Integers that can be written as the sum of two rational cubes

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Contents

1 Introduction 2

2 Background. 4
   2.1 The $L$-function . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
   2.2 Ring class fields. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
      2.2.1 Idelic interpretation of ring class field . . . . . . . . . . . . . . . . . . . 5
      2.2.2 Ring class field . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
      2.2.3 Characterization of ideals in ring class fields . . . . . . . . . . . . . . . . 8
   2.3 The cubic character . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8
      2.3.1 Relating $\chi_D$ to the Galois conjugates of $D^{1/3}$ . . . . . . . . . . . . . . 11
   2.4 Hecke characters . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
      2.4.1 Converting the characters. . . . . . . . . . . . . . . . . . . . . . . . . . . . 12

3 Computing the value $L(E_D, 1)$ using Tate’s Zeta function 13
   3.0.2 Schwartz-Bruhat functions. . . . . . . . . . . . . . . . . . . . . . . . . . . . 13
   3.0.3 Haar measure. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
   3.1 Zeta functions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
      3.1.1 Computing the finite part of Tate’s Zeta function $Z_f(s, \chi_D, \Phi)$ . . . . . . . . 15
      3.1.2 Adelic representatives for $\text{Cl}({\mathcal{O}}_{3D})$ . . . . . . . . . . . . . . 17
      3.1.3 Connection to the Eisenstein series . . . . . . . . . . . . . . . . . . . . . . 17
      3.1.4 Fourier expansion of the Eisenstein series $E_\epsilon(s, z)$ at $s = 0$. . . . . . . . 20
      3.1.5 Connection to the theta function $\Theta_K(z)$. . . . . . . . . . . . . . . . . . 22
      3.1.6 Final formula for $L(1, \chi_D)$ . . . . . . . . . . . . . . . . . . . . . . . . 23
      3.1.7 Turning the formula into a trace. . . . . . . . . . . . . . . . . . . . . . . . . 24
      3.1.8 $S_D$ is an integer . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 25

4 Shimura reciprocity law in the classical setting. 26
   4.1 Applying Shimura reciprocity law to $K = \mathbb{Q}[\sqrt{-3}]$. . . . . . . . . . . . . . 29
      4.1.1 $f(\omega)$ is in the ring class field $H_{3D}$. . . . . . . . . . . . . . . . . . . 29
      4.1.2 Galois conjugates of $f(\omega)$. . . . . . . . . . . . . . . . . . . . . . . . . . 31
5 Writing $S_D$ as a square.
5.1 Factorization Formula
5.2 Ratios of $\theta_r$ and $\theta_0$
5.3 Applying the factorization lemma to get a square
  5.3.1 Representatives of $\text{Cl}(O_{3D})$
  5.3.2 Using the factorization formula
5.4 Shimura reciprocity applied to $\theta_r$
  5.4.1 $\theta_r$ as an automorphic form
  5.4.2 Galois action on modular functions (Shimura reciprocity)
5.5 The square is invariant under Galois action

6 Appendix A: properties of $\Theta_K$
6.1 Properties of $\Theta_K((-b + \sqrt{3})/6)$
6.2 About $\Theta_K(D(-3 + \sqrt{3})/6)$
  6.2.1 Traces of theta functions

Abstract

We are interested in finding for which positive integers $D$ we have rational solutions for the equation $x^3 + y^3 = D$. The aim of the paper is to compute the value of the $L$-function $L(E_D, 1)$, for $E_D : x^3 + y^3 = D$. For the case of prime $p \equiv 1 \mod 9$, two formulas have been computed by Rodriguez-Villegas and Zagier in [14]. We compute several formulas using automorphic methods.

1 Introduction

In the current paper we are interested in finding an algorithm to decide which positive integers can be written as the sum of two rational integers cubes:

$$x^3 + y^3 = D, \quad x, y \in \mathbb{Q}$$

After making the equation homogeneous, we get the equation $x^3 + y^3 = Dz^3$ that has a rational point at $\infty = [1, -1, 0]$. Moreover, after a change of coordinates $X = 12Dz^2/(x + y)$,

$$Y = 36D^2x - y \quad x + y$$

the equation becomes

$$E_D : Y^2 = X^3 - 432D^2$$

which defines an elliptic curve over $\mathbb{Q}$.

We will assume $D$ is cube free and $D \neq 1, 2$ (trivial cases) throughout the paper. It is known that $E_D(\mathbb{Q})$ has trivial torsion for $D \neq 1, 2$ (see [16]). Thus, (1) has a solution iff $E_D(\mathbb{Q})$ has positive rank. From the BSD conjecture, this is equivalent to the vanishing of $L(E_D, 1)$.

