A}}} = D^{1/3}\chi_D(\mathcal{A}) = D^{1/3}\chi_D(\mathcal{A})$.

Remark 2.3. Note that for the complex conjugate character $\overline{\chi_D}$ we have a similar result:

$$(D^{2/3})^{\sigma_{\mathcal{A}}} = D^{2/3}\overline{\chi_D(\mathcal{A})}. \tag{7}$$

2.4 Hecke characters

There are two equivalent ways of defining a Hecke character: classically and adelically. We define the **classical Hecke character** over K to be $\widetilde{\chi} : I(f) \to \mathbb{C}^\times$ a character from the set of fractional ideals prime to f, where f is a nonzero ideal of $\mathcal{O}_K$. We further say that $\widetilde{\chi}$ has

infinity type $\widetilde{\chi}_\infty$ if it is characterized by the condition that on the set of principal ideals $P(f)$ prime to f it satisfies the condition:

$$\widetilde{\chi}((\alpha)) = \widetilde{\epsilon}(\alpha)\widetilde{\chi}_\infty^{-1}(\alpha),$$

where:

- $\widetilde{\varepsilon} : (\mathcal{O}_K/f\mathcal{O}_K)^\times \to \mathbb{T}$ is called the $(\mathcal{O}_K/f\mathcal{O}_K)^\times$-type character i.e. $\tilde{\varepsilon}$ is a character taking values in a finite group $\mathbb{T}$.
- $\widetilde{\chi}_\infty$ is an infinity type continuous character i.e. $\widetilde{\chi}_\infty : \mathbb{C}^\times \to \mathbb{C}^\times$ is a continuous character.

We define the **idelic Hecke character** to be a continuous character $\chi : \mathbb{A}^\times/K^\times \to \mathbb{C}^\times$.

There is a unique correspondence between the idelic and the classical Hecke characters. The correspondence can be explicitly constructed in the following way:

- $\tilde{\chi}(\mathcal{O}_v^\times \varpi_v) := \chi(\mathfrak{p}_v)$, $v \nmid f$
- $\tilde{\chi}_\infty$ is determined by χ_∞
- $\tilde{\chi}_v$ with $v|f$ is determined by Weak Approximation Theorem.

2.4.1 Converting the characters.

We want to compute a formula for $L(s,\chi)$, where $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ is the Hecke character defined by $\chi = \chi_{3D}\varphi$. Here $\chi_{3D}\varphi$ are the adelic correspondent Hecke characters of the classical Hecke characters:

1. $\chi_{3D} : I(3D) \to \{1, \omega, \omega^2\}$ is the cubic character.
2. $\varphi : I(3) \to \mathbb{C}^\times$ is the Hecke character defined by $\chi((\alpha)) = \alpha$ for $\alpha \equiv 1 \mod 3$.

By abuse of notation, I will use φ, χ_{3D} both for the classical and the adelic Hecke characters. This should be clear from the context. We can rewrite the two characters adelically:

1. $\varphi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ such that:

$$\begin{cases} \varphi_v(p) = -p,\ \varphi_v(\mathcal{O}_{K_v}^\times) = 1, & \text{if } v|p, p \equiv 2 \mod 3, \\ \varphi_v(\varpi_v) = \varpi_v,\ \varphi_v(\mathcal{O}_{K_v}^\times), \text{ where } \omega_v \text{ uniformizer of } \mathcal{O}_{K_v}, \varpi_v \equiv 1 \mod 3, & \text{if } v|p, p \equiv 1 \mod 3, \\ \varphi_\infty(x_\infty) = x_\infty^{-1}, & v = \infty \\ \varphi_v(\varpi_v) \text{ can be determined from the Weak approximation theorem,} & \text{if } v = \sqrt{-3}|3 \end{cases}$$

2. Note that χ_{3D} is trivial on $P_{\mathbb{Z},3D}$, thus χ_{3D} is a character on $Cl(\mathcal{O}_{3D})$. We will define the character by making it trivial on $\mathbb{C}^\times$, $U(3D)$ and $K^\times$. Then we can define using Lemma 2.2:

$$\chi_{3D}(l) = \chi_{3D}(l_1) = \chi_{3D}((k_1)).$$

More precisely, this will be:

$$\begin{cases} \chi_{3D,v}(\varpi_v) = \chi_{3D}((\omega_v)),\ \chi_{3D,v}(\mathcal{O}_{K_v}^\times) = 1, & \text{if } v \nmid 3D \\ \chi_{3D,\infty}(x_\infty) = 1, & v = \infty \\ \chi_{3D,v}(\varpi_v) \ \text{ can be determined from the Weak approximation theorem,} & \text{if } v|3D \end{cases}$$

We can generally compute $\chi_f(l_f)$ in the following way:

Lemma 2.7. *If $\chi = \chi_{3D}\varphi$, let $l_f = kl_1$, $k \in K^\times, l_1 \in \prod_v \mathcal{O}_{K_v}^\times$. Note that this decomposition is unique up to a unit of $\mathcal{O}_K^\times$ and pick k such that $l_{1,3} \equiv 1 \mod 3$. Moreover take $k_1 \in K^\times$ such that $l_1 \equiv k_1 \mod 3D\mathcal{O}_{K_v}$. Then:*

$$\chi_f(l_f) = k\overline{\tilde{\chi}_{3D}((k_1))}$$

Proof: We start by writing:

$$\chi_f(l_f) = \chi_f(k)\chi_f(l_1) = \chi_\infty(k)^{-1}\chi_{v|3D}(l_{1,v})$$

Moreover, from the Chinese remainder theorem, we can find $k_1 \in K^\times$ such that $k_1 \equiv l_{1,v} \mod 3D\mathcal{O}_{K_v}$. As we have $k_1^{-1}l_1 \in 1 \mod 3D\mathcal{O}_{K_v}$ and χ is trivial on $(\mathbb{Z} + 3D\mathcal{O}_{K_v})^\times$ for $v|3D$, we get $\chi_v(k_1) = \chi(l_{1,v})$. This implies:

$$\chi_f(l_f) = k\chi_{v|3D}(k_1) = k\chi_{v\nmid 3D}(k_1)^{-1}\chi_\infty(k_1)^{-1}$$

Note that if we write $k_1 = u\prod_v \omega_v^{e_v}$, where $u \in \mathcal{O}_{K_v}^\times$, we get:

$$\prod_{v\nmid 3D} \chi_v(k_1) = \prod_{v\nmid 3D} \chi_v(\omega_v)^{e_v} = \prod_{v\nmid 3D} \tilde{\chi}(\mathfrak{p}_v)^{e_v} = \tilde{\chi}((k_1))$$

This moreover implies:

$$\chi_f(l_f) = k\tilde{\chi}((k_1))^{-1}k_1 = kk_1^{-1}k_1\overline{\tilde{\chi}_{3D}((k_1))} = k\overline{\tilde{\chi}_{3D}((k_1))}$$

3 Computing the value $L(E_D, 1)$ using Tate's Zeta function

In this section we will compute the value of $L(E_D, 1) = L(1, \chi_D\varphi)$, working with χ_D, φ as automorphic Hecke characters. We will show the following result:

3.0.2 Schwartz-Bruhat functions.

We take $V = K$ a quadratic vector space over $\mathbb{Q}$ and $V_{\mathbb{A}_K} = \mathbb{A}_\mathbb{Q} \otimes_\mathbb{Q} K$. Then we can define the Schwartz-Bruhat functions $\Phi = \prod_v \Phi_v$, $\Phi_v \in S(V_{\mathbb{A}_K})$ to be:

$$\begin{cases} \Phi_v = \mathrm{char}_{\mathcal{O}_{K_v}}, & \text{if } v \nmid 3D \\ \Phi_p = \sum\limits_{(a,D)=1} \mathrm{char}_{(a+D\mathcal{O}_{K_p})}, & \text{if } v|3D, v \nmid 3, \mathcal{O}_{K_p} = \prod_{v|p} \mathcal{O}_{K_v} \\ \Phi_p = \mathrm{char}_{(1+3\mathcal{O}_{K_v})}, & \text{if } v = \sqrt{-3} \\ \Phi_\infty(z) = ze^{-\pi q(\det z)}, & \text{where } z \in \mathbb{C} \end{cases}$$

Here $q(z) = |z|^2$ the usual absolute value on $\mathbb{C}$.

Remark 3.1. $\mathrm{char}_{(a+p\mathcal{O}_{K_p})}(m) = \prod_{v|p} \mathrm{char}_{(a+D\mathcal{O}_{K_v})}(m) = \prod_{v|p} \mathrm{char}_{(1+p\mathcal{O}_{K_v})}(a^{-1}m)$ and each $\mathrm{char}_{(1+p\mathcal{O}_{K_v})}$ is a locally constant function with compact support. We are taking a linear combination of these Schwartz-Bruhat functions, thus we do get a Schwartz-Bruhat function.

3.0.3 Haar measure.

We pick the usual additive Haar measure:

$$\begin{cases} d_v^\times x_v = \frac{dx_v}{|x_v|_v}, \text{ normalized such that } \mathrm{vol}(\mathcal{O}_{K_v}^\times) = 1, & \text{if } v \nmid \infty \\ d^\times z = \frac{dz}{|z|_\mathbb{C}}, dz \text{ usual Lebesgue measure,} & z \in \mathbb{C}, |z|_\mathbb{C} = x^2 + y^2, \text{for } z = x + yi \end{cases}$$

We also define the multiplicative Haar measure:
XXXXXX

3.1 Zeta functions

We recall Tate's zeta function. For a Hecke character $\chi_v : K_v^\times \to \mathbb{C}^\times$ and a Schwartz-Bruhat function $\Phi_v \in \mathcal{S}(K_v)$, it is defined locally to be:

$$Z_v(s, \chi_v, \Phi_v) = \int\limits_{K_v^\times} \chi_v(\alpha_v)|\alpha_v|_v^s \Phi_v(\alpha_v) d^\times \alpha_v,$$

where $d^\times \alpha_v$ is the multiplicative Haar measure defined above.
We define globally $Z(s, \chi, \Phi) = \prod_v Z_v(s, \chi_v, \Phi_v)$. As a global integral, this is:

$$Z(s, \chi, \Phi) = \int\limits_{\mathbb{A}_K^\times} \chi(\alpha)|\alpha|^s \Phi(\alpha) d^\times \alpha,$$

Lemma 3.1. *For all s and Φ Schwartz-Bruhat functions chosen as above, we have:*

$$L_f(s, \chi_D\varphi) = Z_f(s, \chi_D\varphi, \Phi)V_{3D},$$

where $V_{3D} = \mathrm{vol}(1 + 3\mathbb{Z}_3[\omega])\,\mathrm{vol}(\mathbb{Z} + D\prod\limits_{p|D} \mathbb{Z}_p[\omega])^\times = \dfrac{1}{6}\prod\limits_{p|D} \dfrac{1}{(p - (\frac{p}{3}))}$

Proof: From Tate's thesis, we have $L_f(s, \chi_D\varphi) = Z_f(s, \chi_D\varphi)\dfrac{\prod_{p|3D} L_p(s, \chi_{D,p}\varphi_p)}{\prod_{p|3D} Z_f(s, \chi_{D,p}\varphi_p, \Phi_p)}$.
Since $\chi_D\varphi$ is ramified at $3D$, we have $L_p(s, \chi_{D,p}\varphi_p) = 1$. We need to compute:

$$Z_p(s, \chi_D\varphi, \Phi_p) = \int\limits_{\mathbb{Q}_p[\omega]^\times} \chi_{D,p}(\alpha_p)\varphi_p(\alpha_p)|\alpha_p|_p^s \Phi_p(\alpha_p) d^\times \alpha_p$$

From the definition for $p|D$, we have $\Phi_p = \mathrm{char}_{(\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times}$, the integral reduces to $Z_p(s, \chi_D\varphi, \Phi_p) = \int\limits_{(\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times} \chi_{D,p}(\alpha_p)\varphi_p(\alpha_p)|\alpha_p|_p^s d^\times \alpha_p$. Note that for $p \neq 3$, all the characters χ_D, φ and $|\cdot|_p$ are unramified, thus we just get the volume $\mathrm{vol}\left((\mathbb{Z} + 3D\mathbb{Z}_p[\omega])^\times\right)$.

For $p = 3$, we have $\Phi_p = \mathrm{char}_{(1+3\mathbb{Z}_3[\omega])}$. Similarly, we get $\mathrm{vol}\left((1 + 3\mathbb{Z}_3[\omega])^\times\right)$.

We compute the volumes. For D a product of primes, we have

$$\mathrm{vol}\left((\mathbb{Z} + 3D\mathbb{Z}_p[\omega])^\times\right) = \mathrm{vol}\left((\mathbb{Z} + p\mathbb{Z}_p[\omega])^\times\right) = (p-1)\,\mathrm{vol}\,(1 + p\mathbb{Z}_p[\omega]) = \frac{1}{(p - (\frac{p}{3}))}$$

Note that $\mathrm{vol}\,(1 + p\mathbb{Z}_p[\omega]) = \frac{1}{p^2-1}\,\mathrm{vol}(\mathbb{Z}_p[\omega]^\times)$ when p is nonsplit and $\mathrm{vol}\,(1 + p\mathbb{Z}_p[\omega]) = \frac{1}{(p-1)^2}\,\mathrm{vol}(\mathbb{Z}_p^\times)^2$ when p is split. This is computed by writing:

- p nonsplit: $\mathrm{vol}(\mathbb{Z}_p[\omega]^\times) = \sum \mathrm{vol}(a + b\omega + p\mathbb{Z}_p[\omega])$, where the sum is taken over all $a + b\omega$ prime to p and $0 \leqslant a, b \leqslant p - 1$. We count $p^2 - 1$ of them and we get $\mathrm{vol}(\mathbb{Z}_p[\omega]^\times) = (p^2 - 1)\,\mathrm{vol}(1 + p\mathbb{Z}_p[\omega])$.
- p split: $\mathrm{vol}(\mathbb{Z}_p[\omega]^\times) = \sum \mathrm{vol}(a + b\omega + p\mathbb{Z}_p[\omega])$. We count similarly $p^2 - 2p + 1$ such terms, as p splits and we have to discard the divisors of p.

For $p = 3$, we have $\mathrm{vol}\,(1 + 3\mathbb{Z}_3[\omega]) = \frac{1}{5}$.

We compute:

- $\mathbb{Z}_3[\omega] = \mathbb{Z}_3[\sqrt{-3}] = \{a_0 + a_1\sqrt{-3} + a_2(-3) + \ldots, 0 \leqslant a_i \leqslant 2\}$
- $\mathrm{vol}(\mathbb{Z}_3[\omega])^\times = 1$
- $(\mathbb{Z}_3[\omega])^\times = \bigcup(a_0 + a_1\sqrt{-3})(1 + 3\mathbb{Z}_3[\omega])$, where $a_0 + a_1\sqrt{-3}$ is prime to 3. Then we have 6 possibilities and thus $\mathrm{vol}(1 + 3\mathbb{Z}_3[\omega]) = \frac{1}{6}$.

By plugging in $s = 1$ in the above Lemma, we get:

Corollary 3.1. *The finite part of the L-function at $s = 1$ equals:*

$$L_f(1, \chi_D\varphi) = \frac{1}{6}\prod_{p|D} \frac{1}{(p - (\frac{p}{3}))} Z_f(1, \chi_D\varphi, \Phi),$$

3.1.1 Computing the finite part of Tate's Zeta function $Z_f(s, \chi_D\varphi, \Phi)$

In this section we will compute the value of $Z_f(s, \chi_D\varphi, \Phi)$. We begin by rewriting Tate's zeta function $Z_f(s, \chi_D\varphi, \Phi)$ as a linear combination of Hecke characters:

Lemma 3.2. *For all $s \in \mathbb{C}$ and the Schwartz-Bruhat functions $\Phi_f \in \mathcal{S}(\mathbb{A}_{K,f})$, we have:*

$$Z_f(s, \chi_D\varphi, \Phi_f) = V_{3D} \sum_{\alpha_f \in U(3D)\backslash \mathbb{A}_{K,f}^\times / K^\times} I(s, \alpha_f, \Phi_f)\chi_D(\alpha)\varphi(\alpha),$$

where $I(s, \alpha_f, \Phi_f) = \sum_{k \in K^\times} \frac{k}{|k|_{\mathbb{C}}^{2s}} \Phi_f(k\alpha_f)$.

Proof: By definition, we have $Z_f(s, \chi_D\varphi, \Phi_f) = \int_{\mathbb{A}_{K,f}^\times} \chi_D(\alpha_f)\varphi(\alpha_f)|\alpha_f|_f^s \Phi_f(\alpha_f) d^\times \alpha_f$. We rewrite the integral by taking a quotient by $K^\times$:

$$Z_f(s, \chi_D\varphi, \Phi_f) = \int_{\mathbb{A}_{K,f}^\times / K^\times} \sum_{k \in K^\times} \chi_{D,f}(k\alpha'_f)\varphi_f(k\alpha'_f)|k\alpha_f|_f^s \Phi_f(k\alpha'_f) d^\times \alpha'_f$$

Note that from the definition of Hecke characters, we have $\chi_{D,f}(k\alpha'_f) = \chi_{D,\infty}^{-1}(k)\chi_{D,f}(\alpha'_f) = \chi_{D,f}(\alpha'_f)$, $\varphi_f(k\alpha'_f) = \varphi_\infty^{-1}(k)\varphi_f(\alpha'_f) = k\varphi_f(\alpha'_f)$ and $|k\alpha'_f|_f^s = |k|_\infty^{-s}|\alpha_f|_f^s = |k|_{\mathbb{C}}^{-2s}|\alpha'_f|_f^s$, where $|\cdot|_{\mathbb{C}}$ is the usual absolute value over $\mathbb{C}$. Then the integral reduces to:

$$Z_f(s, \chi_D\varphi, \Phi_f) = \int_{\mathbb{A}_{K,f}^\times / K^\times} \left(\sum_{k \in K^\times} \frac{k}{|k|_{\mathbb{C}}^{2s}} \chi_{D,f}(\alpha'_f)\Phi_f(k\alpha'_f) \right) \varphi_f(\alpha'_f)|\alpha'_f|_f^s \; d^\times \alpha'_f$$

Furthermore, note that our choice of Schwartz-Bruhat functions $\Phi_f(k\alpha'_f)$ are invariant over $U(3D)$. Similarly:

- $|\cdot|_f$ is trivial on units, thus on $U(3D)$
- χ_D is invariant on $U(3D)$ by definition
- φ is trivial on all the units at all the unramified places. At 3 it is invariant under $1+3\mathbb{Z}_3[\omega]$, thus it is trivial on all of $U(3D)$

Thus we can take the quotient by $U(3D)$ as well. Note that the integral is now a finite sum:

$$Z_f(s, \chi_D\varphi, \Phi_f) = \mathrm{vol}(U(3D)) \sum_{\alpha''_f \in U(3D)\backslash \mathbb{A}_{K,f}^\times / K^\times} \left(\sum_{k \in K^\times} \frac{k}{|k|_{\mathbb{C}}^{2s}} \chi_{D,f}(\alpha''_f)\Phi_f(k\alpha''_f) \right) \varphi_f(\alpha''_f)|\alpha''_f|_f^s$$

Moreover, note that $\mathrm{vol}(U(3D)) = \mathrm{vol}(1 + 3\mathbb{Z}_3\omega) \prod_{p|D} \mathrm{vol}(\mathbb{Z} + D\mathbb{Z}_p[\omega]) = V_{3D}$.

By denoting $I(s, \alpha_f, \Phi_f) = \sum_{k \in K^\times} \frac{k}{|k|_{\mathbb{C}}^s} \Phi_f(k\alpha_f)$, we get the conclusion of the Lemma.

Combining the Lemma **??** and Lemma **??**, we get:

Corollary 3.2. *For all $s \in \mathbb{C}$ and the Schwartz-Bruhat functions $\Phi_f \in \mathcal{S}(\mathbb{A}_{K,f})$ chosen above, we have:*

$$L_f(s, \chi_D\varphi) = \sum_{\alpha_f \in U(3D)\backslash \mathbb{A}_{K,f}^\times / K^\times} I(s, \alpha_f, \Phi_f)\chi_D(\alpha)\varphi(\alpha),$$

3.1.2 Adelic representatives for $\mathrm{Cl}(\mathcal{O}_{3D})$

From the Strong approximation theorem, we can write $\alpha_f \in \mathbb{A}_K^\times = \mathbb{C}^\times K^\times \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times$ in the form $\alpha_f = \gamma_\infty k_\alpha \beta_f$, where $k_\alpha \in K^\times$, $\gamma_\infty \in \mathbb{C}^\times$ and $\beta_f \in \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times$. Then we can take representatives in $\alpha_f \in U(3D)\backslash\mathbb{A}_{K,f}^\times/K^\times$ such that $\alpha_f \in \prod_{v\nmid\infty} \mathcal{O}_{K,v}^\times$. Moreover, since we are taking the quotient by the cube roots of six $\{\pm 1, \pm\omega, \pm\omega^2\}$, we can pick α_f such that $\alpha_3 \equiv 1 \mod 3$. This can be done by replacing α_f by $\pm\alpha_f\omega^i$ for some i, $0 \leqslant i \leqslant 2$.

Furthermore, note that representatives α_f, α_f' are in the same class in $U(3D)$ iff $\alpha_f\alpha_f^{-1} \equiv a \mod D\mathbb{Z}_p[\omega]$, for some integer a such that $(a, D) = 1$.

Moreover, we can define an ideal $\mathcal{A}_\alpha$ that is generated by $k_\alpha \in \mathcal{O}_K$ such that

$$\alpha_p \equiv k_\alpha \mod 3D\mathbb{Z}_p[\omega].$$

Note that this ideal is unique only as a class in $\mathrm{Cl}(\mathcal{O}_{3D})$.

3.1.3 Connection to the Eisenstein series

Using the above representatives, note that φ_f and $|\cdot|_f$ are trivial for the representatives l_f and the Corollary **??** becomes:

$$L_f(s, \chi_D\varphi) = \sum_{\alpha_f \in U(3D)\backslash\mathbb{A}_{K,f}^\times/K^\times} I(s, \alpha_f, \Phi_f)\chi_D(\alpha_f)$$

We will now connect $I(s, \alpha_f, \Phi_f)$ to an Eisenstein series. We define the following classical Eisenstein series of weight 1:

$$E_\varepsilon(s, z) = \sum_{m,n} \frac{\varepsilon(n)}{(3mz+n)|3mz+n|^s},$$

where the sum is taken over all $m, n \in \mathbb{Z}$ except for the pair $(0, 0)$, and $\varepsilon = \left(\frac{\cdot}{3}\right)$ is the quadratic character associated to the field extension $K/\mathbb{Q}$.