Without assuming BSD, from the work of Coates-Wiles [2], or more generally Gross-Zagier [6] and Kolyvagin [10], when $L(E_D, 1) \neq 0$, we have $\text{rank } E_D(\mathbb{Q}) = 0$, thus no rational solutions in (1). We define an invariant $S_D$ of $E_D$ as follows:

$$S_D = \frac{L(E_D, 1)}{\Omega_{D,x} \mathcal{R}_{E_D}}.$$
where:

- $\Omega_{D,\infty}$ is the real period
- $R_{E_D}$ is the regulator

The definition is made such that in the case of $L(E_D, s) \neq 0$ we expect to get from the full BSD conjecture:

$$S_D = \#\Sha(E_D) \prod_{p \mid D} c_p,$$

where $\#\Sha$ is the order of the Tate-Shafarevich group and $c_p$ are the Tamagawa numbers corresponding to the elliptic curve $E_D$. From work of Cassels [1], using the Cassels-Tate pairing, we have that when $\Sha$ is finite, its order is going to be a square. Thus we expect that $S_D$ to be an integer square up to the Tamagawa numbers.

For the case of prime numbers, Sylvester conjectured that the answer is affirmative in the case of $p \equiv 4, 7, 8 \mod 9$. In the cases of $p \equiv 2, 3, 5 \mod 9$ we have $L(E_p, 1) \neq 0$ and $p$ is not the sum of two cubes. This follows either from a 3-descent argument (Sylvester, Lucas and Pepin) or from the theorem of Coates-Wiles [2].

In [14], Rodriguez-Villegas and Zagier computed formulas for $L(E_p, 1)$ in the case of primes $p \equiv 1 \mod 9$. In this case it is predicted by BSD that the rank of $E_D(\mathbb{Q})$ is either 0 or 2. They compute two formulas for $S_D$. In the current paper, we are extending their results to all integers $D$. If we let $K = \mathbb{Q}[\sqrt{-3}]$, we have:

**Theorem 1.1.** For all integers $D$, $S_D$ is an integer and we have the formula:

$$S_D = \text{Tr}_{H_{3D}/K} \left( D^{1/3} \frac{\Theta_K(D\omega)}{\Theta_K(\omega)} \right),$$

where:

- $H_{3D}$ is the ring class field associated to the order $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$,
- $\omega = \frac{-1 + \sqrt{-3}}{2}$ is a third root of unity, and
- $\Theta_K(z) = \sum_{a, b \in \mathbb{Z}} e^{2\pi i (a^2 + b^2 - ab)}$ is the theta function associated to the number field $K$.

A second result makes $S_D$ more easily computable. We also hope to extend this result to show that $S_D$ is an integer square up to Tamagawa numbers:

**Theorem 1.2.** In the case of $D = \prod_{p_i \equiv 1 \mod 3} p_i^{e_i}$, $S_D$ is an integer and we have:

$$S_D = \left| \text{Tr}_{H_D/H_0} \frac{\theta_1(z_0)}{\theta_0(z_0)} D^{-1/3} \right|^2$$

where:

- $\theta_1(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n+1/3)^2}z^{2n}$ a 1/2-weight modular form
- $z_0 = \frac{-b + \sqrt{-3}}{2}$ a CM-point, with $b^2 \equiv -3 \mod 4D^2$. 


• $H_\mathcal{O}$ is the ray class field of modulus $3D$ and $H_0$ is an intermediate field $K \subset H_0 \subset H_\mathcal{O}$ that is the fixed field of a certain Galois subgroup $G_0 \cong \text{Cl}(\mathcal{O}_{3D})$.

**Conjecture 1.1.** We conjecture that the term $I_0 = \text{Tr}_{H_\mathcal{O}/H_3D} \frac{\theta_r(z_0)}{\theta_0(z_0/D)} D^{1/3}$ is an integer and $I_0^2$ is the order of the Tate-Shafaravich group.

Using similar methods, we obtain a general formula for all integers $N$, for which Theorem 1.2 is a particular case.

**Theorem 1.3.** Using the same notation as in Theorem 1.2, we have for all integers $D$:

$$S_D = \sum_{r=0}^{D-1} \left| \text{Tr}_{H_\mathcal{O}/H_3D} \frac{\theta_r(Dz_0)}{\theta_0(z_0/D)} D^{-1/3} \right|^2,$$

where:

• $\theta_r(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n+r/D-1/6)^2 z}$ is a $1/2$-weight modular form

• $z_0 = \frac{-b+\sqrt{-3}}{2}$ a CM-point

**Conjecture 1.2.** We conjecture that all terms $I_r = \left| \text{Tr}_{H_\mathcal{O}/H_3D} D^{-1/3} \frac{\theta_r(Dz_0)}{\theta_0(z_0/D)} \right|^2$ are equal for all $r$ such that $(r, D) = 1$. This is indeed the case when $D$ is a product of primes that split.