Note that the Eisenstein series does not converge absolutely. However, we can still compute its value at 0 using the Hecke trick in order for it to converge. We will compute its Fourier expansion in the following section.

Recall that for $\alpha_f \in \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times$, we have the corresponding ideal class $[\mathcal{A}_\alpha]$ in $\mathrm{Cl}(\mathcal{O}_{3D})$. Such a representative is $\mathcal{A}_{\alpha_f} = (k_\alpha)$, where $k_\alpha \in \mathcal{O}_K$ is chosen such that $k_\alpha \equiv \alpha_p \mod 3D\mathbb{Z}_p[\omega]$ for $p|3D$. Note that we can pick $\mathcal{A}_\alpha$ to be a primitive ideal.

We can further write $\mathcal{A}_\alpha$ as a $\mathbb{Z}$-lattice $\mathcal{A}_\alpha = [a, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$, where $a = \mathrm{Nm}\,\mathcal{A}_\alpha$ and b is chosen (not uniquely) such that $b^2 \equiv -3 \mod 4a$. Then we can take the corresponding CM point $z_{\mathcal{A}_\alpha} := \frac{-b+\sqrt{-3}}{2a}$

Using this notation, we have the following equality:

Lemma 3.3. *For $\alpha_f \in \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times$ and any choice of $z_{\mathcal{A}_\alpha}$ as above, we have:*

$$I(s,\alpha_f,\Phi_f)=\frac{1}{2}\frac{(\mathrm{Nm}\,\mathcal{A}_\alpha)^{1-s}}{k_\alpha}E_\varepsilon(s,z_{\overline{\mathcal{A}_\alpha}})$$

Remark 3.2. Note that the variable $z_{\overline{\mathcal{A}_\alpha}}$ on the left hand side is not uniquely defined. However, the function is going to be invariant on the class $[\mathcal{A}_\alpha]$ in $\mathrm{Cl}(\mathcal{O}_{3D})$.

Proof: Recall that $I(s,\alpha_f,\Phi_f)=\sum_{k\in K^\times}\frac{k}{|k|_{\mathbb{C}}^{2s}}\Phi_f(k\alpha_f)$. We need to compute $\Phi_f(k\alpha_f)$. Note that for all finite places v we have $\Phi_v(k\alpha_v)\neq 0$ only for $k\alpha_v\in\mathcal{O}_{K_v}$, and since $\alpha_v\in\mathcal{O}_{K_v}^\times$, we must have $k\in\mathcal{O}_{K_v}$ for all $v\nmid\infty$. This implies $k\in\mathcal{O}_K$ and for all $v\nmid 3D$ we get $\Phi_v(k\alpha_v)=1$ for $k\in\mathcal{O}_K$. Thus we can rewrite:

$$I(s,\alpha_f,\Phi_f)=\sum_{k\in\mathcal{O}_K}\frac{k}{|k|_{\mathbb{C}}^{2s}}\Phi_{3D}(k\alpha_{3D}),$$

where $\Phi_{3D}=\prod_{v|3D}\Phi_v$ and $\alpha_{3D}=(\alpha_v)_{v|3D}$.

We can further compute $\Phi_v(k\alpha_v)$ for $v|3D$. Recall that for $p|D$ we defined $\Phi_p=\mathrm{char}_{(\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times}$ and $\Phi_3=\mathrm{char}_{(1+3\mathbb{Z}_3[\omega])^\times}$. Then we have $\Phi_{3D}(k\alpha_{3D})\neq 0$ iff $k\alpha_p\in a+3D\mathbb{Z}_p[\omega]$ for some integer $(a,p)=1$ and for $p=3$ we need $k\alpha_3\in 1+3\mathcal{O}_{K_3}$.

Recall that we can define k_α such that $k_\alpha\equiv\alpha_p \mod 3D\mathbb{Z}_p[\omega]$ for all $p|3D$. Then the we have $kk_\alpha\in a+3D\mathbb{Z}_p[\omega]$ for $(a,p)=1$ and $kk_\alpha\in 1+3\mathbb{Z}_3[\omega]$ as well. Furthermore, for $k\in\mathcal{O}_K$ we actually have $\Phi_{3D}(k\alpha_{3D})=\Phi_{3D}(kk_\alpha)$. Then we can rewrite $I(s,\alpha_f,\Phi_f)$ using k_α in the form:

$$I(s,\alpha_f,\Phi_f)=\sum_{k\in\mathcal{O}_K}\frac{k}{|k|_{\mathbb{C}}^{2s}}\Phi_{3D}(kk_\alpha),$$

We can rewrite this further:

$$I(s,\alpha_f,\Phi_f)=\frac{|k_\alpha|_{\mathbb{C}}^{2s}}{k_\alpha}\sum_{k\in\mathcal{O}_K}\frac{kk_\alpha}{|kk_\alpha|_{\mathbb{C}}^{2s}}\Phi_{3D}(kk_\alpha),$$

Finally, we will make this explicit. Note that we must have $kk_\alpha\in\mathcal{A}_\alpha$, where $\mathcal{A}_\alpha=(k_\alpha)$, we well as $kk_\alpha\in a_p+D\mathbb{Z}_p[\omega]$ for some integer a_p, $(a_p,p)=1$ as well as $kk_\alpha\in 1+3\mathbb{Z}_3[\omega]$. By the Chinese remainder theorem, we can find an integer a such that $a\equiv a_p \mod D$ and $a\equiv 1 \mod 3$. Then we have $kk_\alpha\in a+D\prod_{p|3D}\mathbb{Z}_p[\omega]\cap\mathcal{O}_K$, thus $kk_\alpha\in P_{\mathbb{Z},3D}\cap P_{1,3}$. Here $P_{\mathbb{Z},3D}=\{k\in K: k\equiv a \mod 3D\mathcal{O}_K$ for some integer $a,(a,3D)=1\}$ and $P_{1,3}=\{k\in K: k\equiv 1 \mod 3\}$. We rewrite:

$$I(s,\alpha_f,\Phi_f)=\frac{|k_\alpha|_{\mathbb{C}}^{2s}}{k_\alpha}\sum_{k\in\mathcal{A}_\alpha\cap P_{\mathbb{Z},D}\cap P_{1,3}}\frac{k}{|k|_{\mathbb{C}}^{2s}},$$

Finally, we want to write the elements of $\mathcal{A}_\alpha\cap P_{\mathbb{Z},D}\cap P_{1,3}$ explicitly.

Recall that we can write $\mathcal{A}_\alpha$ as a $\mathbb{Z}$-lattice $\mathcal{A}_\alpha=[a,\frac{b+\sqrt{-3}}{2}]$. Then all of the elements of $\mathcal{A}$ are of the form $ma+n\frac{b+\sqrt{-3}}{2}$ for some integers $m,n\in\mathbb{Z}$. Moreover, note that the intersection of $\mathcal{A}$ and $P_{\mathbb{Z},3D}=\{k\in\mathcal{O}_K: k\equiv n \mod 3D,$ for some integer $n,(n,3D)=1\}$ is $\{ma+3Dn\frac{b+\sqrt{-3}}{2}: m,n\in\mathbb{Z}\}$. Further taking the intersection with $P_{1,3}$, we must have $ma\equiv 1$, thus we must have $m\equiv 1 \mod 3$. Thus we can rewrite $I(s,\alpha_f,\Phi_f)$ in the form:

$$I(s,\alpha_f,\Phi_f)=\frac{a^s}{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 1(\text{mod }3)}\frac{ma+n\frac{-b+\sqrt{-3}}{2}}{|ma+n\frac{-b+\sqrt{-3}}{2}|_{\mathbb{C}}^{2s}},$$

Here we have also used the fact that $|k_\alpha|_{\mathbb{C}}=a$. Note that we can further rewrite this as:

$$I(s,\alpha_f,\Phi_f)=a^{s-1}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 1(\text{mod }3)}\frac{1}{(ma+n\frac{-b-\sqrt{-3}}{2})|ma+n\frac{-b+\sqrt{-3}}{2}|_{\mathbb{C}}^{2s-2}},$$

Furthermore, by changing $n\to -n$ and taking out a factor of a^{1-2s}, we have:

$$I(s,\alpha_f,\Phi_f)=a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 1(\text{mod }3)}\frac{1}{(m+n\frac{b+\sqrt{-3}}{2a})|m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}},$$

Note that for $Re(s)>1$ the integral converges absolutely, thus we can rewrite it in the form:

$$I(s,\alpha_f,\Phi_f)=\frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 1(\text{mod }3)}\frac{1}{(m+n\frac{b+\sqrt{-3}}{2a})|m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}+\frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 2(\text{mod }3)}\frac{}{(-m+n\frac{b+\sqrt{-3}}{2a})|}$$

Changing $n\to -n$ in the second sum, we get:

$$\begin{aligned}I(s,\alpha_f,\Phi_f)=&\frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 1(\text{mod }3)}\frac{1}{(m+n\frac{b+\sqrt{-3}}{2a})|m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}-\\&-\frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 2(\text{mod }3)}\frac{1}{(m+n\frac{b+\sqrt{-3}}{2a})|m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}\end{aligned}$$

Thus we can write for $Re(s)>1$ we can rewrite:

$$I(s,\alpha_f,\Phi_f)=\frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 1(\text{mod }3)}\frac{\varepsilon(m)}{(m+n\frac{b+\sqrt{-3}}{2a})|m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}$$

On the right hand side we can recognize the Eisenstein series $E_\varepsilon(2s-2,z_{\overline{\mathcal{A}_\alpha}})$, thus we get:

$$I(s,\alpha_f,\Phi_f)=\frac{1}{2}a^{-s}\overline{k_\alpha}E_\varepsilon(2s-2,z_{\overline{\mathcal{A}_\alpha}})=\frac{1}{2}\frac{a^{1-s}}{k_\alpha}E_\varepsilon(2s-2,z_{\overline{\mathcal{A}_\alpha}})$$

By analytic continuation, we can extend the equality to all $s\in\mathbb{C}$.

Using this Lemma, we can rewrite the Corollary **??** in the form:

Corollary 3.3. *For all s, we have:*

$$L_f(s,\chi_D\varphi)=\frac{1}{2}\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})}E_\varepsilon(2s-2,z_{\mathcal{A}})\overline{\chi_D(\mathcal{A})}\frac{(\mathrm{Nm}\,\mathcal{A})^{1-s}}{\overline{k_{\mathcal{A}}}},$$

Proof: Recall that in the Corollary **??** we got

$$L_f(s,\chi_D\varphi)=\sum_{\alpha_f\in U(3D)\backslash\mathbb{A}_{K,f}^\times/K^\times}I(s,\alpha_f,\Phi_f)\chi_D(\alpha)\varphi(\alpha),$$

We can rewrite $I(s,\alpha_f,\Phi_f) = \frac{1}{2}\frac{a^{1-s}}{k_\alpha}E_\varepsilon(2s-2,z_{\mathcal{A}_\alpha})$ and $\varphi(\alpha)=1$, $\chi_D(\alpha)=\chi_D(k_\alpha)=\chi_D(\mathcal{A}_{\alpha)}$. Then we get:

$$L_f(s,\chi_D\varphi) = \sum_{\alpha_f\in U(3D)\backslash\mathbb{A}^\times_{K,f}/K^\times} \frac{1}{2}\frac{a^{1-s}}{k_\alpha}E_\varepsilon(2s-2,z_{\mathcal{A}_\alpha})\chi_D(\mathcal{A}_\alpha)$$

Finally, consider $\mathcal{A}$ as representatives of $\mathrm{Cl}(\mathcal{O}_{3D})$. Note that by changing $\mathcal{A}\to\overline{\mathcal{A}}$ we just invert the classes of $\mathrm{Cl}(\mathcal{O}_{3D})$. Thus we get the result of the Corollary:

$$L_f(s,\chi_D\varphi) = \sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})} \frac{1}{2}\frac{a^{1-s}}{\overline{k_\mathcal{A}}}E_\varepsilon(2s-2,z_\mathcal{A})\overline{\chi_D(\mathcal{A})}.$$

3.1.4 Fourier expansion of the Eisenstein series $E_\varepsilon(s,z)$ at $s=0$.

We want to connect the Eisenstein series $E_\varepsilon(s,z)$ to the theta function $\Theta_K(z)$. In order to do this, we will compute the Fourier expansion of $E_\varepsilon(s,z)$ at $s=0$.

We will use the Hecke trick to compute the Fourier expansion of the Eisenstein series:

$$E_\varepsilon(s,z) = \sum_{c,d}{}' \frac{\varepsilon(d)}{(3cz+d)|3cz+d|^{2s}}$$

We will follow closely the proof of Pacetti [12]. This is also done by Hecke in [7]. We rederive the formula:

$$E_1(z,s) = \sum_{d}{}' \frac{\varepsilon(d)}{d^{1+2s}} + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2}\sum_{d\in\mathbb{Z}} \frac{\varepsilon(r)}{(3cz+(3d+r))|3cz+(3d+r)|^{2s}}$$

We divide by 3^{2s+1} and get:

$$E_1(z,s) = 2L(\varepsilon,1+2s) + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2}\frac{\varepsilon(r)}{3^{2s+1}}\sum_{d\in\mathbb{Z}}\frac{\varepsilon(r)}{(\frac{3cz+r}{3}+d)|\frac{3cz+r}{3}+d|^{2s}}$$

We define for z in the upper-half plane:

$$H(z,s) = \sum_{m\in\mathbb{Z}}{}' \frac{1}{(z+m)|z+m|^{2s}}$$

Following Shimura **??**, for $z=x+yi$ and $s>0$ we have the Fourier expansion:

$$H(z,s) = \sum_{n=-\infty}^{\infty}\tau_n(y,s+1,s)e^{2\pi inx},$$

where $\tau_n(y,s+1,s)\dfrac{i\Gamma(s+1)\Gamma(s)}{(2\pi)^{2s+1}} = \begin{cases} n^{2s}e^{-2\pi ny}\sigma(4\pi ny,s+1,s), & \text{if } n>0\\ |n|^{2s}e^{-2\pi|n|y}\sigma(4\pi|n|y,s,s+1), & \text{if } n<0\\ \Gamma(2s)(4\pi y)^{-2s}, & \text{if } n=0,\end{cases}$

where $\gamma(Y,\alpha,\beta) = \displaystyle\int_0^\infty (t+1)^{\alpha-1}t^{\beta-1}e^{-Yt}dt$

For any $s > 0$, $H(z,s)$ converges, thus we can compute the limits of each of its Fourier coefficients:

- $n = 0$: $\lim_{s\to 0} \frac{(2\pi)^{2s+1}}{i\Gamma(s+1)} \frac{\Gamma(2s)}{\Gamma(s)} (4\pi y)^{-2s} = -2\pi i \lim_{s\to 0} \frac{\Gamma(2s)}{\Gamma(s)}$
- $n < 0$: $\lim_{s\to 0} \dfrac{(2\pi)^{2s+1}}{i\Gamma(s+1)\Gamma(s)} |n|^{2s} e^{-2\pi|n|y} \int_0^\infty (t+1)^{s-1} t^s e^{-4\pi|n|yt} dt = -2\pi i e^{-2\pi|n|y} \lim_{s\to 0} \dfrac{1}{\Gamma(s)} \int_0^\infty (t+1)^{s-1} t^s e^{-4\pi|n|yt} dt$
- $n > 0$: $\lim_{s\to 0} \dfrac{(2\pi)^{2s+1}}{i\Gamma(s+1)\Gamma(s)} n^{2s} e^{-2\pi ny} \int_0^\infty (t+1)^s t^{s-1} e^{-4\pi nyt} dt$

(COMPUTATION)

We should get:

$$\lim_{s\to 0} H(s,z) = -\pi i - 2\pi i \sum_{n=1}^{\infty} q^n$$

Finally, note that:

$$E_1(s,z) = 2L(\varepsilon,s) + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} H\left(\frac{3dz+r}{3}, s\right)$$

Using the Fourier expansion of $H(z,s)$, we get:

$$E_1(s,z) = 2L(\varepsilon,s) + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} \sum_{n\in\mathbb{Z}} \tau_n(yn, s+1, s) e^{2\pi i n \frac{3xc+r}{3}}$$

Taking the limit as $s \to 0$, and the Fourier expansion above, we get:

$$E_1(s,z) = 2L(\varepsilon,s) + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2} \frac{\varepsilon(r)}{3} \left(-\pi i - 2\pi i \sum_{n=1}^{\infty} e^{2\pi i n z c} \omega^{nr}\right)$$

We compute separately the inner sum:

$$\sum_{r=0}^{2} \frac{\varepsilon(r)}{3} \left(-\pi i + \sum_{n=1}^{\infty} e^{2\pi i n z c} \omega^{nr}\right) = -2\pi i \sum_{n=1}^{\infty} e^{2\pi i n z c} \varepsilon(n) \sum_{r=0}^{2} \omega^{nr} \varepsilon(rn) = -\frac{2\pi i}{3} G(\varepsilon) \sum_{n=1}^{\infty} e^{2\pi i n z c} \varepsilon(n),$$

where $G(\varepsilon) = \sum_{r=0}^{2} \varepsilon(r)\omega^r = \sqrt{-3}$ is the Gaussian quadratic sum corresponding to ε.
Then we get:

$$E_1(0,z) = 2L(\varepsilon,1) - \frac{4\pi i\sqrt{-3}}{3} \sum_{c=1}^{\infty}\sum_{n=1}^{\infty} e^{2\pi i n z c} \varepsilon(n) = 2L(\varepsilon,1) + \frac{4\pi\sqrt{3}}{3} \sum_{N=1}^{\infty} \left(\sum_{m|N} \varepsilon(m)\right) e^{2\pi i N z}$$

Since ε is a quadratic character, we can compute $L(1,\varepsilon) = \frac{\pi\sqrt{3}}{9}$ (see Kowalski [**?**]). This gives us the Fourier expansion:

$$E_1(0,z) = \frac{2\pi\sqrt{3}}{9}\left(1 + 6\sum_{N=1}^{\infty}\left(\sum_{m|N}\varepsilon(m)\right)e^{2\pi i N z}\right)$$

3.1.5 Connection to the theta function $\Theta_K(z)$.

Recall the theta function Θ_K associated to the number field K:

$$\Theta_K(z) = \sum_{a,b\in\mathbb{Z}} e^{2\pi i(a^2-ab+b^2)z}.$$

Equivalently, we can rewrite the theta function in the form: $\Theta_K(z) = 1 + 6\sum_{\mathcal{A}} e^{2\pi i \operatorname{Nm}\mathcal{A} z}$, where we sum over all ideals $\mathcal{A}$. Thus we have the Fourier expansion for Θ_K:

$$\Theta_K(z) = 1 + 6\sum_{n\geqslant 1} c(n)q^n,$$

where $c(n)$ is the number of ideals of norm n. We will show the following version of Siegel-Weil theorem:

Theorem 3.1. *For $E_\varepsilon(s,z)$ defined in the previous section and ε the quadratic character corresponding to to the extension $K/\mathbb{Q}$, we have:*

$$E_\varepsilon(0,z) = 2L(0,\varepsilon)\Theta_K(z)$$

The proof consists of comparing the Fourier expansions of the two sides. This is mainly going to be based on the lemma below:

Lemma 3.4. *For $n \geqslant 1$ then for the ideals in $\mathcal{O}_K$ we have:*

$$\sum_{d|n}\varepsilon(d) = \#\textit{ideals of norm } n$$

Proof: We first show the result for powers of primes p^e. We consider three cases:

If $p \equiv 1 \mod 3$, then there are two ideals of norm p: $(a+b\omega)$ and $(a-b\omega)$ such that $a^2 - ab + b^2 = p$. Then we have $k+1$ ideals of norm p^k: $(a+b\omega)^i(a+b\omega)^{k-i}$ for $0 \leqslant i \leqslant k$. Moreover, since $\varepsilon(p) = 1$, we have $(1 + \varepsilon(p) + \ldots \varepsilon(p^k)) = k+1$.

If $p \equiv 2 \mod 3$, then there are no ideals of norm p. Thus, if k is even, we have exacly one ideal of norm p^k: $\mathcal{A} = (p^{k/2})$. In this case $(1 + \varepsilon(p) + \ldots \varepsilon(p^k)) = 1 - 1 + \cdots + 1 = 1$. If k is odd, we have no ideals of norm p^{2k+1}. Moreover $(1 + \varepsilon(p) + \ldots \varepsilon(p^k)) = 1 - 1 + \cdots - 1 = 0$.

If $p = 3$, then we have exactly one ideal of norm 3^k, namely the ideal $(\sqrt{-3}^k)$. Moreover $\varepsilon(3) = 0$, thus $(1 + \varepsilon(3) + \ldots \varepsilon(3^k)) = 1$.

It is easy to extend the result to all integers. As ε is a character, we have:

$$\sum_{d|n}\varepsilon(d) = \prod_{p_i|n}(1 + \varepsilon(p_i) + \cdots + \varepsilon(p_i)^{c_i}),$$

where $n = \prod_i p_i^{c_i}$, $e_i \geqslant 1$ and p_i are primes. If we have any ideal $\mathcal{A}$ of norm n, then $\mathcal{A} = \prod_{p_v} \mathfrak{p}_v^{e_v}$, and we must have $n = \prod_v \operatorname{Nm} \mathfrak{p}_v^{e_v}$. Moreover, we have #ideals of norm $n = \prod_{p_i|n}$ #ideals of norm $(\operatorname{Nm} p_i)^{c_i}$, which finishes the proof.

We are ready to state the proof of the theorem. Using the above Lemma we can rewrite the Fourier expansion of Θ_K as:

$$\Theta_K(z) = 1 + 6 \sum_{N=1}^{\infty} \left(\sum_{m|N} \varepsilon(m) \right) e^{2\pi i N z}$$

Multiplying by a factor of $\frac{2\pi\sqrt{3}}{9}$, we recognize the Eisenstein series $E_\varepsilon(0, z)$. Thus it implies $E_\varepsilon(0, z) = \frac{2\pi\sqrt{3}}{9}\Theta_K(z)$. Note that this is the same as:

$$E_\varepsilon(0, z) = 2L(1, \varepsilon)\Theta_K(z)$$

3.1.6 Final formula for $L(1, \chi_D\varphi)$

Applying Corollary **??** for $s = 1$ we get:

$$L_f(1, \chi_D\varphi) = \frac{1}{2} \sum_{\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_{3D})} \frac{1}{\overline{k}_{\mathcal{A}}} E_\varepsilon(0, Dz_{\mathcal{A}}) \overline{\chi_{3D}(\mathcal{A})}$$

Furthermore, from Theorem 3.1 this is the same as:

$$L_f(1, \chi_D\varphi) = \frac{\pi\sqrt{3}}{9} \sum_{\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_{3D})} \frac{1}{\overline{k}_{\mathcal{A}}} \Theta_K(Dz_{\mathcal{A}}) \overline{\chi_{3D}(\mathcal{A})} \tag{8}$$

We need one more step before rewriting the formula as a trace. This is going to be the following lemma:

Lemma 3.5. *For* $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]$ *a primitive ideal of norm* $\operatorname{Nm}\mathcal{A} = a$*, with generator* $\mathcal{A} = (k_{\mathcal{A}})$*, where* $k_{\mathcal{A}} \equiv 1 \mod 3$*, we have:*

$$\Theta_K\left(\frac{-b+\sqrt{-3}}{2a}\right)\frac{k_{\mathcal{A}}}{a} = \Theta_K\left(\frac{-1+\sqrt{-3}}{2}\right)$$

Proof: Since $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$ as a $\mathbb{Z}$-lattice, we can write its generator $k_{\mathcal{A}}$in the form $k_{\mathcal{A}} = ma + 3n\frac{-b+\sqrt{-3}}{2}$ for some integers m, n. Moreover, $\overline{k}_{\mathcal{A}} = m - 3n\frac{b+\sqrt{-3}}{2}$ and $\frac{\overline{k}_{\mathcal{A}}}{a} = m - 3n\frac{b+\sqrt{-3}}{2a}$. Moreover, since $k_{\mathcal{A}}$ is the generator of a primitive ideal, we have $\gcd(m, 3n) = 1$. Then we can find through the Euclidean algorithm integers A, B such that $mA + 3nB = 1$, which makes $\begin{pmatrix} A & B \\ -3n & m \end{pmatrix}$ a matrix in $\Gamma_0(3)$. Since Θ is a modular form of weight 1 for $\Gamma_0(3)$, we have:

$$\Theta_K\left(\frac{A\frac{-b+\sqrt{-3}}{2a} + B}{-3n\frac{-b+\sqrt{-3}}{2a} + m}\right) = \left(m - 3n\frac{-b+\sqrt{-3}}{2a}\right)\Theta_K\left(\frac{-b+\sqrt{-3}}{2a}\right)$$

Noting that $-3n\frac{-b+\sqrt{-3}}{2a}+m = k_{\mathcal{A}}/a = 1/\overline{k}_{\mathcal{A}}$, we can compute $\frac{A\frac{-b+\sqrt{-3}}{2a}+B}{-3n\frac{-b+\sqrt{-3}}{2a}+m} = \frac{(A\frac{-b+\sqrt{-3}}{2}+Ba)\overline{k}_{\mathcal{A}}}{a}$. This is $(aB + A\frac{-b+\sqrt{-3}}{2})(ma + 3n\frac{b+\sqrt{-3}}{2})/a$. After expanding, we get:

$$-3nA\frac{b^2+3}{4a} + abB + \frac{b(-mA+3nB)}{2} + \frac{\sqrt{-3}}{2}$$

Note that $mA + 3nB = 1$ implies that mA and $3nB$ have different parities. Also we chose b odd, since $b^2 + 3 \equiv 0 \mod 4a$. Then we note that $-3nA\frac{b^2+3}{4a} + abB + \frac{b(-mA+3nB)+1}{2} \in \mathbb{Z}$ and thus using the period 1 of Θ_K we get:

$$\Theta_K\left(\frac{A\frac{-b+\sqrt{-3}}{2a}+B}{-3n\frac{-b+\sqrt{-3}}{2a}+m}\right) = \Theta_K\left(\frac{-1+\sqrt{-3}}{2}\right)$$

This finishes the proof.

Note that the Lemma above is equivalent to $\Theta_K(\tau_{\mathcal{A}}) = \overline{k_{\mathcal{A}}}\Theta_K(\omega)$, where $\tau_{\mathcal{A}} = \frac{-b+\sqrt{-3}}{2a}$. Then we can rewrite (**??**) as:

Proposition 3.1.

$$L_f(E_D,1) = \frac{\pi\sqrt{3}}{9}\Theta_K(\omega)\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})}\frac{\Theta_K(D\tau_{\mathcal{A}})}{\Theta_K(\tau_{\mathcal{A}})}\overline{\chi_{3D}(\mathcal{A})} \tag{9}$$

3.1.7 Turning the formula into a trace.

We will rewrite (9) as a trace. First, let $f(z) = \dfrac{\Theta_K(Dz)}{\Theta_K(z)}$. This is a modular function for $\Gamma_0(3D)$. We will prove in the following section **??** the following proposition:

Proposition 3.2. *Take $\mathcal{A}$ representative ideals for $\mathrm{Cl}(\mathcal{O}_{3D})$. We can take all $\mathcal{A}$ to be primitive and we can write them in the form $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$. Then the Galois conjugates of $f(\omega)$ are:*

$$f(\omega)^{\sigma_{\mathcal{A}}^{-1}} = \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}$$

We will also rewrite the character χ_D to include a trace. In the Introduction we have also showed in Lemma **??** that $(D^{1/3})^{\sigma_{\mathcal{A}}^{-1}} = D^{1/3}\overline{\chi_D(\mathcal{A})}$.

Then the formula (9) becomes:

$$L_f(E_D,1) = \frac{\pi\sqrt{3}}{9}D^{-1/3}\Theta_K(\omega)\sum_{\mathcal{A}\in Cl(\mathcal{O}_{3D})}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)^{\sigma_{\mathcal{A}^{-1}}} \tag{10}$$

Moreover, we also have $D^{1/3} \in H_{3D}$. See Cohn [3] for a proof. Thus we can rewrite the sum on the left hand side as $\mathrm{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)$. We can compute the extra terms as well.

- Rodriguez-Villegas and Zagier in [] cite $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = -3\Gamma\left(\frac{1}{3}\right)^3/(2\pi)^2$. We will use several properties of Θ_K proved in section **??**. We can rewrite $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right)$ as $\Theta_K\left(\frac{-3+\sqrt{-3}}{18}-\frac{1}{3}\right)$ and using formula **??**, we get:

$$\Theta_K\left(\frac{-3+\sqrt{-3}}{18}-\frac{1}{3}\right) = (1-\omega^2)\Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right) + \omega^2\Theta_K\left(\frac{-3+\sqrt{-3}}{18}\right)$$

  Using $\Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right) = 0$, we get $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = \omega^2\Theta_K\left(\frac{-3+\sqrt{-3}}{18}\right)$.

  Furthermore, the functional equation $\Theta(-1/3z) = -\sqrt{-3}z\Theta(z)$ for $z = \frac{3+\sqrt{-3}}{2}$, we get $-\sqrt{-3}\frac{3+\sqrt{-3}}{2}\Theta(\omega) = \Theta_K\left(\frac{-3+\sqrt{-3}}{18}\right)$. Note that $-\sqrt{-3}\frac{3+\sqrt{-3}}{2} = 3\omega$, thus we get $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = 3\Theta(\omega)$.

  This gives us the value $\Theta(\omega) = \Gamma\left(\frac{1}{3}\right)^3/(2\pi)^2$

- $L_\infty(s,\chi_D\varphi) = L_\infty(s,\varphi_\infty)$, where $\varphi_\infty(z) = z^{-1}$. Then we can compute:

$$L_\infty(s,\varphi_\infty) = L_\infty(s-1/2, |\cdot|_\infty^{1/2}\varphi_\infty) = 2(2\pi)^s\Gamma(s).$$

  This gives us $L_\infty(1,\chi_D\varphi) = 4\pi$.

  XXXX Should have 2 instead here.

- The real period Ω_D of the elliptic curve E_D. The real period of E_1 is $\dfrac{\Gamma\left(\frac{1}{3}\right)^3}{9\pi}$ (see []). Then to compute the real period of E_D we twist by a factor of $D^{-1/3}$ (see []) and get:

$$\Omega_D = D^{-1/3}\frac{\Gamma\left(\frac{1}{3}\right)^3}{18\pi}$$

  Check this.

Multiplying all the terms, we get:

$$L(E_D,1) = 2\frac{\pi\sqrt{3}}{9}D^{-1/3}\frac{\Gamma\left(\frac{1}{3}\right)^3}{(2\pi)^2}\operatorname{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)$$

This gives us the theorem:

Theorem 3.2.

$$L(E_D,1) = \frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)^3}{18\pi}D^{-1/3}\operatorname{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)$$

3.1.8 S_D is an integer

In the previous section we showed that $S_D \in K$. Note that $\overline{D^{1/3}\Theta(D\omega)/\Theta(\omega)} = D^{1/3}\Theta(-D+D\omega)/\Theta(-1+\omega) = D^{1/3}\Theta(D\omega)/\Theta(\omega)$. Thus $S_D \in \mathbb{Q}$. We would like to show that $S_D \in \mathbb{Z}$.

First we look at the Fourier expansion of $f(z) = \Theta(Dz)/\Theta(z)$:

$$\Theta(z) = 1 + 6 \sum_{N \in \mathbb{Z}_{\geqslant 1}} c(N) q^N,$$

where $c(N) = \#$ ideals with norm N in K and, $q = e^{2\pi i z}$. Then we also have the Fourier expansion of $\Theta(Dz)$:

$$\Theta(Dz) = 1 + 6 \sum_{N \in \mathbb{Z}_{\geqslant 1}} c(N) q^{DN},$$

By taking their ratio we get $\dfrac{\Theta(Dz)}{\Theta(z)} = \sum_{n \in \mathbb{Z}} a_n q^n$, $a_n \in \mathbb{Z}$. This is easy to see just by straight computation. The minimal polynomial of $D^{1/3} f(\omega)$ is:

$$\prod_{\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_{3D})} (X - D^{1/3} \chi_D(\mathcal{A}) (f(\omega))^{\sigma_{\mathcal{A}}}) \in \mathbb{Z}[\omega, D^{1/3}](X, q)$$

This implies that $\mathrm{Tr}_{H_{3D}/K}\, D^{1/3} f(\omega) \in \mathbb{Z}[\omega, D^{1/3}]$. We already know that $\mathrm{Tr}_{H_{3D}/K}\, D^{1/3} f(\omega) \in \mathbb{Q}$, thus $\mathrm{Tr}_{H_{3D}/K}\, D^{1/3} f(\omega) \in \mathbb{Z}$.

4 Shimura reciprocity law in the classical setting.

Let $\mathcal{F}$ be the field of modular functions over $\mathbb{Q}$. From CM theory (see [], for example), it is known that if $\tau \in K \cap \mathcal{H}$ and $f \in \mathcal{F}$, then we have $f(\tau) \in K^{ab}$, where K^{ab} is the maximal abelian extension of K. Shimura reciprocity law gives us a way to compute the Galois conjugates of $f(\tau)^\sigma$ when acting with $\sigma \in \mathrm{Gal}(K^{ab}/K)$. We will follow the exposition of Stevenhagen []. For more details see Gee [].

We recall that $\mathcal{F} = \bigcup_{N \geqslant 1} \mathcal{F}_N$, where $\mathcal{F}_N$ is the space of modular functions of level N. Moreover, we can think of $\mathcal{F}_N$ as the function field of the modular curve $X(N) = \Gamma(N) \backslash \mathcal{H}^*$ over $\mathbb{Q}(\zeta_N)$, where $\zeta_N = e^{2\pi i/N}$ and $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. We can compute explicitly $\mathcal{F}_N = \mathbb{Q}(j, j_N)$, where j is the j-invariant and $j_N(z) = j(Nz)$. In particular, we have $\mathcal{F}_1 = \mathbb{Q}(j)$.

When working over $\mathbb{Q}$, one has an isomorphism:

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}.$$

More precisely, if we denote by γ_σ the Galois action corresponding to the matrix $\gamma \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ under the isomorphism above, it is enough to define the Galois action for $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and for $G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}$. We state explicitly the two actions below.

- **Action of $\alpha \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mathcal{F}_N$.**

  $$(f(\tau))^{\sigma_\alpha} = f^\alpha(\tau) := f(\alpha\tau),$$

  where α is acting on the upper half plane via fractional linear transformations.

- **Action of $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^\times$ on $\mathcal{F}_N$.** Note that for $f \in \mathcal{F}_N$ we have a Fourier expansion

$f(z) = \sum_{n \geqslant 0} a_n q^{n/N}$ with coefficients $a_n \in \mathbb{Q}(\zeta_N)$, $q = e^{2\pi i z}$. If we denote $u_d := \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, then the action of σ_{u_d} is given by

$$(f(\tau))^{\sigma_{u_d}} = f^{u_d}(\tau) := \sum_{n \geqslant 0} a_n^{\sigma_d} q^{n/N},$$

where σ_d is the Galois action in $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ that sends $\zeta_N \to \zeta_N^d$.

As the restriction maps between the fields $\mathcal{F}_N$ are in correspondence with the natural maps between the groups $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ we can take the projective limit to get the isomorphism:

$$\mathrm{Gal}(\mathcal{F}/\mathcal{F}_1) \cong \mathrm{GL}_2(\widehat{\mathbb{Z}})/\{\pm 1\}.$$

To further get all the automorphisms of $\mathcal{F}$ we need to consider the action of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$. We get the exact sequence:

$$1 \to \{\pm 1\} \to \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \to \mathrm{Aut}(\mathcal{F}) \to 1$$

For this to make sense, we need to extend the action from $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ to $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$. We do this by using the action of $\mathrm{GL}_2(\mathbb{Q})^+$:

- **Action of $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$ on $\mathcal{F}$.**

$$f^{\alpha}(\tau) = f(\alpha\tau),$$

where α acts by fractional linear transformations.

We extend the action of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ to $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ by writing the elements $\gamma \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ in the form $\gamma = u\alpha$, where $u \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$. Note that this decomposition is not uniquely determined. However, by combining the two actions of u and α, a well defined action is given by:

$$f^{u\alpha} = (f^u)^{\alpha}.$$

We want to look at the action of $\mathrm{Gal}(K^{ab}/K)$ inside $\mathrm{Aut}(\mathcal{F})$. From class field theory we have the exact sequence:

$$1 \to K^{\times} \to \mathbb{A}_{K,f}^{\times} \xrightarrow{[\cdot,K]} \mathrm{Gal}(K^{ab}/K) \to 1,$$

where $[\cdot, K]$ is the Artin map.

We are going to embed $\mathbb{A}_{K,f}^{\times}$ into $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ such that the Galois action of $\mathbb{A}_{K,f}^{\times}$ through the Artin map and the action of the matrices in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ are compatible. We do this by constructing a matrix $g_\tau(x)$ for the idele $x \in \mathbb{A}_{K,f}^{\times}$.

Let $\mathcal{O}$ be the order of K generated by τ i.e. $\mathcal{O} = \mathbb{Z}[\tau]$. We define the matrix $g_\tau(x)$ to be the unique matrix in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ such that $x \begin{pmatrix} \tau \\ 1 \end{pmatrix} = g_\tau(x) \begin{pmatrix} \tau \\ 1 \end{pmatrix}$. We can compute it explicitly. To do that, consider the minimal polynomial of τ:

$$p(X) = X^2 + BX + C$$

Then if we write $x_p \in \mathbb{Q}_p^\times$ in the form $x_p = s_p\tau + t_p \in \mathbb{Q}_p^\times$ with $s_p, t_p \in \mathbb{Q}_p$, we can compute:

$$g_\tau(x_p) = \begin{pmatrix} t_p - s_pB & -s_pC \\ s_p & t_p \end{pmatrix}$$

Shimura reciprocity law is going to make the following diagram commute:

$$\begin{tikzcd}
1 \arrow[r] & K^\times \arrow[r] & \mathbb{A}_{K,f}^\times \arrow[r, "{[\cdot, K]}"] \arrow[d, "g_\tau"] & \mathrm{Gal}(K^{ab}/K) \arrow[r] & 1 \\
1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \arrow[r] & \mathrm{Aut}(\mathcal{F}) \arrow[r] & 1
\end{tikzcd}$$

We make the statement explicit below:

Theorem 4.1. (Shimura reciprocity law) *For $f \in \mathcal{F}$ and $x \in \mathbb{A}_{K,f}^\times$, we have:*

$$(f(\tau))^{[x,K]} = f^{g_\tau(x^{-1})}(\tau),$$

where $[x, K]$ is the Galois action corresponding to the idele x via the Artin map, g_τ is defined above and the action of $g_\tau(x)$ is the action in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$.

Remark 4.1. Note that the elements of $K^\times$ have trivial action. This can be easily seen by embedding $K^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q})^+$ given by $k \hookrightarrow g_\tau(k)$. Noting that τ is fixed by the action of the torus $K^\times$, we have:

$$f^{g_\tau(k^{-1})}(\tau) = f(g_\tau(k^{-1})\tau) = f(\tau)$$

Remark 4.2. We can also rewrite the theorem for ideals in K. Let $f \in \mathcal{F}_N$ and $\mathcal{O} = \mathbb{Z}[\tau]$ of conductor M. Going through the Artin map, we can restate Shimura reciprocity in this case in the form:

$$f(\tau)^{\sigma_\mathcal{A}} = f^{g_\tau(\mathcal{A})^{-1}}(\tau), \tag{11}$$

where $\sigma_\mathcal{A}$ is the Galois action corresponding to the ideal $\mathcal{A}$ through the Artin map and

$$g_\tau(\mathcal{A}) := g_\tau((\alpha)_{p|\,\mathrm{Nm}(\mathcal{A})}).$$

Note that $g_\tau(\mathcal{A})$ is unique up to multiplication by roots of unity in K. However, these have trivial action on f. This can be easily seen by multiplying by an element of $(\pm\omega^j)_v \in K^\times$ and noticing that we get trivial action at the unramified places $p \nmid MN$.

Remark 4.3. Note that the action of $g_\tau(\mathcal{A})$ is the same as the action of $g_\tau((\alpha)_{p|MN})^{-1}$.

Remark 4.4. Note that the maps above are based on the map between the ideals $\mathcal{A}$ prime to MN and the ideles:

$$I(MN) \to \mathbb{A}_{K,f}^{\times}/K^{\times}$$
$$\mathcal{A} = \prod_v \mathfrak{p}_v^{e_v} \to (\varpi_v)_v^{e_v},$$

where ϖ_v is the uniformizer of the ideal $\mathfrak{p}_v$ at the place $v \nmid \infty$.

4.1 Applying Shimura reciprocity law to $K = \mathbb{Q}[\sqrt{-3}]$.

Lemma 4.1. *The function $f(z) = \dfrac{\Theta_K(Dz)}{\Theta_K(z)}$ is a modular function of level $3D$ with integer Fourier coefficients at the cusp ∞.*

Proof: Since $\Theta_K(z)$ is a modular form of weight 1 for $\Gamma_0(3)$, it can be easily seen that $\Theta(Dz)$ is a modular form of weight 1 for $\Gamma(3D)$. Furthermore, their ratio is modular function for $\Gamma_0(3D)$. We check this below. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(3D)$, we have:

$$f(\gamma z) = \frac{\Theta\left(\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)}{\Theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)} = \frac{\Theta\left(\begin{pmatrix} a & bD \\ c/D & d \end{pmatrix}(Dz)\right)}{\Theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)} = \frac{(cz+d)\Theta(Dz)}{(cz+d)\Theta(z)} = f(z)$$

To find the Fourier expansion of $f(z)$ at ∞, it is enough to write the Fourier expansions of $\Theta(Dz)$ and $\Theta(z)$:

$$\frac{\Theta(Dz)}{\Theta(z)} = \frac{1 + \sum\limits_{N \geqslant 1} c(N) q^{ND}}{1 + \sum\limits_{N \geqslant 1} c(N) q^{N}} = \sum_{M \geqslant 0} a_M q^M$$

We can compute the Fourier coefficients explicitly from the equality:

$$1 + \sum_{N \geqslant 1} c(N) q^{ND} = (1 + \sum_{N \geqslant 1} c(N) q^{N})(\sum_{M \geqslant 0} a_M q^M)$$

Note that we have $a_0 = 1$ and $a_M = -a_{M-1}c(1) - a_{M-2}c(2) - \cdots - a_1 c(M-1) - a_0 c(M)$ if $D \nmid M$ and $a_M = c(M/D) - a_{M-1}c(1) - a_{M-2}c(2) - \cdots - a_1 c(M-1) - a_0 c(M)$ if $D|M$. By induction, since $c(N) \in \mathbb{Z}$, we get all the coefficients $a_M \in \mathbb{Z}$.

4.1.1 $f(\omega)$ is in the ring class field H_{3D}.

From CM-theory, we have that if $f \in \mathcal{F}_{3D}$ and τ generating $\mathcal{O}_K$, we have $f(\tau) \in H_{3D,\mathcal{O}_K}$ the ray class field of conductor $3D$. We claim that $f(\omega) \in H_{3D}$. Recall that we have $\mathrm{Gal}(K^{ab}/H_{3D}) \cong U(3D)\backslash\mathbb{A}_{K,f}^{\times}/K^{\times}$. Thus in order to show that $f(\omega) \in H_{3D}$, we need to check that $f(\omega)$ is invariant under the action of $U(3D)$.

Lemma 4.2. *For $\omega = \frac{-1+\sqrt{-3}}{2}$ and $f(z) = \frac{\Theta_K(Dz)}{\Theta_K(z)}$ we have $f(\omega) \in H_{3D}$.*

Proof: In order for $f(\omega) \in H_{3D}$, we need to show that it is invariant under $\mathrm{Gal}(K^{ab}/H_{3D})$. Using Shimura reciprocity law, we need to show:

$$f(\omega) = f^{r_\omega(s)}(\omega),$$

for all $s \in K^\times U(3D)$. From Remark 4.1, the action of $K^\times$ is trivial. Thus it is enough to show the result for all elements $l = (A_p + B_p\omega)_p \in U(3D)$. By the definition of $U(3D)$, this implies that $A_p + B_p\omega \in (\mathbb{Z}_p[\omega])^\times$ for all p and $A_3 \equiv 1 \mod 3$, $B_3 \equiv 1 \mod 3$, $B_p \equiv 0 \mod D$ for all $p|D$. Since the action for $p \nmid 3D$ is trivial, s has the same action $l_D = (A_p + B_p\omega)_{p|3D} \in U(3D)$. Moreover, this has the same action as $l_0 = (A + B\omega)_{p|3D}$, where $A + B\omega \in \mathcal{O}_K$ and $A \equiv A_p \mod 3D\mathbb{Z}_p$ and $B \equiv B_p \mod 3D\mathbb{Z}_p$ for all $p|3D$.

Note further that we can pick A, B such that $(A + B\omega)$ generates a primitive ideal $\mathcal{A}$ in $\mathcal{O}_K$. Moreover, from above we have $3D|B$ and $A \equiv 1 \mod 3$. Recall that we can rewrite any primitive ideal in the form $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$, where $a = \mathrm{Nm}\,\mathcal{A}$ and $b^2 \equiv -3 \mod 4a$. Then the generator is $A + B\omega = ta + s\frac{-b+\sqrt{-3}}{2}$ for $t, s \in \mathbb{Z}$, $3D|s$.

Now observe that $f(\omega) = f(\tau)$, where $\tau = \frac{-b+\sqrt{-3}}{2}$, thus from Shimura reciprocity law, we have:

$$(f(\tau))^{\sigma_{l^{-1}}} = f^{r_\tau(l)}(\omega).$$

Here $r_\tau(l) = \begin{pmatrix} A_p - bB_p & -B_p c \\ B_p & A_p \end{pmatrix}_p$ and $r_\tau(l)$ has the same action as $r_\tau(l_0)$, where $l_0 = (A + B\omega)_{p|3D}$ and $A + B\omega = ta + s\frac{-b+\sqrt{-3}}{2}$. Then we need to compute the action of:

$$(f(\tau))^{\sigma_{l^{-1}}} = f^{r_\tau(l_0)}(\tau).$$

Note that $r_\tau(l_0) = \begin{pmatrix} ta-sb & -sc \\ s & ta \end{pmatrix}_{p|3D}$, where $c = \frac{b^2+3}{4}$. Then we can rewrite the action of $r_\tau(l_0)$:

$$f^{r_\tau(l_0)}(\tau) = f^{\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix}_{p|3D} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|3D}}(\tau) = f^{\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|3D}}\left(\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix}\tau\right)$$

Since $a|c$, the matrix $\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and we can rewrite:

$$f\left(\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} z\right) = \frac{\Theta_K\left(\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} z\right)}{\Theta_K\left(\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} z\right)}. = \frac{\Theta_K\left(\begin{pmatrix} ta-sb & -scD/a \\ s/D & t \end{pmatrix}(Dz)\right)}{\Theta_K\left(\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} z\right)}.$$

Note that since $3D|s$, we actually have $\begin{pmatrix} ta-sb & -scD/a \\ s/D & t \end{pmatrix}, \begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} \in \Gamma_0(3)$ and we can apply the properties of the modular form Θ_K:

$$\frac{\Theta_K\left(\begin{pmatrix} ta-sb & -scD/a \\ s/D & t \end{pmatrix}(Dz)\right)}{\Theta_K\left(\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} z\right)}. = \frac{(sz+t)^{-1}\Theta_K(Dz)}{(sz+t)^{-1}\Theta_K(z)}. = f(z)$$

Finally, note that since $(a, 3D) = 1$ and f has rational coefficients, the action of $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|3D}$ is trivial. This finishes the proof that $f(\omega)$ is invariant under the Galois action coming from $U(3D)$, thus $f(\omega) \in H_{3D}$.

Remark 4.5. A different proof is shown in the in Appendix **??**, where we reinterpret the classical Shimura reciprocity law in the setting of Shimura curves following Hida [].

4.1.2 Galois conjugates of $f(\omega)$.

Let $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$ be a primitive ideal prime to $3D$. For $\tau_1 = \frac{-b+\sqrt{-3}}{2}$, let $\mathcal{O}_D = \mathbb{Z} + D\tau\mathbb{Z}$.

Lemma 4.3. *Let $f \in \mathcal{F}_N$ be a modular function of level N with rational Fourier coefficients in its Fourier expansion. Let $\tau_1 = \frac{-b+\sqrt{-3}}{2}$ be a CM point and let $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]$ be a primitive ideal prime to N. Then we have the Galois action:*

$$f(\tau)^{\sigma_{\mathcal{A}}^{-1}} = f(\tau/a)$$

Proof: From Shimura reciprocity (11), we have:

$$f(\tau)^{\sigma_{\mathcal{A}}^{-1}} = f^{g_\tau(\mathcal{A})}(\tau).$$

Note that the minimum polynomial of τ is $p_{\tau(X)} = X^2 + bX + \frac{b^2+3}{4}$. Now let $\alpha = ta + s\frac{-b+\sqrt{-3}}{2} = ta + s\tau$ be a generator of $\mathcal{A}$. Then we have $g_\tau(\mathcal{A}) = \left(\begin{smallmatrix} ta-sb & -s\frac{b^2+3}{4} \\ -s & ta \end{smallmatrix}\right)_{p|a}$. We can rewrite the matrix in the form:

$$g_\tau(\mathcal{A}) = \left(\begin{smallmatrix} ta-sb & \frac{b^2+3}{4a} \\ -s & t \end{smallmatrix}\right)_{p|a} \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{p|a}$$

As $\left(\begin{smallmatrix} ta-sb & -\frac{b^2+3}{4a} \\ -s & t \end{smallmatrix}\right)_{p|a} \in \mathrm{SL}_2(\mathbb{Z}_p)$ for $p \nmid ND$, it has a trivial action. Then:

$$f^{g_\tau(\mathcal{A})}(\tau) = f^{\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{p|a}}(\tau)$$

We rewrite the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{p|a} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{p\nmid a} \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{\mathbb{Q}}$, where $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/a \end{smallmatrix}\right)_p \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{\mathbb{Q}} \in \mathrm{GL}_2(\mathbb{Q})^+$.

Note that the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/a \end{smallmatrix}\right)_p$ is only given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/a \end{smallmatrix}\right)_{p|NM}$. However, since f has rational Fourier coefficients in its Fourier expansion, this action is trivial. Thus we are left with:

$$f^{g_\tau(\mathcal{A})}(\tau) = f^{\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{\mathbb{Q}}}(\tau)$$

This is just $f^{g_\tau(\mathcal{A})}(\tau) = f(\tau/a)$.

Lemma 4.4. *Take the primitive ideals $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$ to be the representatives of the ring class field H_{3D} such that all norms $\mathrm{Nm}\,\mathcal{A}$ are relatively prime to each other and $b^2 \equiv -3 \mod 4a$ for all the $a = \mathrm{Nm}\,\mathcal{A}$ chosen.*

Then the only Galois conjugates of $f(\omega) = \dfrac{\Theta_K(D\omega)}{\Theta_K(\omega)}$ are the following:

$$\left(\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)^{\sigma_{\mathcal{A}}^{-1}} = \frac{\Theta_K\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta_K\left(\frac{-b+\sqrt{-3}}{2a}\right)}$$

Proof: Note that $\dfrac{\Theta_K(D\omega)}{\Theta_K(\omega)} = \dfrac{\Theta_K\left(D\frac{-b+\sqrt{-3}}{2}\right)}{\Theta_K\left(\frac{-b+\sqrt{-3}}{2}\right)}$ and apply lemma 4.3 to $\tau = \frac{-b+\sqrt{-3}}{2}$ and $f(z) = \dfrac{\Theta_K(Dz)}{\Theta_K(z)}$. These are the only Galois conjugates from Lemma **??**.

5 Writing S_D as a square.

In this section we will show the following result:

Theorem 5.1. *For $D = \prod_{p_i \equiv 1 \mod 3} p_i^{e_i}$, let $\tau = \frac{-b+\sqrt{-3}}{2}$ such that $b^2 \equiv -3 \mod 12D^2$. Moreover, let $b^* \equiv b^{-1} \mod D$.*

Let $H_{\mathcal{O}}$ be the ray class field of conductor $3D$ and let $H_0 \subset H_{\mathcal{O}}$ be the subfield of $H_{\mathcal{O}}$ that is the fixed field of $G_0 = \{r \in (\mathbb{Z}/D\mathbb{Z})^\times, r \equiv 1 \mod 6 : \mathcal{A}_r^\circ = \left(1 + b^*(1-r)\frac{-b+\sqrt{-3}}{2}\right)\}$. Then we have

$$S_D = |\operatorname{Tr}_{H_{\mathcal{O}}/H_0}(f_1(\tau)D^{2/3})|^2$$

and $S_D \in \mathbb{Z}$.

The main tool in proving Theorem 5.1 is a Factorization Formula of Rodriguez-Villegas and Zagier [13]. We will apply the Factorization Formula 12 to the formula for the L-function $L(E_D, 1)$ in Theorem **??**.

5.1 Factorization Formula

We recall the version of Factorization Formula ([13], Theorem, page 7) simplified to the case of $\alpha = p = 0$:

Theorem 5.2. (Factorization formula.) *For $a \in \mathbb{Z}_{>0}$, $\mu, \nu \in \mathbb{Q}$, $z = x + yi \in \mathbb{C}$ and $Q_z(m,n) = \frac{|mz-n|^2}{2y}$, we have:*

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n\mu)} e^{\pi(imn - \frac{|mz-n|^2}{2y})/a} = \sqrt{2ay}\,\theta\begin{bmatrix} a\mu \\ \nu \end{bmatrix}(a^{-1}z)\cdot\theta\begin{bmatrix} \mu \\ -a\nu \end{bmatrix}(-a\bar{z}), \tag{12}$$

where $\theta\begin{bmatrix} \mu \\ \nu \end{bmatrix}(z) = \sum_{n\in\mathbb{Z}+\mu} e^{\pi i n^2 z + 2\pi i \nu n}$ is a theta function of half integral weight.

Using the formula above, we will prove the following Proposition:

Proposition 5.1. *For $a \equiv 1 \mod 6$, $D \equiv 1 \mod 6$ and $b^2 \equiv -3 \mod 4D^2a^2a_1$, $b \equiv 1 \mod 16$, we have:*

$$\frac{3}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right) - \frac{1}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right) = \sum_{r\in\mathbb{Z}/D\mathbb{Z}} \frac{\sqrt[4]{3}}{D\sqrt{a_1}} e^{\pi i(a-1)/6}\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right), \tag{13}$$

where $\theta_s(z) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+s/D-1/6)^2 z}(-1)^n$ *is a theta function of weight* $1/2$ *for* s *non-negative integer. Here we use the notation* $r \in \mathbb{Z}/D\mathbb{Z}$ *to mean any representatives* r *for the residues* mod D.

We start by applying the Factorization Formula (12) several times for $\mu := \frac{\mu+r}{D}$, $0 \leqslant r \leqslant D-1$ and $z := z/D$. Summing up the formulas, we are going to get:

Lemma 5.1. *We have the following factorization lemma:*

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}} \frac{\sqrt{2ay}}{\sqrt{D}}\theta\begin{bmatrix} a(\mu+r)/D \\ \nu \end{bmatrix}\left(D\frac{z}{a}\right)\theta\begin{bmatrix} (\mu+r)/D \\ -a\nu \end{bmatrix}(-aD\overline{z}) = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n\mu)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{D}{a}}$$

Proof: Plugging in $\mu := \frac{\mu+r}{D}$, $z := z/D$ in 12, we get:

$$\sqrt{2ay}\theta\begin{bmatrix} a\frac{(\mu+r)}{D} \\ \nu \end{bmatrix}\left(\frac{z}{a}\right)\theta\begin{bmatrix} \frac{(\mu+r)}{D}) \\ -a\nu \end{bmatrix}(-a\overline{z}) = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}}$$

We sum from r in $\mathbb{Z}/D\mathbb{Z}$:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}} \sqrt{2ay}\theta\begin{bmatrix} a\frac{(\mu+r)}{D} \\ \nu \end{bmatrix}\left(\frac{z}{a}\right)\theta\begin{bmatrix} \frac{(\mu+r)}{D}) \\ -a\nu \end{bmatrix}(-a\overline{z}) = \sum_{r\in\mathbb{Z}/D\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}}$$

We change the two sums on the RHS:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}} = \sum_{m,n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}/D\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}}$$

Note that the LHS can be rewritten as $\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}} \sum_{r\in\mathbb{Z}/D\mathbb{Z}} e^{2\pi inr/D}$ and note further that:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}} e^{2\pi inr/D} = \sum_{r=0}^{D-1} e^{2\pi inr/D} = \begin{cases} 0, & \text{for } D \nmid n \\ D, & \text{for } D|n \end{cases}$$

Thus we are only summing over the n's that are multiples of D:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}} = D\sum_{m,n'\in\mathbb{Z}} e^{2\pi i(m\nu+n'(\mu+r))}e^{\pi(mn'i-\frac{|n'-m(z/D)|^2}{2(y/D)})\frac{D}{a}}$$

Going back to our initial equality, we can replace $z' = z/D$ and get:

$$\sqrt{2aDy'}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\theta\begin{bmatrix} a\frac{(\mu+r)}{D} \\ \nu \end{bmatrix}\left(D\frac{z'}{a}\right)\theta\begin{bmatrix} \frac{(\mu+r)}{D}) \\ -a\nu \end{bmatrix}\left(-aD\overline{z'}\right) = D\sum_{m,n'\in\mathbb{Z}} e^{2\pi i(m\nu+n'(\mu+r))}e^{\pi(mn'i-\frac{|n'-mz'|^2}{2y'})\frac{D}{a}}$$

Corollary 5.1. *Another version of the factorization lemma above is:*

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}} \frac{\sqrt{2ay}}{\sqrt{D}}\theta\begin{bmatrix} a\mu + ar/D \\ \nu \end{bmatrix}\left(D\frac{z}{a}\right)\theta\begin{bmatrix} \mu + r/D \\ -a\nu \end{bmatrix}(-aD\overline{z}) = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+nD\mu)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{D}{a}} \tag{14}$$

Proof: We apply the previous factorization lemma for $\mu := D\mu$.

We will apply Corollary 5.1 for $\mu = -1/6$ and $\nu = 1/2$, D odd, $z = \frac{-b+\sqrt{-3}}{2aa_1}$, where $b^2 \equiv -3 \mod 4a^2a_1D$ and $b \equiv 1 \mod 16$. This gives us:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}} \sqrt{2ay/D}\theta\begin{bmatrix} -\frac{a}{6}+\frac{ar}{D} \\ 1/2 \end{bmatrix}\left(D\frac{-b+\sqrt{-3}}{2a^2a_1D}\right)\theta\begin{bmatrix} -\frac{1}{6}+\frac{r}{D} \\ -a/2 \end{bmatrix}\left(D\frac{b+\sqrt{-3}}{2Da_1}\right) =$$

$$= \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-nD/6)}e^{\pi(mni-\frac{|nDaa_1-m\frac{-b+\sqrt{-3}}{2}|^2}{Daa_1\sqrt{3}})\frac{D}{a}}$$

We will analyze firs the LHS of the equation. Note that from the definition of $\theta\begin{bmatrix} \mu \\ \nu \end{bmatrix}$ we have:

$$\theta\begin{bmatrix} -\frac{1}{6}+\frac{r}{D} \\ -a/2 \end{bmatrix}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{r}{D}-\frac{1}{6}\right)^2 z}e^{-a\pi i\left(n+\frac{r}{D}-\frac{1}{6}\right)} = e^{-a\pi ir/D}e^{a\pi i/6}\theta_r(z).$$

Similarly, as $a \equiv 1 \mod 6$, we have $\theta\begin{bmatrix} -\frac{a}{6}+\frac{ar}{D} \\ 1/2 \end{bmatrix}(z) = \theta\begin{bmatrix} -\frac{1}{6}+\frac{ar}{D} \\ 1/2 \end{bmatrix}(z)$. Then from the definition of θ_r we get:

$$\theta\begin{bmatrix} -\frac{a}{6}+\frac{ar}{D} \\ 1/2 \end{bmatrix}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{ar}{D}-\frac{1}{6}\right)^2 z}e^{\pi i\left(n+\frac{ar}{D}-\frac{1}{6}\right)} = e^{\pi iar/D}e^{-\pi i/6}\theta_{ar}(z)$$

Also, since $D \equiv 1 \mod 6$ we also have: $e^{-2\pi inD/6} = e^{-2\pi in/6}$. We can also compute: $\frac{\sqrt{2ay}}{\sqrt{D}} = \frac{\sqrt{2a\frac{\sqrt{3}}{2aa_1D}}}{\sqrt{D}} = \frac{\sqrt[4]{3}}{D\sqrt{a_1}}$. Thus we can rewrite the formula:

$$\frac{\sqrt[4]{3}}{D\sqrt{a_1}}\sum_{r\in\mathbb{Z}/D\mathbb{Z}} e^{\pi i(a-1)/6}\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right) = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)}e^{\pi(mni-\frac{|nDaa_1-m\frac{-b+\sqrt{-3}}{2}|^2}{Daa_1\sqrt{3}})\frac{D}{a}} \tag{15}$$

Now we are going to analyze below the RHS of the equation (15):

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)}e^{\pi(mni-\frac{|nDaa_1-m\frac{-b+\sqrt{-3}}{2}|^2}{Daa_1\sqrt{3}})\frac{D}{a}}$$

First note that we have the following lemma:

Lemma 5.2. *For $b \equiv 1 \mod 16, b \equiv 0 \mod 3, b^2 \equiv -3 \mod 4a^2a_1D$, we have:*

$$e^{2\pi i(m/2+n/2)}e^{\pi(mni-\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}} = e^{2\pi i\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D}}D^{\frac{-b+\sqrt{-3}}{6a}}$$

Proof: We only need to show that:

$$2\pi i\left(\frac{m}{2}+\frac{n}{2}+\frac{Dmn}{2a}\right)\equiv -2\pi i\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D}D\frac{b}{6a} \mod 2\pi i\mathbb{Z}.$$

After dividing by $2\pi i$, we compute the RHS of the identity:

$$\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D}D\frac{b}{6a}=\left(Dm^2\frac{b(b^2+3)}{24a^2a_1}-D\frac{b^2mn}{6a}+\frac{Dba_1n^2}{6}\right)$$

Thus our claim turns into:

$$\left(\frac{m}{2}+\frac{n}{2}+\frac{Dmn}{2a}\right)\equiv\left(Dm^2\frac{b(b^2+3)}{24a^2a_1}-D\frac{b^2mn}{6a}+\frac{Dba_1n^2}{6}\right) \mod \mathbb{Z}$$

Equivalently:

$$\frac{m}{2}+\frac{n}{2}\equiv\left(D\frac{m^2}{2}\frac{b}{3}\frac{(b^2+3)}{4a^2a_1}-D\frac{(b^2+3)mn}{6a}+\frac{n^2}{2}\frac{b}{3}Da_1\right) \mod \mathbb{Z}$$

We have $b^2\equiv -3 \mod 4aa_1^2$, $b\equiv 1 \mod 16$, $b\equiv 0 \mod 3$. Note that this implies that b is odd and that $b^2+3\equiv 4 \mod 8$, as well as $b^2+3\equiv 0 \mod 3$. Then, since a, a_1, D are odd, we get:

- $m/2\equiv m^2/2\equiv D\frac{m^2}{2}\frac{b}{3}\frac{(b^2+3)}{4a^2a_1} \mod \mathbb{Z}$
- $n/2\equiv n^2/2\equiv \frac{n^2}{2}\frac{b}{3}Da_1 \mod \mathbb{Z}$
- $-D\frac{(b^2+3)mn}{6a}\in\mathbb{Z}$

This finishes the proof.

Lemma 5.3. *Under the same conditions as above we have:*

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi in/3}e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}=\frac{3}{2}\Theta(3z)-\frac{1}{2}\Theta(z),$$

where $z\in\mathcal{H}$, $\mathcal{A}\mathcal{A}_1=[aa_1,\frac{-b+\sqrt{-3}}{2}]$ *and* $b\equiv 0 \mod 3$, $b^2\equiv -3 \mod 4aa_1$.

Proof: Note first that by changing $m\to -m$ and $-m\cdot\frac{-b+\sqrt{-3}}{2}+naa_1$ to its conjugate, we have $\sum_{m,n\in\mathbb{Z}} e^{2\pi in/3}e^{2\pi i\frac{|-m\cdot\frac{-b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}z}=\sum_{m,n\in\mathbb{Z}} e^{2\pi in/3}e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}z}$.

We can split the sum in three terms, depending on $n \mod 3$:

$$\sum_{m,3|n\in\mathbb{Z}} e^{\frac{2\pi i|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}+\omega\sum_{m,n\in\mathbb{Z},n\equiv 1 \mod 3} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}+\omega^2\sum_{m,n\in\mathbb{Z},n\equiv 2 \mod 3} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$$

Note that the first term equals $\sum_{m,n\in\mathbb{Z}} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+n3aa_1|^2}{3aa_1}\cdot 3z}=\Theta_K(3z)$.

Also note that the two terms $\sum_{m,n\in\mathbb{Z},n\equiv 1 \mod 3} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$ and $\sum_{m,n\in\mathbb{Z},n\equiv 2 \mod 3} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$ equal each other, by changing in the latter $n\to -n$ and $m\to -m$. Thus we got so far:

$$\Theta(3z)+(\omega+\omega^2)\sum_{m,n\in\mathbb{Z},n\equiv 1 \mod 3} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$$

Furthermore, we have:

$$\sum_{m,n\in\mathbb{Z},n\equiv 1 \mod 3} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}=\frac{1}{2}\sum_{m,n\in\mathbb{Z},(n,3)=1} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$$

Finally, this is just:

$$\frac{1}{2}\sum_{m,n\in\mathbb{Z}} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}-\frac{1}{2}\sum_{m,n\in\mathbb{Z}} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+3naa_1|^2}{3aa_1}\cdot 3z}=\frac{1}{2}(\Theta(z)-\Theta(3z))$$

Finally, we get $\sum_{m,n\in\mathbb{Z}} e^{2\pi in/3}e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}=\Theta(3z)-\frac{1}{2}(\Theta(z)-\Theta(3z))=\frac{3}{2}\Theta(3z)-\frac{1}{2}\Theta(z)$.

From the previous two lemmas, we get the following corollary:

Corollary 5.2. *Under the above conditions, we have:*

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)}e^{\pi(mni-\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}}=\frac{3}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)-\frac{1}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)$$

Proof: Note that we can rewrite the LHS in the form:

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)}e^{\pi(mni-\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}}=\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/2+n/3)}e^{\pi(mni-\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}}$$

Then, from Lemma 5.2, we have:

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)}e^{\pi(mni-\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}}=\sum_{m,n\in\mathbb{Z}} e^{2\pi in/3}e^{\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D})D\frac{-b+\sqrt{-3}}{6a}}$$

Now apply Lemma 5.3 for $z=D\frac{-b+\sqrt{-3}}{6a}$, we get:

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi in/3}e^{\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D}D\frac{-b+\sqrt{-3}}{6a}}=\frac{3}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)-\frac{1}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)$$

Finally, from (15) and Corollary 5.2 we get the result of Proposition 5.1:

$$\frac{3}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)-\frac{1}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)=\frac{\sqrt[4]{3}}{D\sqrt{a_1}}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}e^{\pi i(a-1)/6}\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right).$$

A particular case of Proposition 5.1 is going to be the following result:

Corollary 5.3. *For $b^2\equiv -3 \mod 12a^2a_1$, $b\equiv 1 \mod 16$, we have:*

$$\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)=\frac{\sqrt[4]{3}}{\sqrt{a_1}}e^{\pi i(a-1)\frac{1}{6}}\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2a_1}\right),$$

where $\theta_0(z)=\sum_{n\in\mathbb{Z}}e^{\pi i(n-1/6)^2z}(-1)^n$.

Proof: Applying the Proposition 5.1 for $D=1$ we get:

$$\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)-\frac{1}{2}\Theta\left(\frac{-b+\sqrt{-3}}{6a}\right)=\frac{\sqrt[4]{3}}{\sqrt{a_1}}e^{\pi i(a-1)\frac{1}{6}}\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2a_1}\right).$$

Furthermore, using the result from Appendix A, Lemma 6.3 that $\Theta\left(\frac{-b+\sqrt{-3}}{6a}\right)=0$, we get the result of the Corollary.

We can rewrite further the ratios:

Corollary 5.4. *Under the same conditions as above, we have:*

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}-\frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}=\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\frac{\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2a_1}\right)}$$

Proof: We begin by writing the ratio of the formulas in Proposition 5.1 and Corollary 5.3:

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}-\frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}=\frac{\frac{\sqrt[4]{3}}{a_1}e^{\pi i(a-1)/6}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right)}{\frac{\sqrt[4]{3}}{a_1}e^{\pi i(a-1)/6}\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2a_1}\right)}$$

Simplifying, we get the result of the Corollary.

Remark 5.1. Note that $\theta_0(z)=\eta(z/3)$, where η is the Dedekind eta function.

5.2 Ratios of θ_r and θ_0

Now we will apply the Factorization Lemma once more to connect the theta functions θ_r to the theta function θ_0. We do this by applying the Factorization Formula (12) twice and comparing the results.

Note first that any primitive ideal $\mathcal{A}$ in $\mathcal{O}_K$ prime to 6 has a generator $(n_a a+m_a\frac{-b+\sqrt{-3}}{2})$ such that $a=\mathrm{Nm}(\mathcal{A})$, $b^2\equiv -3 \mod 12a$ and $n_a\equiv 1 \mod 3$. Moreover, note that $a=n_a'^2a^2+m_a^2\frac{b^2+3}{4}-m_an_aab$, thus $m_an_ab\equiv 1 \mod a$, as $a|(b^2+3)/4$.

Using this notation, we have:

Lemma 5.4. *For $b \equiv 0 \mod 3$, $b^2 \equiv -3 \mod 4D^2aa'$, $n_{a'} \equiv 1 \mod 3$, we have:*

$$\theta_r\left(\frac{-b+\sqrt{-3}}{2aa'}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2aa'}\right) = \frac{1}{\sqrt{a'}}\theta_{n_{a'}r}\left(\frac{-b+\sqrt{-3}}{2a}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2a}\right)$$

Proof: We write the generator of $\mathcal{A}'$ in the form $(n_{a'}a' + m_{a'}\frac{-b+\sqrt{-3}}{2})$, where $b^2 \equiv -3 \mod 4aa'D^2$. Moreover, we can pick $n_{a'} \equiv 1 \mod 3$. Then, using the Factorization Formula (12) for $\mu = -\frac{1}{6} + \frac{r}{D}$, $\nu = \frac{1}{2}$, $a := D$ and $z = \frac{-b+\sqrt{-3}}{2aa'D}$, we have:

$$\frac{\sqrt[4]{3}}{2\sqrt{aa'}}\theta\begin{bmatrix}-\frac{1}{6}+\frac{r}{D}\\ D/2\end{bmatrix}\left(D\frac{-b+\sqrt{-3}}{2aa'D}\right)\theta\begin{bmatrix}-\frac{D}{6}\\ -1/2\end{bmatrix}\left(\frac{b+\sqrt{-3}}{2D^2aa'}\right) = \sum_{m,n} e^{2\pi i\frac{nr}{D}}e^{2\pi i\frac{n}{3}}e^{2\pi i(\frac{mn}{2D}+\frac{m}{2}+\frac{n}{2})}e^{2\pi i\frac{|m\frac{-b+\sqrt{-3}}{2}+naa'D|^2}{aa'D}\frac{\sqrt{-3}}{6D}}$$

Note that on the LHS we have $\theta\begin{bmatrix}-\frac{1}{6}+\frac{r}{D}\\ D/2\end{bmatrix}(z) = e^{-\pi i/6}e^{\pi ir}\theta_r(z)$ and $\theta\begin{bmatrix}-\frac{D}{6}\\ -1/2\end{bmatrix} = e^{\pi i/6}\theta_r(z)$. Furthermore, using Lemma 5.2, the RHS equals:

$$\sum_{m,n} e^{2\pi i\frac{nr}{D}}e^{2\pi i\frac{n}{3}}e^{2\pi i\frac{|m\frac{-b+\sqrt{-3}}{2}+naa'D|^2}{aa'D}\frac{-b+\sqrt{-3}}{6D}}.$$

Thus we got:

$$\frac{\sqrt[4]{3}}{2\sqrt{aa'}}e^{\pi ir}\theta_r\left(D\frac{-b+\sqrt{-3}}{2aa'D}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2aa'}\right) = \sum_{m,n} e^{2\pi i\frac{nr}{D}}e^{2\pi i\frac{n}{3}}e^{2\pi i\frac{|m\frac{-b+\sqrt{-3}}{2}+naa'D|^2}{aa'D}\frac{-b+\sqrt{-3}}{6D}}. \tag{16}$$

Note that if we write any element of $\mathcal{A}\mathcal{A}'D$, we can write it as an element of $\mathcal{A}D$ multiplied by the generator of $\mathcal{A}'$. Thus if we write an element of $\mathcal{A}\mathcal{A}'D$, in the form $m\frac{-b+\sqrt{-3}}{2} + naa'D$, it is going to equal an element $m_0\frac{-b+\sqrt{-3}}{2} + n_0aD \in \mathcal{A}D$ times the generator $m_{a'}\frac{-b+\sqrt{-3}}{2} + n_{a'}a'$ of $\mathcal{A}'$:

$$m\frac{-b+\sqrt{-3}}{2} + naa'D = (m_0\frac{-b+\sqrt{-3}}{2} + n_0aD)(m_{a'}\frac{-b+\sqrt{-3}}{2} + n_{a'}a')$$

This gives us:

$$\begin{cases} m = m_0m_{a'} + n_0n_{a'} - m_0m_{a'}b \\ n = n_0n_{a'} - m_0m_{a'}\frac{b^2+3}{4aa'D} \end{cases}$$

Since $b^2 + 3 \equiv 0 \mod 4D^2$, it implies that $n \equiv n_0n_{a'} \mod D$. Then we have:

$$\sum_{m,n} e^{2\pi i\frac{nr}{D}}e^{2\pi i\frac{n}{3}}e^{2\pi i\frac{|m\frac{-b+\sqrt{-3}}{2}+naa'D|^2}{aa'D}\frac{-b+\sqrt{-3}}{6D}} = \sum_{m_0,n_0} e^{2\pi i\frac{n_0n_{a'}r}{D}}e^{2\pi i\frac{n_0n_{a'}}{3}}e^{2\pi i\frac{|m_0\frac{-b+\sqrt{-3}}{2}+n_0aD|^2}{aD}\frac{-b+\sqrt{-3}}{6D}}$$

Since we picked $n_{a'} \equiv 1 \mod 3$, this is the same as $\sum_{m_0,n_0} e^{2\pi i\frac{n_0n_{a'}r}{D}}e^{2\pi i\frac{n_0}{3}}e^{2\pi i\frac{|m_0\frac{-b+\sqrt{-3}}{2}+n_0aD|^2}{aD}\frac{-b+\sqrt{-3}}{6D}}$.

Then applying the Factorization Formula (12) again for $\mu := -\frac{1}{6} + \frac{n'_ar}{D}$, $\nu := \frac{1}{2}$, $a := D$ and

$z := \frac{-b+\sqrt{-3}}{2aD}$, we get:

$$\sum_{m_0,n_0} e^{2\pi i \frac{n_0 n_{a'} r}{D}} e^{2\pi i \frac{n_0}{3}} e^{2\pi i \frac{|m_0 \frac{-b+\sqrt{-3}}{2} + n_0 aD|^2}{aD} \frac{-b+\sqrt{-3}}{6D}} = \frac{\sqrt[4]{3}}{2\sqrt{a}} \theta \begin{bmatrix} -\frac{1}{6} + \frac{n'_a r}{D} \\ D/2 \end{bmatrix} \left(D \frac{-b+\sqrt{-3}}{2aD} \right) \theta \begin{bmatrix} -\frac{D}{6} \\ 1/2 \end{bmatrix} \left(\frac{-b+\sqrt{-3}}{2D^2 a}. \right)$$

Moreover, on the RHS we have the theta functions $\theta \begin{bmatrix} -\frac{1}{6} + \frac{n'_a r}{D} \\ D/2 \end{bmatrix} (z) = e^{-\pi i/6} e^{\pi i n'_a r} \theta_r(z)$ and $\theta \begin{bmatrix} -\frac{D}{6} \\ 1/2 \end{bmatrix} (z) = e^{\pi i/6} \theta_0(z)$. Thus we can rewrite the equality as:

$$\sum_{m_0,n_0} e^{2\pi i \frac{n_0 n_{a'} r}{D}} e^{2\pi i \frac{n_0}{3}} e^{2\pi i \frac{|m_0 \frac{-b+\sqrt{-3}}{2} + n_0 aD|^2}{aD} \frac{-b+\sqrt{-3}}{6D}} = \frac{\sqrt[4]{3}}{2\sqrt{a}} e^{\pi i n'_a r} \theta_{n'_a r} \left(D \frac{-b+\sqrt{-3}}{2aD} \right) \theta_r \left(\frac{-b+\sqrt{-3}}{2D^2 a}. \right) \tag{17}$$

Comparing the two relations (16) and (17), we get:

$$\frac{1}{\sqrt{a'}} e^{\pi i r} \theta_r \left(\frac{-b+\sqrt{-3}}{2aa'} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2D^2 aa'} \right) = e^{\pi i n'_a r} \theta_{n_{a'} r} \left(\frac{-b+\sqrt{-3}}{2a} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2D^2 a} \right)$$

Lemma 5.5. *Under the same conditions as above, we have:*

$$\frac{e^{\pi i r} \theta_r \left(\frac{-b+\sqrt{-3}}{2aa'} \right)}{\theta_0 \left(\frac{-b+\sqrt{-3}}{2aa' D^2} \right)} = \frac{e^{\pi i n_{a'} r} \theta_{n_{a'} r} \left(\frac{-b+\sqrt{-3}}{2a} \right)}{\theta_0 \left(\frac{-b+\sqrt{-3}}{2aa' D^2} \right)}$$

Proof: Note that from Corollary 5.3, we have $\frac{3}{2}\Theta \left(\frac{-b+\sqrt{-3}}{2} \right) = \frac{\sqrt[4]{3}}{D\sqrt{aa'}} \theta_0 \left(\frac{-b+\sqrt{-3}}{2aa' D^2} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aa' D^2} \right)$.

Moreover, we also have from the same corollary that $\frac{3}{2}\Theta \left(\frac{-b+\sqrt{-3}}{2} \right) = \frac{\sqrt[4]{3}}{D\sqrt{a}} \theta_0 \left(\frac{-b+\sqrt{-3}}{2aD^2} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aD^2} \right)$, thus:

$$\frac{1}{\sqrt{a'}} \theta_0 \left(\frac{-b+\sqrt{-3}}{2aa' D^2} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aa' D^2} \right) = \theta_0 \left(\frac{-b+\sqrt{-3}}{2aD^2} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aD^2} \right)$$

Recall from the previous Lemma that we also have:

$$\frac{1}{\sqrt{a'}} e^{\pi i r} \theta_r \left(\frac{-b+\sqrt{-3}}{2aa'} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aa' D^2} \right) = e^{\pi i n_{a'} r} \theta_{n_{a'} r} \left(\frac{-b+\sqrt{-3}}{2a} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aD^2} \right)$$

Dividing the two relations, we get exactly:

$$\frac{e^{\pi i r} \theta_r \left(\frac{-b+\sqrt{-3}}{2aa'} \right)}{\theta_0 \left(\frac{-b+\sqrt{-3}}{2aa' D^2} \right)} = \frac{e^{\pi i n_{a'} r} \theta_{n_{a'} r} \left(\frac{-b+\sqrt{-3}}{2a} \right)}{\theta_0 \left(\frac{-b+\sqrt{-3}}{2aD^2} \right)}$$

5.3 Applying the factorization lemma to get a square

We would like to apply the factorization lemma for the formula in Theorem **??** for certain ideals that are representatives of the ring class field $\mathrm{Cl}(\mathcal{O}_{3D})$. We will pick this ideals below.

5.3.1 Representatives of $\mathrm{Cl}(\mathcal{O}_{3D})$.

Recall that, using Cox [4], for the ideal class group of conductor $3D$, we have:

$$\mathrm{Cl}(\mathcal{O}_{3D}) = (\mathcal{O}_{3D}/3D\mathcal{O}_K)^\times/(\mathbb{Z}/3D\mathbb{Z})^\times(\mathcal{O}_K^\times/\{\pm 1\})$$

Moreover,we can compute explicitly that for $D = \prod_{p_i \equiv 1 \mod 3} p_i$ we have $\mathrm{Cl}(\mathcal{O}_{3D}) \cong (\mathbb{Z}/D\mathbb{Z})^\times$ which also gives us $\#\,\mathrm{Cl}(\mathcal{O}_{3D}) = \phi(D)$, where ϕ is Euler's totient function.

Furthermore, we are claiming that we can take as representatives of $\mathrm{Cl}(\mathcal{O}_{3D})$ ideals with norm $\mathrm{Nm}\,\mathcal{A}_k \equiv k \mod D$ for $k \in (\mathbb{Z}/D\mathbb{Z})^\times$. We construct these ideals in the following lemma:

Lemma 5.6. *We can take as representatives of* $\mathrm{Cl}(\mathcal{O}_{3D})$ *the ideals:*

$$\mathcal{A}_k = \left(n_k a_k + m_k \frac{-b+\sqrt{-3}}{2}\right),$$

where $\mathrm{Nm}\,\mathcal{A}_k = a_k \equiv k \mod D$ *for* $k \in (\mathbb{Z}/D\mathbb{Z})^\times$, $a_k \equiv 1 \mod 6$ *and* $n_k \equiv 1 \mod D$. *We can pick such an ideal if we take* $m_k \equiv b^{-1}(k+1) \mod D$. *We can further put the conditions* $n_k, m_k \equiv 1 \mod 3$ *to determine the ideal uniquely modulo* $3D$.

Proof: Note first that two ideals $\mathcal{A}, \mathcal{B}$ are in the same class in $\mathrm{Cl}(\mathcal{O}_{3D})$ if we can find generators α, β for $\mathcal{A}$ and $\mathcal{B}$, respectively, such that $\alpha\beta^{-1} \equiv m \mod 3D$, where m is an integer prime to $3D$. Note that this implies $\alpha\beta^{-1} \equiv \pm 1 \mod 3$.

Let us assume that $\mathcal{A}_k$ and $\mathcal{A}_l$ are in the same class in $\mathrm{Cl}(\mathcal{O}_{3D})$. Then we must have $\pm\omega^i\left(n_k a_k + m_k \frac{-b+\sqrt{-3}}{2}\right) \equiv \pm\omega^j R\left(n_l a_l + m_l \frac{-b+\sqrt{-3}}{2}\right) \mod 3D$ for some i, j. Since we chose $n_k, m_k, n_l, m_l \equiv 1 \mod 3$ and b is odd we actually have $n_k a_k + m_k \frac{-b+\sqrt{-3}}{2} \equiv n_l a_l + m_l \frac{-b+\sqrt{-3}}{2} \equiv \omega \mod 3$, which determines the choice of $\pm\omega^i = \pm\omega^j$ on both sides. We further need the condition:

$$n_k a_k + m_k \frac{-b+\sqrt{-3}}{2} \equiv R(n_l a_l + m_l \frac{-b+\sqrt{-3}}{2}) \mod D$$

Note that this is equivalent to:

$$k + b^{-1}(k+1)\frac{-b+\sqrt{-3}}{2} \equiv R(l + b^{-1}(l+1)\frac{-b+\sqrt{-3}}{2}) \mod D$$

Furthermore, this can be rewritten as:

$$\frac{kb + (k+1)\sqrt{-3}}{2} \equiv R\frac{lb + (l+1)\sqrt{-3}}{2} \mod D$$

This implies $k \equiv lR \mod D$ and $k + 1 \equiv lR + R \mod D$, thus $R \equiv 1 \mod D$ and $k \equiv l \mod D$.

Finally, we have $\#(\mathbb{Z}/D\mathbb{Z})^\times$ such ideals, all in different classes of $\mathrm{Cl}(\mathcal{O}_{3D})$, thus we have representatives in every class of $\mathrm{Cl}(\mathcal{O}_{3D})$.

5.3.2 Using the factorization formula

We will pick representatives as in the above Lemma to rewrite the Proposition 5.1 and apply Corollary 5.4. We will denote by $\{s \in (\mathbb{Z}/D\mathbb{Z})^\times, s \equiv 1 \mod 6\}$ the norms of the ideals chosen in Lemma 5.6. Furthermore, we are going to choose in Proposition 5.1 all r to be even. We will use the notation $\{r \in \mathbb{Z}/D\mathbb{Z}, r \text{ even}\}$ to express this.

Lemma 5.7. *Picking representatives of $s \in (\mathbb{Z}/D\mathbb{Z})^\times$ such that $s \equiv 1 \mod 6$ and $r \in \mathbb{Z}/D\mathbb{Z}$ also such that $r \equiv 0 \mod 2$, we have*

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \frac{\theta_{sr}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_r\mathcal{A}_s)} \cdot \frac{\theta_r\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)}\chi(\mathcal{A}_r)$$

Proof: We fix $\phi(D)$ ideals $\mathcal{A}_k$ as in Lemma 5.6. Recall that we pick $\mathcal{A}_k$ such that $\operatorname{Nm}\mathcal{A}_k = a_k \equiv k \mod D$ for $k \in (\mathbb{Z}/D\mathbb{Z})^\times$, $a_k \equiv 1 \mod 6$ and $n_k \equiv 1 \mod D$. We can pick such an ideal if we take $\mathcal{A}_k = (n_k a_k + m_k \frac{-b+\sqrt{-3}}{2})$ with $m_k \equiv b^{-1}(k+1) \mod D$. We will try to compute:

$$S = \sum_{\substack{k\in(\mathbb{Z}/D\mathbb{Z})^\times \\ k\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_k}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_k}\right)}\overline{\chi(\mathcal{A}_k)}$$

Recall that from Corollary 5.4, we have:

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)} - \frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{6a_s}\right)} = \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \frac{\theta_{a_s r}\left(\frac{-b+\sqrt{-3}}{2a_s^2}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2a_s^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)}$$

Moreover since $r, a_r s$ are both even, we have $e^{\pi i n_{a_s^2} rs} = e^{\pi i rs} = 1$ and thus in Lemma 5.5 we have:

$$\frac{\theta_{rs}\left(\frac{-b+\sqrt{-3}}{2a_s^2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{-2a_s^2D^2}\right)} = \frac{\theta_{rs}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}$$

Then our sum can be written in the form:

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)} - \frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)} = \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \frac{\theta_{sr}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)} \tag{18}$$

Now summing up for all $s \in (\mathbb{Z}/D\mathbb{Z})^\times$, we get the result of the lemma:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} - \frac{1}{3}\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} =$$

$$= \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \frac{\theta_{rs}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_r\mathcal{A}_s)} \cdot \frac{\theta_r\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)}\chi(\mathcal{A}_r)$$

From Lemma 6.7 in Appendix A, we have $\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = 0$. This give us the result of the Lemma.

Proposition 5.2. *Under the conditions above, we have:*

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \left|\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\theta_s\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_s)}\right|^2$$

Proof: Only for the purpose of this proposition we will use the following notation for θ_r, to emphasize how it depends on D:

$$\theta_{r/D}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{r}{D}-\frac{1}{6}\right)^2 z}(-1)^n$$

Using the new notation, in the previous Lemma we have proved:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \frac{\theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)} \cdot \frac{\theta_{r/D}\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_s)}.$$

Note that using Corollary 5.3 for $a = D^2$ can rewrite:

$$\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right) = \frac{D}{\sqrt[4]{3}}\Theta\left(\frac{b+\sqrt{-3}}{2}\right)$$

Thus the equation becomes:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \frac{1}{\frac{D}{\sqrt[4]{3}}\Theta\left(\frac{b+\sqrt{-3}}{2}\right)} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_{r/D}\left(\frac{b+\sqrt{-3}}{2}\right)\overline{\chi(\mathcal{A}_s)} \tag{19}$$

Let $R \equiv R' \mod D$, R even and $S \equiv 1 \mod 6$. Then we have by definition:

$$\theta_{RS}(z_1)\Theta_R(z_2) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+RS/D-1/6)^2 z_1} e^{\pi i n} \sum_{m\in\mathbb{Z}} e^{\pi i(m+R/D-1/6)^2 z_2} e^{\pi i m}$$

By changing $n \to n+S$ and $m \to m+1$, we change $R \to D+R$ and $R+D \equiv R' \mod 2D$. We get

$$\theta_{RS}(z_1)\Theta_R(z_2) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+R'S/D-1/6)^2 z_1} e^{\pi i n}(-1)^S \sum_{m\in\mathbb{Z}} e^{\pi i(m+R'/D-1/6)^2 z_2} e^{\pi i m}(-1) = \theta_{R'S}(z_1)\Theta_{R'}(z_2')$$

Thus we can choose in the formulas above all r to be actually odd. Furthermore, by making a change of $r \pm 2D$ we can also choose $r \equiv 1 \mod 3$. Then we can rewrite the equation as:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \frac{1}{\frac{D}{\sqrt[4]{3}}\Theta\left(\frac{b+\sqrt{-3}}{2}\right)} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r\equiv 1 \mod 6}} \theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_{r/D}\left(\frac{b+\sqrt{-3}}{2}\right)\overline{\chi(\mathcal{A}_s)} \tag{20}$$

Denote $\tau_D = \frac{-b+\sqrt{-3}}{2}$. Note that we are summing over all residues $r \mod D$. We can separate the terms, depending on whether a prime divisor p_i divides both D and r. We do this by using the Inclusion-Exclusion principle and note that the sum gets rewritten as:

$$\begin{aligned}
\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r\equiv 1 \mod 6}} \theta_{sr/D}(\tau_D)\theta_{r/D}(-\overline{\tau_D})\overline{\chi(\mathcal{A}_s)} = & \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in(\mathbb{Z}/D\mathbb{Z})^\times \\ r\equiv 1 \mod 6}} \theta_{sr/D}(\tau_D)\theta_{r/D}(-\overline{\tau_D})\overline{\chi(\mathcal{A}_s)} \\
& + \sum_{p_i|D} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/(D/p_i)/\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{\frac{sr}{(D/p_i)}}(\tau_D)\theta_{\frac{r}{(D/p_i)}}(-\overline{\tau_D})\overline{\chi(\mathcal{A}_s)} \\
& - \sum_{p_ip_j|D} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/(D/p_ip_j)\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{\frac{sr}{(D/p_ip_j)}}(\tau_D)\theta_{\frac{r}{(D/p_ip_j)}}(-\overline{\tau_D})\overline{\chi(\mathcal{A}_s)} \\
& \dots \\
& + (-1)^{n-1} \sum_{p_1\dots p_n|D} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/(D/p_1\dots p_n)\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{\frac{sr}{(D/p_1\dots p_n)}}(\tau_D)\theta_{\frac{r}{(D/p_1\dots p_n)}}(-
\end{aligned}$$

Using Lemma 5.8 proved below, all of the terms except for the first one equal 0. Thus getting back to the equation (19), we get:

$$\begin{aligned}
\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} &= \sum_{\substack{s,r\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv r\equiv 1 \mod 6}} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{6D^2}\right)}\overline{\chi(\mathcal{A}_r\mathcal{A}_s)} \cdot \frac{\theta_{r/D}\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{6D^2}\right)}\chi(\mathcal{A}_r) \\
&= \sum_{s,r} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_r\mathcal{A}_s)} \cdot \overline{\frac{\theta_{r/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_r)}} \\
&= \left| \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\theta_{s/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_s)} \right|^2
\end{aligned}$$

Below we prove Lemma 5.8 used in the proof of Proposition 20:

Lemma 5.8. *If $D = p_1 \dots p_n$ and $D' = D/(p_{i_1} \dots p_{i_k})$, then:*

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D'\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{sr/D'}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_{r/D'}\left(\frac{b+\sqrt{-3}}{2}\right)\overline{\chi(\mathcal{A}_s)} = 0$$

Proof: Note that first that we can rewrite the sum in the form:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D'\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{sr/D'}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_{r/D'}\left(\frac{b+\sqrt{-3}}{2}\right)\overline{\chi(\mathcal{A}_s)} =$$

$$= \theta_0\left(\frac{-b+\sqrt{-3}}{2D'^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D'^2}\right) \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D'\mathbb{Z} \\ r\equiv 1 \mod 6}} \frac{\theta_{rs/D'}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D'^2}\right)} \cdot \frac{\theta_{r/D'}\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{2D'^2}\right)}\overline{\chi(\mathcal{A}_s)}$$

Using (18) for $D := D'$, we recognize the sum on the LHS to be:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D'\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)}$$

Denote $m = D/D' = p_{i_1} \dots p_{i_k}$. Moreover, recall that from the definition of the cubic character we have:

$$D^{1/3}\overline{\chi_D(\mathcal{A}_s)} = (D^{1/3})^{\sigma_{\mathcal{A}_s}} = (D'^{1/3})^{\sigma_{\mathcal{A}_s}}(m^{1/3})^{\sigma_{\mathcal{A}_s}} = D'^{1/3}\overline{\chi_{D'}(\mathcal{A}_s)}m^{1/3}\overline{\chi_m(\mathcal{A}_s)}$$

Then we can rewrite the sum as:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D'\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi_D(\mathcal{A}_s)} = \sum_{s'\in(\mathbb{Z}/D'\mathbb{Z})^\times, s'\equiv 1 \mod 6} \frac{\Theta\left(D'\frac{-b+\sqrt{-3}}{2a'_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a'_s}\right)}\overline{\chi_{D'}(\mathcal{A}_s)} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times, \\ s\equiv s'\equiv 1 \mod 6 \\ s\equiv s' \mod D'}} \overline{\chi_m(\mathcal{A}_s)}$$

Note that as $D = p_1 \cdots p_n$, we have $\{s \in (\mathbb{Z}/D\mathbb{Z})^\times, s \equiv s' \mod D'\} \cong (\mathbb{Z}/m\mathbb{Z})^\times$. Moreover, note that $\chi_m(\mathcal{A}_s)$ depends only on $s \mod m$. Thus we are summing the character $\chi_m(\mathcal{A}_s) = \chi_m(\mathcal{A}_{s''})$ over $s'' \in (\mathbb{Z}/m\mathbb{Z})^\times$.

Moreover, $\overline{\chi_m(\mathcal{A}_s)}$ is a nontrivial character as a function of s, as $m^{1/3}\overline{\chi_m(\mathcal{A}_s)} = (m^{1/3})^{\sigma_{\mathcal{A}_s}} = m^{1/3}$ for all $\mathcal{A}_s$ iff $m^{1/3} \in \mathbb{Q}[\sqrt{-3}]$. As we are summing a non-trivial character over a group, the sum is just 0:

$$\sum_{s''\in(\mathbb{Z}/m\mathbb{Z})^\times} \overline{\chi_m(\mathcal{A}_{s''})} = 0,$$

thus the whole sum is zero.

We left out the case $r \equiv 0 \mod D$. In this case we have:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2}\right)}{\frac{D}{\sqrt[4]{3}}\Theta\left(\frac{b+\sqrt{-3}}{2}\right)}\overline{\chi(\mathcal{A}_s)} = \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{1}{D}\overline{\chi(\mathcal{A}_s)} = 0$$

5.4 Shimura reciprocity applied to θ_r

We define:

$$f_r(z) = \frac{\theta_r(z)}{\theta_0(z)} = \frac{\sum\limits_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{r}{D}-\frac{1}{6}\right)^2 z} e^{\pi i n}}{\sum\limits_{n\in\mathbb{Z}} e^{\pi i\left(n-\frac{1}{6}\right)^2 z} e^{\pi i n}}$$

Then we can rewrite Proposition 5.2 as:

$$\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \left|\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} f_s(\tau)\overline{\chi(\mathcal{A}_s)}\right|^2 \left|\frac{\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\right|^2$$

Note that from the Corollary 5.3, we can compute $\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2}\right) = \frac{\sqrt[4]{3}}{1}\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2}\right)$ as well as $\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2}\right) = \frac{\sqrt[4]{3}}{D}\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)$. Taking the ratio of the two relations, gives us:

$$\left|\frac{\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\right|^2 = D$$

Thus we get:

$$\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = D\left|\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} f_s(\tau)\overline{\chi(\mathcal{A}_s)}\right|^2.$$

By further multiplying by $D^{1/3}$, we have:

$$\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}D^{1/3}\overline{\chi(\mathcal{A}_s)} = \left|\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} f_s(\tau)\overline{\chi(\mathcal{A}_s)}D^{2/3}\right|^2. \tag{21}$$

Our goal in this section is to show that all the terms $f_s(\tau)\overline{\chi(\mathcal{A}_s)}D^{2/3}$ are Galois conjugates of each other.

5.4.1 θ_r as an automorphic form

We will first look closer at the function θ_r. We will rewrite θ_r as an automorphic theta function $\Theta : \mathrm{SL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$:

$$\Theta(g) = \sum_{m \in \mathbb{Q}} r(g)\Phi(m),$$

where $\Phi \in (\mathbb{A}_{\mathbb{Q}})$ is a Schwartz-Bruhat function and r is the Weil representation defined by:

- $r\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \Phi\right)(x) = \chi_0(a)|a|^{1/2}\Phi(ax)$
- $r\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Phi\right)(x) = \psi(bx^2)\Phi(x)$
- $r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi\right)(x) = \gamma\widehat{\Phi}(x),$

where $\psi_p(x) = e^{-2\pi i \operatorname{Frac}_p(x)}$ and $\psi_\infty(x) = e^{2\pi i x}$, γ is an 8th root of unity, and χ_0 is a character.

XXX define χ_0 and γ

We define the following Schwartz-Bruhat functions for θ. Let $f_\mu = \prod_{v_p(\mu)<0} p$. We define $\Phi_\mu = \prod_v \Phi_{\mu,v}$, where:

$$\begin{cases} \Phi_p^{(r)} = \operatorname{char}_{\mathbb{Z}_p}, & \text{if } p \nmid D \\ \Phi_p^{(r)} = \operatorname{char}_{\mathbb{Z}_p - \frac{r}{D}}, & \text{if } p|D, p \nmid 2,3 \\ \Phi_3^{(r)} = \operatorname{char}_{\mathbb{Z}_3 + \frac{1}{3}}, & \\ \Phi_2^{(r)}(n) = e^{\pi i \operatorname{Frac}_2(n)} \operatorname{char}_{\mathbb{Z}_2 + \frac{1}{2}}(n), & \\ \Phi_\infty^{(r)}(x) = e^{-2\pi q(x)}. & \end{cases}$$

We define the theta function:

$$\Theta_{\Phi^{(r)}}(g) = \sum_{n \in \mathbb{Q}} r(g)\Phi^{(r)}(n)$$

Note that $\Phi_f^{(r)}(n) \neq 0$ for $n \in \mathbb{Q}$ implies $n - \frac{r}{D} + \frac{1}{6} \in \mathbb{Z}_p$ for all p. This implies $n - \frac{r}{D} + \frac{1}{6} \in \mathbb{Z}$, thus $n \in \mathbb{Z} + \frac{r}{D} - \frac{1}{6}$ Also note that for $g_z = \left(\begin{smallmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2} \end{smallmatrix}\right)$, we have $r(g_z)\Phi_\infty(n) = r\left(\left(\begin{smallmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & xy^{-1} \\ 0 & 1 \end{smallmatrix}\right)\Phi\right)(n) = y^{1/2}e^{2\pi i(x+yi)n^2}$. Then we can compute:

$$\Theta_{\Phi^{(r)}}(g_z, 1_f) = \sum_{n \in \mathbb{Z} + \frac{r}{D} - \frac{1}{6}} e^{2\pi i z n^2} e^{\pi i \operatorname{Frac}_2(n)} = y^{1/2}\theta_r(2z)$$

Note that: $\theta_r(2z) = y^{-1/2}\Theta_{\Phi^{(r)}}(g_z, 1_f)$ and $\theta_0(2z) = y^{-1/2}\Theta_{\Phi^{(0)}}(g_z, 1_f)$, which implies:

$$\frac{\theta_r(z)}{\theta_0(z)} = \frac{\Theta_{\Phi^{(r)}}(g_{z/2}, 1_f)}{\Theta_{\Phi^{(0)}}(g_{z/2}, 1_f)}$$

5.4.2 Galois action on modular functions (Shimura reciprocity)

Recall the function f_r:

$$f_r(z) = \frac{\theta_r(z)}{\theta_0(z)} = \frac{\Theta_{\Phi^{(r)}}(g_{z/2}, 1_f)}{\Theta_{\Phi^{(3)}}(g_{z/2}, 1_f)}$$

Lemma 5.9. *The theta function $\theta_r(z)$ is modular form of weight $1/2$ for $\Gamma(72D^2)$.*

Proof: Recall that $\theta_r(z) = \Theta_{\Phi^{(r)}}(g_{z/2})$. We will compute $\theta_r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)$, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(72D^2)$. Note first that:

$$\theta_r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} g_z\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} g_z\right)$$

As $\Theta_{\Phi^{(r)}}$ is invariant under $\mathrm{SL}_2(\mathbb{Q})$, we can rewrite $\Theta_{\Phi^{(r)}}$ as:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} g_z, 1\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}^{-1}\right)$$

We will compute separately the two terms, using the Weil representations. For the RHS, note that we have to compute $r\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}^{-1}\Phi_f^{(r)} = r\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}\Phi_f^{(r)}$. We will show:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}^{-1}\right) = \prod_{p|6D} \gamma_p^2 \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} g_z,\right)$$

We rewrite the matrix as:

$$\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix} = \begin{pmatrix} 1/a & -b/2 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2c/a & 1 \end{pmatrix}$$

At $p \nmid 6D$, the action of $\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}$ is trivial, as it belongs to $\mathrm{SL}_2(\mathbb{Z}_p)$ and $\Phi_p^{(r)}$ is the characteristic function of $\mathbb{Z}_p$. For $p|6D$, we compute:

First we compute $r\begin{pmatrix} 1 & 0 \\ -2c/d & 1 \end{pmatrix}\Phi_p^{(r)}(x) = \gamma_p^2\Phi_p^{(r)}(x)$. We rewrite the matrix as

$$\begin{pmatrix} 1 & 0 \\ -2c/d & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2c/d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and compute the Weil representation action:

- $r\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Phi_p^{(r)}(x) = \gamma_p\widehat{\Phi_p^{(r)}}(x)$. Note that we can compute:

$$\widehat{\Phi_p^{(r)}}(x) = \int_{\mathbb{Q}_p} e^{-2\pi i\,\mathrm{Frac}_p(2xy)}\,\mathrm{char}_{\mathbb{Z}_p+\frac{r}{D}}(y)dy = \int_{\mathbb{Z}_p} e^{-2\pi i\,\mathrm{Frac}_p(2x(y+r/D))}dy = e^{-2\pi i Frac_p(2rx/D}\,\mathrm{char}_{\mathbb{Z}_p}(x)$$

- $r\begin{pmatrix}1 & 2c/d\\ 0 & 1\end{pmatrix}\widehat{\Phi^{(r)}}_p(x) = e^{-2\pi i\,\mathrm{Frac}_p(2c/dx^2}e^{-2\pi i Frac_p(2rx/D}\,\mathrm{char}_{\mathbb{Z}_p}(x)$. As $v_p(c/d)\geqslant 0$, we have $e^{-2\pi i\,\mathrm{Frac}_p(2c/dx^2}=1$, thus the action is trivial on $\widehat{\Phi_p^{(r)}}(x)$
- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\widehat{\Phi_p^{(r)}}(x) = \gamma_p\widehat{\widehat{\Phi_p^{(r)}}}(x)$. By the choice of the self-dual Haar measure, this equals $\gamma_p\Phi_p^{(r)}(-x)$.
- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Phi_p^{(r)}(-x) = \Phi_p^{(r)}(x)$

Now we also want to compute the action of $r\begin{pmatrix}1/a & -b/2\\ 0 & a\end{pmatrix}\Phi_p^{(r)}(x)$. We rewrite the matrix as:

$$\begin{pmatrix}1/a & -b/2\\ 0 & a\end{pmatrix} = \begin{pmatrix}1/a & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}1 & -ba/2\\ 0 & 1\end{pmatrix}$$

and compute the action:

- $r\begin{pmatrix}1 & -ba/2\\ 0 & 1\end{pmatrix}\Phi_p^{(r)}(x) = e^{2\pi i\,\mathrm{Frac}_p(ba/2x^2)}\,\mathrm{char}_{\mathbb{Z}_p+r/D}(x)$. As $D^2|ba/2$, we have $e^{2\pi i\,\mathrm{Frac}_p(ba/2x^2)}$, thus we have trivial action.
- $r\begin{pmatrix}1/a & 0\\ 0 & a\end{pmatrix}\Phi_p^{(r)}(x) = \chi_0(a)|a|_p^{1/2}\Phi_p^{(r)}(x/a)$. As $a\equiv 1 \mod D$, we get $\Phi_p^{(r)}(x/a)=\Phi_p^{(r)}(x)$, as well as $\chi_0(a)=|a|_p^{1/2}=1$.

For $p=3$ the computation is similar. For $p=2$, we compute again the action of$r\begin{pmatrix}1 & 0\\ -2c/d & 1\end{pmatrix}\Phi_p^{(r)}(x) = \gamma_p^2\Phi_p^{(r)}(x)$:

- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Phi_2^{(r)}(x) = \gamma_2\widehat{\Phi_p^{(r)}}(x)$. Note that we can compute:
  $\widehat{\Phi_2^{(r)}}(x) = \int\limits_{\mathbb{Q}_2} e^{-2\pi i\,\mathrm{Frac}_2(2xy)}\,\mathrm{char}_{\mathbb{Z}_2+\frac{1}{2}}(y)e^{\pi i\,\mathrm{Frac}_2(y)}dy = e^{\pi i/2}\int\limits_{\mathbb{Z}_2} e^{-2\pi i\,\mathrm{Frac}_2(2x(y+1/2))}e^{\pi i\,\mathrm{Frac}_2(y)}dy = e^{\pi i/2}e^{-2\pi i\,\mathrm{Frac}_2(x)}\int\limits_{\mathbb{Z}_2} e^{-2\pi i\,\mathrm{Frac}_2((2x-1/2)y)}dy = e^{\pi i/2}e^{-2\pi i\,\mathrm{Frac}_2(x)}\,\mathrm{char}_{\frac{1}{2}(\mathbb{Z}_2+1/2)}(x)$
- $r\begin{pmatrix}1 & 2c/d\\ 0 & 1\end{pmatrix}\widehat{\Phi^{(r)}}_2(x) = e^{-2\pi i\,\mathrm{Frac}_2(2c/dx^2}\widehat{\Phi^{(r)}}_2(x)$. As $v_p(2c/d)\geqslant 4$, we have $e^{-2\pi i\,\mathrm{Frac}_p(2c/dx^2}=1$, thus the action is trivial on $\widehat{\Phi_p^{(r)}}(x)$
- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\widehat{\Phi_p^{(2)}}(x) = \gamma_2\widehat{\widehat{\Phi_2^{(r)}}}(x)$. By the choice of the self-dual Haar measure, this equals $\gamma_2\Phi_2^{(r)}(-x)$.
- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Phi_2^{(r)}(-x) = \Phi_2^{(r)}(x)$

We compute similarly the action of $r\begin{pmatrix}1/a & -b/2\\ 0 & a\end{pmatrix}\Phi_2^{(r)}(x)$:

- $r\begin{pmatrix}1 & -ba/2\\ 0 & 1\end{pmatrix}\Phi_2^{(r)}(x) = e^{2\pi i\,\mathrm{Frac}_p(ba/2x^2)}e^{\pi i\,\mathrm{Frac}_2(x)}\,\mathrm{char}_{\mathbb{Z}_2+1/2}(x)$. As $4|ba/2$, we have $e^{2\pi i\,\mathrm{Frac}_2(ba/2x^2)}=1$, thus we have trivial action.
- $r\begin{pmatrix}1/a & 0\\ 0 & a\end{pmatrix}\Phi_2^{(r)}(x) = \chi_0(a)|a|_2^{1/2}\Phi_2^{(r)}(x/a)$. As $a\equiv 1 \mod 8$, we get $\Phi_2^{(r)}(x/a)=\Phi_2^{(r)}(x)$, as well as $\chi_0(a)=|a|_2^{1/2}=1$.

This finishes the computation of the finite part. We got:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix}1/\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}g_z,\begin{pmatrix}a & b/2\\ 2c & d\end{pmatrix}^{-1}\right) = 2^{-1/4}y^{1/2}\sum_{m\in ZZ+\frac{r}{D}-\frac{1}{6}} e^{\pi i m^2 z}(-1)^m = 2^{-1/4}y^{1/2}\theta_r(z) \tag{22}$$

We will compute now the infinite part. Note first that $r(g_z)\Phi_\infty(m) = y^{1/4}e^{2\pi i z|m|^2}$ We rewrite the matrix:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & b/d\\ 0 & 1\end{pmatrix}\begin{pmatrix}1/d & 0\\ 0 & d\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & -c/d\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

We compute the Weil representation action:

- $F_1(m) := \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}e^{2\pi i z m^2} = \gamma_\infty\frac{1}{\sqrt{z}}e^{-2\pi i\frac{1}{z}}$
- $F_2(m) := r\begin{pmatrix}1 & -c/d\\ 0 & 1\end{pmatrix}F_1(m) = e^{-2\pi i\frac{c}{d}m^2}F_1(m) = \gamma_\infty\frac{1}{\sqrt{z}}e^{-2\pi i\frac{cz+d}{dz}}$
- $F_3(m) := r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}F_3(m) = \gamma_\infty\widehat{F_3(m)} = \gamma_\infty^2\frac{1}{\sqrt{z}}e^{2\pi i\frac{dz}{cz+d}}(*)\sqrt{\frac{dz}{cz+d}} = \gamma_\infty^2(*)\sqrt{\frac{d}{cz+d}}e^{2\pi i\frac{dz}{cz+d}m^2}$
- $F_4(m) := r\begin{pmatrix}1/d & 0\\ 0 & d\end{pmatrix}F_3(m) = \mathrm{sgn}(d)d^{1/2}F_3(m/d) = \mathrm{sgn}(d)d^{1/2}F_3(m/d)\gamma_\infty^2(*)\sqrt{\frac{d}{cz+d}}e^{2\pi i\frac{z}{d(cz+d)}m^2} = \mathrm{sgn}(d)F_3(m/d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}e^{2\pi i\frac{z}{d(cz+d)}m^2}$
- $F_5(m) := r\begin{pmatrix}1 & b/d\\ 0 & 1\end{pmatrix}F_4(m) = e^{2\pi i\frac{b}{d}m^2}F_4(m) = \mathrm{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}e^{2\pi i\left(\frac{b}{d}+\frac{z}{d(cz+d)}\right)m^2} = \mathrm{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}e^{2\pi i\left(\frac{(bc+1)z+bd}{d(cz+d)}\right)m^2} = \mathrm{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}e^{2\pi i\left(\frac{az+b}{cz+d}\right)m^2}$
- $r\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}F_5(m) = -F_5(-m) = -\mathrm{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}e^{2\pi i\left(\frac{az+b}{cz+d}\right)m^2}$

We still have to compute the action of $r\begin{pmatrix}1/\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}$ on $-y^{1/2}\,\mathrm{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}e^{2\pi i\left(\frac{az+b}{cz+d}\right)m^2}$. This gives us just:

$$2^{-1/4}y^{1/2}\operatorname{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}e^{\pi i\left(\frac{az+b}{cz+d}\right)m^2}$$

Thus we have:

$$\Theta\left(\begin{pmatrix}1/\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}z,1\right)=2^{-1/4}y^{1/2}\operatorname{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}\sum_{m\in\mathbb{Z}+\frac{r}{D}-\frac{1}{6}}e^{\pi i\left(\frac{az+b}{cz+d}\right)m^2}(-1)^m$$

Note that this is exactly:

$$2^{-1/4}y^{1/2}\operatorname{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}\theta_r\left(\frac{az+b}{cz+d}\right) \tag{23}$$

From (21) and (22) we get that:

$$\operatorname{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}\theta_r\left(\frac{az+b}{cz+d}\right)=\gamma_f^2\theta_r(z)$$

XXXXXX γ should disappear here

Lemma 5.10. *$f\in\mathcal{F}$ is a modular function for $\Gamma(72D^2)$.*

Proof: We need to check that for $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma(3D)$ we have:

$$f_r(\gamma z)=f_r(z).$$

Proof: Using the previous lemma, for $\gamma\in\Gamma(72D^2)$ we have $\operatorname{sgn}(d)\gamma_\infty^2(*)\sqrt{\frac{1}{cz+d}}\theta_r(\gamma z)=\theta_r(z)$. Applying the same computation for $r=0$, we get $\operatorname{sgn}(d)\gamma^2(*)\sqrt{\frac{1}{cz+d}}\theta_0(\gamma z)=\theta_0(z)$. Thus we have:

$$\frac{\theta_r(\gamma z)}{\theta_0(\gamma z)}=\frac{(*)\sqrt{cz+d}\theta_r(z)}{(*)\sqrt{cz+d}\theta_0(z)}=\frac{\theta_r(z)}{\theta_0(z)}$$

Lemma 5.11. *The modular function f_r has rational Fourier coefficients.*

Proof: Note that $\theta_r(z)=q^{(D-r)^2/72}(1+\sum_{M\geqslant 1}a_Mq^{M/(72D^2)})$, where $a_m\in\mathbb{Z}$ and $\theta_0(z)=q^{1/72}(1+\sum_{M\geqslant 1}b_Mq^{M/72})$

Then we can compute $f_r(z)=q^{((D-r)^2-1)/72}(1+\sum_m a_mq^{m/72D^2})$ with $a_m\in\mathbb{Z}$.

From CM-theory we have $f(\tau)\in H_{\mathcal{O}}$, where $H_{\mathcal{O}}$ is the ray class field of modulus $72D^2$. In order to compute its Galois conjugates over K we can use Shimura reciprocity law. In its generality:

Shimura reciprocity law. For $\tau\in K\cap\mathcal{H}$ with minimal polynomial $X^2+BX+C=0$, we have its Galois conjugates:

$$f_r(\tau)^{\sigma_x^{-1}} = f_r^{g_\tau(x)}(\tau),$$

for $x \in \mathbb{A}_{K_f}^\times$, $g_\tau(x) = \begin{pmatrix} t - sB & -sC \\ s & t \end{pmatrix}$.

In our case, we want to compute the Galois conjugates of $f_r(\tau)$, where $\tau = \frac{-b+\sqrt{-3}}{2}$. Note that it has the minimum polynomial $X^2 + bX + \frac{b^2+3}{4}$. Thus we have to compute the action of all $g_\tau((x_p)_p) = \prod_p \begin{pmatrix} t_p - s_p b & -s_p \frac{b^2+3}{4} \\ s_p & t_p \end{pmatrix}_p$.

We will compute all these actions. However, we claim that it is enough to compute the action of the ideals $\mathcal{A}$ through the correspondence:

$$I(3) \to \mathbb{A}_{K,f}^\times / K^\times$$
$$\mathcal{A} = (A + B\omega) \longrightarrow (A + B\omega)_{p|6D},$$

where $A + B\omega \equiv 1 \mod 3$ is the generator of the ideal $\mathcal{A}$.

More precisely, in order to find the Galois conjugates over K, we will compute the action of all Galois actions corresponding to $(A_p + B_p\omega)_p \in \mathbb{A}_K^\times$ and we will prove that the Galois action from Shimura reciprocity law is:

Proposition 5.3. *For $\mathcal{A} = (n_a a + m_a \frac{-b+\sqrt{-3}}{2})$, where $b^2 \equiv -3 \mod 4Db^2$ is an ideal prime to $6D$, we have:*

$$f_1(\tau)^{\sigma_A} = f_{n_a}(\tau)$$

and $f_r(\tau)$ are all the Galois conjugates of $f(\tau)$, where $r \in (\mathbb{Z}/D\mathbb{Z})^\times$. Moreover, this implies that $f_1(\tau) \in H_{6D}$.

Proof: First we note that we do not have to consider the action of all $(x_p)_p \in \mathbb{A}_K^\times$. By applying the Strong Approximation Theorem for GL_1 and the number field K that is a PID, we have:

$$\mathbb{A}_K^\times = K^\times \times \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times \times \mathbb{C}^\times$$

This implies:

$$\mathbb{A}_{K_f}^\times = K^\times \times \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times$$

Then any $x = (x_v) \in \mathbb{A}_{K,f}^\times$ can be written as $x = k(l_v)$, where $k \in K^\times$, $(l_v)_v \in \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times$.

Since $\operatorname{Nm} k > 0$, we have the embedding:

$$k \in K^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q})^+$$

We also have the embedding:

$$(l_v)_v \in \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times \hookrightarrow \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$$

Thus if we know the Galois action of $K^\times$ and of $\widehat{\mathcal{O}}_K^\times$, we will know the Galois action of $\mathbb{A}_{K,f}^\times$.

We recall the way the action of $g_\tau(x)$ is defined for. For $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$, f^α is defined by $f^\alpha(\tau) = f(\alpha\tau)$. In our case we only need to look at the action of $K^\times$. Recall that $k \in K^\times$ embeds into $\mathrm{GL}_2(\mathbb{Q})^+$ under the map:

$$k = t + s\frac{-b+\sqrt{-3}}{2} \hookrightarrow g_\tau(k) = \begin{pmatrix} t - sb & -sc \\ s & t \end{pmatrix}$$

Then the Galois action from Shimura reciprocity is:

$$f(\tau)^{k^{-1}} = f^{g_\tau(k)}(\tau) = f(g_\tau(k)\tau)$$

Note that $t + s\tau \to \left(\begin{smallmatrix} t-sb & -sc \\ s & t \end{smallmatrix}\right)$ is the torus that preserves τ, thus we have:

$$f(\tau)^{k^{-1}} = f(g_\tau(k)\tau) = f(\tau)$$

Now all we have left is to compute the action of $\prod \mathcal{O}_{K_v}^\times$. Note that for all $v \nmid 6D$ the action is trivial. For $v|6D$ we project the action of $(g_\tau(x_v))_v \twoheadrightarrow g_\tau(x') \in \mathrm{GL}_2(\mathbb{Z}/6D^2\mathbb{Z})$.

Remark 5.2. Note that we have for $(\pm\omega^i)_p \hookrightarrow \mathbb{A}_{K_f}^\times$ acting trivially. Thus we have for $x \in \mathbb{A}_{K_f}^\times$:

$$(f_r(\tau))^{\sigma_{\pm\omega^i x}} = ((f_r(\tau))^{\sigma_{\pm\omega^i}})^{\sigma_x} = (f_r^{g_\tau(\pm\omega^i)}(\tau))^{\sigma_x} = f_r(\tau)^{\sigma_x}$$

Lemma 5.12. *For $x \in \prod_v \mathcal{O}_{K_v}^\times$ we can find ω^i, $i = 0, \pm 1$ such that:*

$$(x_2 \pm \omega^i)_2 = (t_2 + s_2\omega)$$

with $v_2(t_2) = 0$, $v_2(s_2) \geqslant 1$ and

$$(x_3 \pm \omega^i)_3 = (t_3 + s_3\omega)$$

with $t_3 + s_3 \equiv 1 \mod 3$.

Proof: Note first that if $v_2(s) \geqslant 1$, then we must have $v_2(t_2) = 0$, as we need $x_2\omega^i \in (\mathbb{Z}_2[\omega])^\times$. Thus we must find $x\omega^i$ such that $v_2(s) \geqslant 1$. We write $x_2 = t_2' + s_2'\omega$. Then:

$$x_2\omega = t_2'\omega + s_2'\omega^2 = (t_2' - s_2')\omega + s_2'$$
$$x_2\omega^2 = t_2'\omega^2 + s_2' = (-t_2')\omega + (s_2' - t_2')$$

One of $t_2', s_2', t_2' - s_2'$ must have positive valuation. Assume this is not true: $v_2(t_2') = v_2(s_2') = 0$. Then $s_2', t_2' \equiv 1 \mod 2$, thus $s_2' - t_2' \equiv 0 \mod 2$ and has positive valuation. Thus we can always pick $x\omega^i$ as claimed above at the place 2.

Now since take $x_3\omega^i = s_3'\omega + t_3' = s_3'\frac{-3+\sqrt{-3}}{2} + (t_3' + s_3')$. Then, since x_3 is a unit in $\mathbb{Z}_3[\omega]$, we must have $v_3(s_3' + t_3') = 0$, thus $s_3' + t_3' \equiv \pm 1 \mod 3$. We pick $x_3\omega$ or $-x_3\omega$ to get the condition $s_3' + t_3' \equiv 1 \mod 3$.

Since from the remark above x and $\pm\omega^i x$ act the same, we can consider the Galois action of $\sigma_{x\omega^i}$ as in the lemma above. We compute it below.

Let $x_p \in \prod_v \mathcal{O}_{K_v}^\times$ chosen as above. Then:

$$x_p = t_p + s_p\frac{-b+\sqrt{-3}}{2} \hookrightarrow g_\tau(x_p) = \begin{pmatrix} t_p - s_p b & -s_p c \\ s_p & t_p \end{pmatrix}$$

Elements of $\prod_p \mathrm{GL}_2(\mathbb{Z}_p)$ project to $\mathrm{GL}_2(\mathbb{Z}/6D^2\mathbb{Z})$, which is the action we care about. From Chinese remainder theorem, we can find $k_0 \in K$ such that $k_0 \equiv x_p \mod 6D^2\mathbb{Z}_p$ for all $p|6D$. Note that k_0 is independent of the choice of τ.

Then we only need to compute the action of:

$$f_r(\tau)^{\sigma_x^{-1}} = f^{g_\tau(x)}(\tau) = f_r^{g_\tau(x_v)_{v|6D}}(\tau) = f^{g_\tau(t+s\tau)_{v|6D}}(\tau)$$

We will now compute $f_r^{g_\tau(x)_{p|6D}}(\tau)$. Note that, for $c' = \frac{b^2+3}{4}$, we have the map :

$$k_0 = s\tau + t \to g_\tau(k_0) = \begin{pmatrix} t - sb' & -sc \\ s & t \end{pmatrix}$$

Let $\mathrm{Nm}(k_0) = a$. We write the action:

$$f(\tau)^{\sigma_x} = f^{\begin{pmatrix} t - sb & -sc/a \\ s & t/a \end{pmatrix}_{p|6D}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|6D}}$$

Note that $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|6D}$ acts trivially on f_r as both functions $\theta\begin{bmatrix} -\frac{1}{6} + \frac{r}{D} \\ \frac{1}{2} \end{bmatrix} e^{-\pi i(r/D - 1/6)}$ and $\theta\begin{bmatrix} -\frac{1}{6} \\ \frac{1}{2} \end{bmatrix} e^{\pi i/6)}$ have rational Fourier coefficients.

Thus we need to compute the action:

$$f_r^{\begin{pmatrix} t - sb & -sc/a \\ s & t/a \end{pmatrix}_{p|6D}}(\tau)$$

Note that $\begin{pmatrix} t - sb & -sc/a \\ s & ta \end{pmatrix}\begin{pmatrix} t - sb & -sca^* \\ s & ta^* \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/6D^2\mathbb{Z})$ and we can lift it to an element of $\mathrm{SL}_2(\mathbb{Z})$.

Lift from $\mathrm{SL}_2(\mathbb{Z}/6D^2)$ to $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 5.13. *We can always lift a matrix in $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z})$.*

Proof: Take $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, $A, B, C, D \in \mathbb{Z}$. We can further assue $(C, D) = 1$. Let $AD - BC = k \in \mathbb{Z}$. Then we can take:

$A_0 = A + NA_1$
$B_0 = B + NB_1$
$C_0 = C + NC_1$
$D_0 = D + ND_1$
We want to have the condition:
$1 = A_0D_0 - B_0C_0 = AB - CD + N(AD_1 + A_1D - BC_1 - B_1C) + N^2(A_1D_1 - B_1C_1) = 1 + Nk + N(AD_1 + A_1D - BC_1 - B_1C) + N^2(A_1D_1 - B_1C_1)$
For example, pick $D_1 = C_1 = 0$. Then we only need:

$$(A_1D - B_1C) = -k$$

Note that since $(C, D) = 1$, we can find $mC + nD = 1$. Then $(-kn)D - kmC = -k$, thus pick $A_1 = -kn$ and $B_1 = km$.

We look at such a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that:

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \equiv \begin{pmatrix} s - tb & -s\frac{b^2+3}{4}a^* \\ s & t \end{pmatrix} \mod 6D^2$$

Conditions obtained:

- $v_2(s) \geqslant 0$ and $v_2(t) = 0$ imply $b_0, c_0 \equiv 0 \mod 2$, $a_0, d_0 \equiv 1 \mod 2$.
- From the choice $3|b$ we also have $a_0 \equiv d_0 \mod 3$ and $b_0 \equiv 0 \mod 3$. Since we picked $k_0 = t_0 + s_0\omega \equiv s\frac{-b+\sqrt{-3}}{2} + t$ with $s_0 + t_0 \equiv 1 \mod 3$, we must have $t \equiv t_0 + s_0 \mod 3$, thus $d_0 \equiv t_0 \equiv 1 \mod 3$.
- From the choice of $t + s\frac{-b+\sqrt{-3}}{2}$ unit in $\prod_{v|6D} \mathcal{O}_{K_v}^\times$, we have $(t, D) = 1$. Otherwise note that the norm is $t^2 - tsb + s^2\frac{b^2+3}{4}$ is divisible by $p|D$, a contradiction.

We will find the action using the following lemma:

Lemma 5.14. *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *such that* $v_p(d) = 0$ *and* $d \equiv 1 \mod 6$*, we have:*

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \Theta_{\Phi^{(d^{-1}r)}}(z/2)$$

Here by d^{-1} *we mean* $d^{-1} \mod D$*.*

Proof: We compute:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} z\right)$$

Moreover, it equals:

$$\Theta_{\Phi^{(r)}}\left[z/2, \begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}\right]$$

Note that for $p \nmid 6D$ we have $\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}_p$ in $\mathrm{SL}_2(\mathbb{Z}_p)$, thus acts trivially.

For $p|3D$, we have $\Phi_r = \text{char}_{\mathbb{Z}_p - \frac{1}{6} + \frac{r}{D}}$. For now, we will call $\mu_r := -\frac{1}{6} + \frac{r}{D}$.
If $v_p(d) = 0$, we rewrite:

$$\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2c/d & 1 \end{pmatrix} \begin{pmatrix} d & -b/2 \\ 0 & d^{-1} \end{pmatrix}$$

We can further write it in the form:

$$\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2c/d & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b/(2d) \\ 0 & 1 \end{pmatrix}$$

- $r \begin{pmatrix} 1 & -b/(2d) \\ 0 & 1 \end{pmatrix} \Phi_p(x) = e^{-2\pi i Frac_p(-b/(2d)x^2)} \Phi_p(x) = \Phi_p(x)$

- $r \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \Phi_p(x) = |d|_p \chi_p(d) \Phi_p(dx) = \Phi_p^{(d^{-1}r)}(x)$

  Note that $\Phi_p(dx) \neq 0$ iff $dx \in \mathbb{Z}_p + \mu_r$ iff $x \in d^{-1}\mathbb{Z}_p + d^{-1}\mu_r = \mathbb{Z}_p + d^{-1}\mu_r$. Note that $d^{-1}\mu_r = d^{-1}r/D - d^{-1}/6$. Since we picked $d \equiv 1 \mod 6$, this is the same as $\mu_{d^{-1}r}$.

  Note: We need to check that the character corresponding to $\mathbb{Q}$ is trivial on units ($\chi_p(d) = 1$).

- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_p^(d^{-1}r)(x) = e^{2\pi i Frac_p(2d^{-1}xr/D)} \text{char}_{\mathbb{Z}_p + 1/2}(x)$

- $r \begin{pmatrix} 1 & 2c/d \\ 0 & 1 \end{pmatrix} (e^{2\pi i Frac_p(2xd^{-1}r/D)} \text{char}_{\mathbb{Z}_p+1/2}(x)) = e^{2\pi i Frac_p(2c/dx^2)}(e^{2\pi i Frac_p(2xd^{-1}r/D)} \text{char}_{\mathbb{Z}_p}(x)$
  $= (e^{2\pi i Frac_p(2xd^{-1}r/D)} \text{char}_{\mathbb{Z}_p}(x))$

- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (e^{2\pi i Frac_p(2xd^{-1}r/D)} \text{char}_{\mathbb{Z}_p}(x)) = \Phi_p^{(d^{-1}r)}(-x)$

- $r \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_p^{(d^{-1}r)}(-x) = \Phi_p^{(d^{-1}r)}(x)$

In here we have used the Fourier transform:

$$\widehat{\Phi_3^{(r)}}(x) = \int_{\mathbb{Q}_p} \Phi_p^{(r)}(y) e^{-2\pi i Frac_p(2xy)} dy = \int_{\mathbb{Z}_p + \frac{r}{D}} \Phi_p^{(r)}(y) e^{2\pi i 2xy} dy = \int_{\mathbb{Z}_p} e^{-2\pi i Frac_p(2x(y+r/D))} dy$$

$$= \int_{\mathbb{Z}_p} e^{-2\pi i Frac_p(2xy))} e^{-2\pi i Frac_p(xr/D)} dy = e^{-2\pi i Frac_p(2xr/D)} \int_{\mathbb{Z}_p} e^{-2\pi i Frac_p(2xy)} dy$$

$$= e^{-2\pi i Frac_p(2xr/D)} \text{char}_{\mathbb{Z}_p - 1/2}(x) = e^{-2\pi i Frac_p(2xr/D)} \text{char}_{\mathbb{Z}_p}(x)$$

Similarly we get $\widehat{\Phi_3^{(r)}}(x) = e^{-2\pi i Frac_3(x/3)} \text{char}_{\mathbb{Z}_p}(x)$

Note that the only difference for $p = 3$ in the action of $\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}$ is that it does not modify r/D. Instead, it leaves $\Phi^{(r)}$ unchanged.

At the place $p = 2$, we have $\Phi_2 = e^{\pi i Frac_2(x)} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x)$. We can compute:

- $r \begin{pmatrix} 1 & -b/(2d) \\ 0 & 1 \end{pmatrix} \Phi_p(x) = e^{-2\pi i Frac_2(-b/(2d)x^2)} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x) = e^{2\pi i b/8d} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x) \Phi_p(x)$

  Note that we picked $2|b$. Then we have $x \in \mathbb{Z}_2 - 1/2$ iff $x = n - 1/2$ for $n \in \mathbb{Z}_2$. Then $-b/2d(n - 1/2)^2 = -b/(2d)n^2 + b/(2d)n - b/(8d) \in \mathbb{Z}_2 - b/8d$.

- $r \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} e^{2\pi i b/8d} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x) = e^{2\pi i b/8d} e^{\pi i d x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(dx) = e^{2\pi i b/8d} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x)$

  Note that we have used above $v_2(d) = 0$.

- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_2^(r)(x) = e^{2\pi i b/8d} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x) = e^{2\pi i b/8d} e^{2\pi i Frac_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4}(x)$

  Below we compute the Fourier transform:
  $\int_{\mathbb{Q}_2} e^{\pi i Frac_2(y)} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(y) e^{-2\pi i Frac_2(2xy)} dy = \int_{\mathbb{Z}_2 - 1/2} e^{2\pi i Frac_2(y/2 + 2xy)} = \int_{\mathbb{Z}_2} e^{2\pi i Frac_2(y/2 + 1/4 + 2xy + x)} dy = e^{2\pi i Frac_2(x+1/4)} \int_{\mathbb{Z}_2} e^{2\pi i Frac_2(y(1/2+2x))} dy = e^{2\pi i Frac_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4}(x)$

- $r \begin{pmatrix} 1 & 0 \\ 2c/d & 1 \end{pmatrix} e^{2\pi i b/8d} e^{2\pi i Frac_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4} = e^{2\pi i b/8d} e^{2\pi i Frac_2(2c/dx^2)} e^{2\pi i Frac_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4}$

  Note that we have the assumptions $2|c$ and $2 \nmid d$. We have $x + 1/4 = 1/2n, n \in \mathbb{Z}_2$. Note $x^2 = (n - 1/2)/4 = (n^2 - n + 1/4)/4$ and then $e^{2\pi i Frac_2(2c/dx^2)} = e^{2\pi i Frac_2(c/2d(n^2 - n + 1/4))} = e^{2\pi i Frac_2(c/8d)}$. Here we have used the fact that $c/2d(n^2 - n) \in \mathbb{Z}_2$. Thus we get:
  $e^{2\pi i Frac_2((c+b)/8d)} e^{2\pi i Frac_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4}(x)$

- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{2\pi i Frac_2((c+b)/8d)} e^{2\pi i Frac_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4}(x) = e^{2\pi i Frac_2((c+b)/8d)} e^{2\pi i Frac_2(-x)} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(-x)$

- $r \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} e^{2\pi i Frac_2((c+b)/8d)} e^{2\pi i Frac_2(-x)} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(-x) = e^{2\pi i Frac_2((c+b)/8d)} e^{2\pi i Frac_2(x)} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x) = e^{2\pi i Frac_2((c+b)/8d)} \Phi_2(x)$

Finally we are ready to prove the proposition. We have showed so far that:

$$f_r(\tau)^{\sigma_x^{-1}} = f_r^{g_\tau(x)}(\tau) = f_r^{(g_\tau(k_0))_{p|6D}}(\tau) = f^{\begin{pmatrix} t - sb & -sc \\ s & t \end{pmatrix}_{p|6D}}(\tau) = f^{\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}_{p|6D}}(\tau)$$

From the above lemma we get immediately:

$$\left(\frac{\Theta^{(r)}(\tau/2)}{\Theta^{(0)}(\tau/2)} \right)^{\sigma_x} = f_r^{\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}}(\tau) = \frac{\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \tau \right)}{\Theta_{\Phi^{(0)}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \tau \right)} = \frac{\Theta_{\Phi^{(d^{-1}r)}}(\tau/2)}{\Theta_{\Phi^{(0)}}(\tau/2)} = f_{d^{-1}r}(\tau)$$

For $\mathcal{A} \in Cl(\mathcal{O}_{3D})$, $\mathcal{A} = (k_{\mathcal{A}}) = (n_a a + m_a \frac{-b+\sqrt{-3}}{2})$, where $a = \operatorname{Nm} \mathcal{A}$, we take the map:

$$x = (k_{\mathcal{A}})_{p|6D} \leftrightarrow \mathcal{A}$$

This gives us:

$$x^{-1} \leftrightarrow \mathcal{A}^{-1}$$

Then we have:

$$f_r(\tau)^{\sigma_{\mathcal{A}^{-1}}} = f_r(\tau)^{\sigma_{x^{-1}}} = f_r^{g_\tau(x_p)_{p|6D}}(\tau) = f_r^{g_\tau(k_{\mathcal{A}})_{p|6D}}(\tau) = f_{n_{\mathcal{A}}^{-1} r}(\tau)$$

This implies for $r \equiv n_{\mathcal{A}} \mod D$ that we have $f_{n_{\mathcal{A}}}(\tau)^{\sigma_{\mathcal{A}^{-1}}} = f_1(\tau)$, or equivalently:

$$f_1(\tau)^{\sigma_{\mathcal{A}}} = f_{n_{\mathcal{A}}}(\tau)$$

Remark. This implies that for $\mathcal{A}_r = (1 \cdot r + b^*(r-1)\frac{-b+\sqrt{-3}}{2})$ we have:

$$f_1(\tau)^{\sigma_{\mathcal{A}_r}} = f_1(\tau)$$

Also it implies that $\mathfrak{a}_r = (r^{-1}) = (r \cdot r^{-2} + 0\frac{-b+\sqrt{-3}}{2})$ we have:

$$f_1(\tau)^{\sigma_{\mathfrak{a}_r}} = f_r(\tau)$$

5.5 The square is invariant under Galois action

We are finally ready to prove Theorem 5.1.

We define $\mathcal{A}_r^\circ = \left(1 + b^*(1-r)\frac{-b+\sqrt{-3}}{2}\right)$. Note $n_r = r^{-1}$. Note that $\mathcal{A}_r^\circ = \mathcal{A}_r(r^{-1})$, thus $\mathcal{A}_r$ and $\mathcal{A}_r^\circ$ are in the same class in $Cl(\mathcal{O}_{3D})$. This implies:

$$\overline{\chi_D(\mathcal{A}_r)} = \overline{\chi_D(\mathcal{A}_r^\circ)}$$

Moreover, from the definition of χ_D we have: $(D^{2/3})^{\sigma_{\mathcal{A}_r^\circ}} = D^{2/3}\overline{\chi_D(\mathcal{A}_r^\circ)}$
Moreover, from Proposition 5.3:

$$f_1(\tau)^{\sigma_{\mathcal{A}_r{}^\circ}} = f_{n_{\mathcal{A}_r^\circ}}(\tau) = f_r(\tau)$$

Then we can rewrite the term in Proposition 20:

$$\kappa := \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} f_r(\tau) D^{2/3} \overline{\chi(\mathcal{A}_r)} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} f_r(\tau) D^{2/3} \overline{\chi(\mathcal{A}_r^\circ)} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} f_1(\tau)^{\sigma_{\mathcal{A}_r{}^\circ}} (D^{2/3})^{\sigma_{\mathcal{A}_r^\circ}}$$

$$= \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau) D^{2/3})^{\sigma_{\mathcal{A}_r{}^\circ}}$$

We want to write κ as a Galois trace of a modular function at a CM-point. Note that the ideals $\{\mathcal{A}_r^{\circ,(r \in \mathbb{Z}/D\mathbb{Z})^\times}\}$ for a group, as we have $\mathcal{A}_r^\circ \mathcal{A}_s^\circ = \mathcal{A}_{rs}^\circ$. Then take $G_0 = \{r \in (\mathbb{Z}/D\mathbb{Z})^\times :$

$\mathcal{A}_r^\circ\} \cong (\mathbb{Z}/D\mathbb{Z})^\times$ that is a subgroup of $\mathrm{Gal}(H_\mathcal{O}/K)$, where $H_\mathcal{O}$ is the ray class field of conductor $3D$.

We define fixed field of G_0 in H:

$$H_0 = \{h \in H_\mathcal{O} : \sigma(h) = h, \forall \sigma \in G_0\}$$

From abelian Galois theory this implies $\mathrm{Gal}(H_\mathcal{O}/H_0) \cong G_0$. Then we got:

$$\kappa = \mathrm{Tr}_{H_\mathcal{O}/H_0}(f_1(\tau)D^{2/3}) \tag{24}$$

Thus we have proved so far that:

$$S_D = |\kappa|^2,$$

where $\kappa \in H_0$. We claim that actually $|\kappa|^2 \in \mathbb{Q}$. To prove this, it is enough to show that $|\kappa|^2 \in K^\times$, as

Lemma 5.15. *We have $\kappa^3 \in K$.*

Proof: We will show that the Galois conjugates of κ over K are $\kappa\omega$ and $\kappa\omega^2$.
Take $\mathcal{A} \in \mathrm{Cl}(\mathcal{O})$. Then we have:

$$\kappa^{\sigma_\mathcal{A}} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau)D^{2/3})^{\sigma_{\mathcal{A}_r{}^\circ \mathcal{A}}}$$

We can write $\mathcal{A} = \mathcal{A}_s^\circ(m)$. Then we have:

$$\kappa^{\sigma_\mathcal{A}} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau)D^{2/3})^{\sigma_{\mathcal{A}_{rs}{}^\circ(m)}}$$

Note that (m) acts trivially on $D^{2/3}$, but acts as $\mathcal{A}_m^\circ$ on $f_1(\tau)$. Then we have:

$$\kappa^{\sigma_\mathcal{A}} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau))^{\sigma_{\mathcal{A}_{rsm}{}^\circ}} D^{2/3}\overline{\chi(\mathcal{A}_{rs}^\circ)} = \chi(\mathcal{A}_m^\circ) \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau))^{\sigma_{\mathcal{A}_{rsm}{}^\circ}} D^{2/3}\overline{\chi(\mathcal{A}_{rsm}^\circ)} = \chi(\mathcal{A}_m^\circ)\kappa$$

Remark 5.3. Recall that $|\kappa|^2 \in \mathbb{Q}$. Let $\kappa^3 = a + b\sqrt{-3} \in K$. Then $|\kappa|^6 = a^2 + 3b^2$ and we must have $a^2 + 3b^2 = m^3$ for some $m \in \mathbb{Q}$. With this notation we have $|\kappa|^2 = m = \sqrt[3]{a^2 + 3b^2}$.

Remark 5.4. If we try to apply $\overline{\kappa}$, this implies:
$\overline{\kappa}^{\sigma_{\mathcal{A}_r^\circ}} = \overline{(\kappa^{\sigma_{(\mathcal{A}_r^\circ)^{-1}}})} = \overline{\kappa}$
$\overline{\kappa}^{\sigma_{\mathcal{A}(r)}} = \overline{(\kappa^{\sigma_{\mathcal{A}_r}})} = \overline{\chi(\mathcal{A}_m^\circ)\overline{\kappa}}$
$\overline{\kappa}^{\sigma_{\mathcal{A}_r}} = \overline{\kappa\chi(\mathcal{A}_r)} = \overline{\kappa}\overline{\chi(\mathcal{A}_r)}$

6 Appendix A: properties of Θ_K

In this appendix we would like to present a few properties of Θ_K. First, we have a functional equation for the theta function (see [9]):

$$\Theta_K(-1/3z) = \frac{3}{\sqrt{-3}} z\Theta_K(z) \tag{25}$$

Furthermore, we can compute the transformation of $\Theta_K(z \pm 1/3)$ in the lemma below:

Lemma 6.1. *We have the following relations:*

(i) $\Theta\left(z + \frac{1}{3}\right) = (1-\omega)\Theta(3z) + \omega\Theta(z)$

(ii) $\Theta\left(z - \frac{1}{3}\right) = (1-\omega^2)\Theta(3z) + \omega^2\Theta(z)$

Proof: We will rewrite the Fourier expansion of $\Theta(z)$ for $z := z + 1/3$:

$$\Theta\left(z + \frac{1}{3}\right) = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)}.$$

We split the sum in two parts, depending on whether the ideal $(m + n\omega)$ is prime to $(\sqrt{-3})$. Then we have:

$$\Theta\left(z + \frac{1}{3}\right) = \sum_{m,n\in\mathbb{Z},(\sqrt{-3})|(m+n\omega)} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} + \sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)}.$$

Note that on the RHS we can rewrite the first term as:

$$\sum_{m,n\in\mathbb{Z},(\sqrt{-3})|(m+n\omega)} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m^2+n^2-mn)(3z+1)} = \Theta(3z+1) = \Theta(3z)$$

Also note that when $3 \nmid m^2 + n^2 - mn$, then we have $m^2 + n^2 - mn \equiv 1 \mod 3$. Then the second term on the RHS can be rewritten as:

$$\sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} = \sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)z}\omega.$$

We rewrite this:

$$\sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} = \omega \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m^2+n^2-mn)z} - \omega \sum_{m,n\in\mathbb{Z},(\sqrt{-3})|(m+n\omega))} e^{2\pi i(m^2+n^2-mn)z}$$

Finally we recognize the two terms as theta functions Θ_K:

$$\sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} = \omega\Theta(z) - \omega\Theta(3z)$$

Now going back to our initial computation, we get:

$$\Theta\left(z+\frac{1}{3}\right)=\Theta(3z)+\omega\Theta(z)-\omega\Theta(3z)=(1-\omega)\Theta(3z)+\omega\Theta(z)$$

This finishes the proof of the first formula. We get the second formula by applying the first formula for $z := z - 1/3$. We get $\Theta(z) = (1-\omega)\Theta(3z-1) + \omega\Theta(z-1/3)$ and this is easily rewritten to give us the second formula.

6.1 Properties of $\Theta_K((-b+\sqrt{3})/6)$.

Lemma 6.2. $\Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right)=0$

Proof: We apply the functional equation 24 for $z=\frac{-3+\sqrt{-3}}{6}$:

$$\Theta\left(\frac{-3+\sqrt{-3}}{6}\right)=(-\sqrt{-3})\frac{-3+\sqrt{-3}}{6}\Theta\left(\frac{3+\sqrt{-3}}{6}\right).$$

Since $\Theta\left(\frac{-3+\sqrt{-3}}{6}\right)=\Theta\left(\frac{3+\sqrt{-3}}{6}\right)$, we get the result of the lemma.

Lemma 6.3. *For the primitive ideal* $\mathcal{A}=[a,\frac{-b+\sqrt{-3}}{a}]_{\mathbb{Z}}$ *prime to* 3*, where* $a=\operatorname{Nm}\mathcal{A}$*,* $b\equiv 0 \mod 3$ *and* $b^2\equiv -3 \mod 4a$*, we have:*

$$\Theta_K\left(\frac{-b+\sqrt{-3}}{6a}\right)=0$$

Proof: The proof is similar to that of Lemma 3.5. We can write the generator of primitive ideal $\mathcal{A}=\left[a,\frac{-b+\sqrt{-3}}{2}\right]$ in the form $k_{\mathcal{A}}=ma+n\dfrac{-b+\sqrt{-3}}{2}$ for some integers m,n. Note that $(m,3)=1$, thus we can find through the Euclidean algorithm integers A,B such that $mA+3nB=1$, which makes $\begin{pmatrix} A & B\\ -3n & m\end{pmatrix}$ a matrix in $\Gamma_0(3)$. Since Θ is a modular form of weight 1 for $\Gamma_0(3)$, we have:

$$\Theta_K\left(\frac{A\frac{-b+\sqrt{-3}}{6a}+B}{-3n\frac{-b+\sqrt{-3}}{6a}+m}\right)=\left(m-n\frac{-b+\sqrt{-3}}{2a}\right)\Theta_K\left(\frac{-b+\sqrt{-3}}{6a}\right)$$

Noting that $-3n\frac{-b+\sqrt{-3}}{6a}+m=k_{\mathcal{A}}/a=1/\overline{k}_{\mathcal{A}}$, we can compute $\frac{A\frac{-b+\sqrt{-3}}{6a}+B}{-n\frac{-b+\sqrt{-3}}{2a}+m}=\frac{(A\frac{-b+\sqrt{-3}}{2}+3Ba)\overline{k}_{\mathcal{A}}}{3a}$. This is $(3aB+A\frac{-b+\sqrt{-3}}{2})(ma+n\frac{b+\sqrt{-3}}{2})/(3a)$. After expanding, we get:

$$-nA\frac{b^2+3}{4a}+abB/3+\frac{b(-mA+3nB)}{6}+\frac{\sqrt{-3}}{6}$$

Note that $mA+3nB=1$ implies that mA and $3nB$ have different parities. Also we chose b odd, since $b^2+3\equiv 0 \mod 4a$. Finally, recall $3|b$ and thus using the period 1 of Θ_K we get:

$$\Theta_K\left(\frac{A\frac{-b+\sqrt{-3}}{6a}+B}{-3n\frac{-b+\sqrt{-3}}{2a}+m}\right)=\Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right)$$

From the previous Lemma, we have $\Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right)$, thus $\Theta_K\left(\frac{-b+\sqrt{-3}}{6a}\right) = 0$ which finishes the proof.

6.2 About $\Theta_K(D(-3+\sqrt{-3})/6)$.

In this section we will show that for D a product of split primes $p \equiv 1 \mod 3$ and for the representative ideals $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]$ of $\mathrm{Cl}(\mathcal{O}_{3D})$with $b \equiv 0 \mod 3$, we have:

$$\sum_{\mathcal{A}\in \mathrm{Cl}(\mathcal{O}_{3D})} \frac{\Theta\left(\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)} \overline{\chi_D(\mathcal{A})} D^{1/3} = 0$$

We will first show that the LHS is equal to the trace of $\frac{\Theta_K\left(D\frac{-b+\sqrt{-3}}{6}\right)}{\Theta(\omega)} D^{1/3}$ with $b \equiv 0 \mod 3$. We will show this by using Shimura reciprocity law. Note first that:

Lemma 6.4. *The modular function* $f_0(z) = \dfrac{\Theta(Dz/3)}{\Theta(z)}$ *is a modular function for* $\Gamma(3D)$ *and* $f_0(z)$ *has rational Fourier coefficients at the cusp* ∞.

Proof: The proof that f_0 is invariant under $\Gamma(3D)$ is straightforward. The proof that the Fourier coefficients are rational is also similar to the proof of Lemma **??**

Lemma 6.5. *For* f_0 *as above and* $\tau = \frac{-b_0+\sqrt{-3}}{2}$, *we have* $f_0(\tau) \in H_{3D}$.

Proof: To show that $f(\tau) \in H_{3D}$, we need to look at action of $U(3D)$. We follow closely the proof of Lemma **??**. We rewrite the primitive ideal $\mathcal{A} = (A + B\omega)$ as $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$ with $b \equiv b_0 \mod 3$. The only difference is computing:

$$f_0(\left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) z) = \frac{\Theta_K\left(\left(\begin{smallmatrix} D & 0 \\ 0 & 3 \end{smallmatrix}\right)\left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) z\right)}{\Theta_K\left(\left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) z\right)} = \frac{\Theta_K\left(\left(\begin{smallmatrix} ta-sb & -scD/(3a) \\ 3s/D & t \end{smallmatrix}\right)(Dz)\right)}{\Theta_K\left(\left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) z\right)}.$$

Note that we still have $\left(\begin{smallmatrix} ta-sb & -scD/(3a) \\ 3s/D & t \end{smallmatrix}\right), \left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) \in \Gamma_0(3)$, thus we simply get $f_0(z)$ and all the arguments from Lemma **??** follow.

Lemma 6.6. *For* $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]$ *a primitive ideal ideal with* $a = \mathrm{Nm}\,\mathcal{A}$ *and* $b^2 \equiv -3 \mod 4a$, *we have:*

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)} = \left(\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6}\right)}{\Theta(\omega)}\right)^{\sigma_{\mathcal{A}}^{-1}}$$

Proof: Note that $f_0(z)$ satisfies the properties of Lemma **??**, thus applying its result for $f_0\left(\frac{-b+\sqrt{-3}}{2}\right)$ gives us the result.

From the previous two lemmas, we immediately get the following Corollary:

Corollary 6.1. *For* $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]$ *primitive ideals that are representatives of* $\mathrm{Cl}(\mathcal{O}_{3D})$ *as in Lemma* **??**, *we have:*

$$\mathrm{Tr}_{H_{3D}/K} \frac{\Theta(D\frac{-b+\sqrt{-3}}{6})}{\Theta\left(\frac{-b+\sqrt{-3}}{2}\right)} D^{1/3} = \sum_{\mathcal{A}\in \mathrm{Cl}(\mathcal{O}_{3D})} \frac{\Theta\left(\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)} \overline{\chi_D(\mathcal{A})} D^{1/3}$$

6.2.1 Traces of theta functions

We will show the following lemma:

Lemma 6.7. *For* $D \equiv 1 \mod 3$, $b_0 \equiv 0 \mod 3$ *as before, we have:*

$$\mathrm{Tr}_{H_{3D}/K} \frac{\Theta\left(D\frac{-3+\sqrt{-3}}{6}\right)}{\Theta(\omega)} D^{1/3} = \sum_{\mathcal{A}\in \mathrm{Cl}(\mathcal{O}_{3D})} \frac{\Theta_K\left(D\frac{-b_0+\sqrt{-3}}{6a}\right)}{\Theta_K\left(D\frac{-b_0+\sqrt{-3}}{6a}\right)} \overline{\chi_D(\mathcal{A})} D^{1/3} = 0.$$

Proof: The method will be to apply Lemma 6.1 two times. We first apply Lemma **??** (i) for $z = \frac{1-2D}{6D}$ to get:

$$\Theta\left(\frac{1+\sqrt{-3}}{6D}\right) = (1-\omega)\Theta\left(\frac{1+\sqrt{-3}}{2D}\right) + \omega\Theta\left(\frac{1-2D+\sqrt{-3}}{6D}\right)$$

This can be rewritten as:

$$\frac{\Theta\left(\frac{1+\sqrt{-3}}{6D}\right)}{\Theta(\omega)} = (1-\omega)\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(\omega)} + \omega\frac{\Theta\left(\frac{1-2D+\sqrt{-3}}{6D}\right)}{\Theta(\omega)}$$

By taking the inverses and denoting $B_1 := -1+2D$, $a_1 := (B_1^2+3)/4$, we have:

$$3D\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{2}\right)}{\Theta(\omega/3)} = (1-\omega)\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(\omega)} + 3D\omega\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{6a}\right)}$$

Note that $B_1 \equiv 1-2D \equiv 1 \mod 3$. Furthermore, noting that $\Theta(\omega/3) = (1-\omega)\Theta(\omega)$ and $\Theta\left(\frac{B_1+\sqrt{-3}}{6a}\right) = (1-\omega^2)\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)$, we get:

$$\frac{3D}{1-\omega}\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{2}\right)}{\Theta(\omega)} = (1-\omega)\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(\omega)} + \frac{3D\omega}{1-\omega^2}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}$$

Multiplying by $D^{1/3}$ and rewriting the first term on the RHS, we have:

$$\frac{3D}{1-\omega}\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{2}\right)}{\Theta(\omega)} D^{1/3} = (1-\omega)(1-\omega^2)\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{(1-\omega^2)\Theta(\omega)} D^{1/3} + \frac{3D\omega}{1-\omega^2}\chi_D(\mathcal{A}_1)^{-1}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)} D^{1/3}\chi_D(\mathcal{A}_1)$$

By taking the trace from H_{3D} to K and denoting by $\mathcal{A}_1 := \left(\frac{B_1+\sqrt{-3}}{2}\right)$, we have:

$$\frac{3D}{1-\omega}\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\omega\right)}{\Theta(\omega)}D^{1/3} = 3\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(-D\omega^2\right)}{\Theta(-\omega^2/3)}D^{1/3}+\frac{3D\omega}{1-\omega^2}\overline{\chi_D(\mathcal{A}_1)}^{-1}\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)}$$

Note that by definition we have $\chi_D(\mathcal{A}_1) = \chi_D\left(\frac{B_1+\sqrt{-3}}{2}\omega\right)$. We can compute the value of the character using Lemma **??**. For each $\mathfrak{p}|D$, we have:

$$\chi_{\mathfrak{p}}\left(\frac{B_1+\sqrt{-3}}{2}\omega\right) = \left(\frac{(1-2D-\sqrt{-3})\omega^2}{(1-2D+\sqrt{-3}})\omega\right)^{(\operatorname{Nm}\mathfrak{p}-1)/3} = \left(\frac{-1}{1}\right)^{(\operatorname{Nm}\mathfrak{p}-1)/3} = 1.$$

Thus we get $\chi_D(\mathcal{A}_1) = 1$, and we can rewrite the equation above as:

$$\frac{3D}{1-\omega}\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\omega\right)}{\Theta(\omega)}D^{1/3} = 3\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(-D\omega^2\right)}{\Theta(-\omega^2/3)}D^{1/3}+\frac{3D\omega}{1-\omega^2}\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)}.$$

Furthermore, from Lemma **??**, we have $\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)} = \left(\frac{\Theta(D\omega)}{\Theta(\omega)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)}\right)^{\sigma_{\mathcal{A}}^{-1}}$, thus:

$$\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)} = \operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\omega\right)}{\Theta\left(\omega\right)}D^{1/3}$$

Denoting $S := \operatorname{Tr}_{H_{3D}/K}\frac{\Theta(D\omega)}{\Theta(\omega)}D^{1/3}$, we get the relation:

$$\frac{3D}{1-\omega}S = 3\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(-\omega^2/3)}D^{1/3} + \frac{3D\omega}{1-\omega^2}S$$

This implies:

$$\frac{3D}{1-\omega^2}S = 3\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(-\omega^2/3)}D^{1/3}$$

This is equivalent to:

$$\frac{D}{1-\omega^2}S = \operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(-\omega^2/3)}D^{1/3}$$

Note that if we apply the transformation $z \to -1/3z$ given by the functional equation (24) to both theta functions on the RHS we get:

$$\frac{1}{1-\omega^2}S = \frac{1}{3}\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{6}\right)}{\Theta(\omega)}D^{1/3}$$

This is equivalent to:

$$(1-\omega)S = \mathrm{Tr}_{H_{3D}/K}\,\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{6}\right)}{\Theta(\omega)}D^{1/3}. \tag{26}$$

We will apply now Lemma 6.1 (ii) for $z = D\frac{-b_1+\sqrt{-3}}{2a}$, where $b_1 \equiv 1 \mod 3$. We denote by b_0 an integer $b_0 \equiv 0 \mod 3$ such that $b_0 \equiv b_1 \mod 4a$. Then we have:

$$\Theta\left(D\frac{-b_0+\sqrt{-3}}{6a}\right) = (1-\omega^2)\Theta\left(D\frac{-1+\sqrt{-3}}{2a}\right) + \omega^2\Theta\left(D\frac{-b_1+\sqrt{-3}}{6a}\right)$$

This can be rewritten as:

$$\frac{\Theta\left(D\frac{-b_0+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b_0+\sqrt{-3}}{2a}\right)}D^{1/3}\chi_D(\mathcal{A}) = (1-\omega^2)\frac{\Theta\left(D\frac{-b_0+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b_0+\sqrt{-3}}{2a}\right)}D^{1/3}\chi_D(\mathcal{A}) + \omega^2\frac{\Theta\left(D\frac{-b_1+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b_0+\sqrt{-3}}{2a}\right)}D^{1/3}\chi_D(\mathcal{A})$$

By taking the sums, we get:

$$M = (1-\omega^2)S + \omega^2(1-\omega)S = 0$$

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