Further results not included in this draft:

In Appendix A we provide a second proof of Theorem 1.1 based on an idea of Xinyi Yuan. In Appendix B we compute a different formula for $S_D$ inspired by the Rallis inner product. This expresses $S_N$ in the following way:

**Theorem 1.4.** For $D$ a product of primes $p \equiv 1 \mod 4$, we have

$$L(E_D, s) = c_D \sum_{A \in \text{Cl}(\mathcal{O}_{3D})} E_0(s, g_A, \Phi) \chi_D(A)(-1)^{\frac{NmA-1}{2}},$$

where $g_A$ is the embedding of the generator of the ideal $A = (a + b\sqrt{-3})$ into $\text{GL}_2(\mathbb{A}_\mathbb{Q})$ and $E_0(s, g_A, \Phi) = \sum_{m=0 \mod D,(n,D)=1} \frac{1}{(mz+n)[mz+n]^{s}}$ is a sum of Eisenstein series defined by Hecke in [7].

2 Background.

Let $K = \mathbb{Q}[\sqrt{-3}]$. Note that $K$ is a PID and has the ring of integers $\mathcal{O}_K = \mathbb{Z}[\omega]$, where $\omega = \frac{-1+\sqrt{-3}}{2}$ is a fixed root of unity. We will denote $K_v$ the localization of $K$ at the place $v$. We will denote by $K_p := \prod_{v \mid p} K_v \cong \mathbb{Q}_p[\sqrt{-3}]$. 

4
2.1 The L-function

Our goal is to compute several formulas for the special value of the L-function $L(E_D, 1)$ of the elliptic curve $E_D : x^3 + y^3 = Dz^3$. The elliptic curve $E_D$ has complex multiplication (CM) by $O_K$. Then $L(E_D, s)$ is the L-function of a Hecke character that is computed explicitly in Ireland and Rosen [8]. We have:

$$L(E_D, s) = L(s, \chi_D^*),$$

where $\chi_D$ and $\varphi$ are classical Hecke characters such that $\varphi \chi_D$ is the Hecke character corresponding to the elliptic curve $E_D$. The Hecke character $\varphi$ is the Hecke character corresponding to $E_1$ and $\chi_D$ is the Hecke character corresponding to the cubic twist. More precisely, the Hecke characters are defined to be:

- $\varphi : I(3) \to \mathbb{K}^*$ is defined on the ideals prime to 3 by $\varphi(A) = \alpha$, where $\alpha$ is the unique generator of the ideal $A$ such that $\alpha \equiv 1 \mod 3$.

- $\chi_D : I(3D) \to \{1, \omega, \omega^2\}$ is the cubic character defined below in Section 2.3; it is defined on the space $I(3D)$ of all fractional ideals of $O_K$ prime to $3D$. Moreover, it is well-defined over $Cl(O_{3D})$ the ring class group corresponding to the order $O_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$.

The L-function can be expanded:

$$L(E_D, s) = \sum_{A \in I(3D)} \frac{\chi_D(A)\varphi(A)}{(Nm A)^s} = \sum_{\alpha \in O_K, \alpha \equiv 1 \mod 3} \frac{\chi_D(\alpha)\alpha}{Nm(\alpha)^s}. $$

2.2 Ring class fields.

Recall that an order $O$ of $K$ is a subring of $O_K$ that is a finitely generated $\mathbb{Z}$-module and such that $O \otimes_{\mathbb{Z}} \mathbb{Q} = K$. As $K$ is a quadratic number field, each order is of the form $O = \mathbb{Z} + fO_K$ and we call $f = \lvert O_K : O \rvert$ the conductor of $O$. We can also write $O$ using a $\mathbb{Z}$-basis in the form $O = \{1, f\omega\}$.

We define the class group $Cl(O)$ of the order $O$ of conductor $f$ is defined to be:

$$Cl(O) := I_O(f)/P_O(f),$$

where $I_O(f)$ is the set of fractional $O$-ideals prime to the conductor $f$, and $P_O(f)$ the subgroup of $I_O(f)$ of principal fractional $O$-ideals.

We define the ring class field to be the abelian extension $H_O$ of $K$ corresponding to the Galois group $Cl(O)$ from class field theory, meaning:

$$Gal(H_O/K) \cong Cl(O).$$

We denote by $I(N)$ the group of fractional ideals in $K$ prime to $N$. We denote the subgroup $P_{\mathbb{Z},N} = \{(a) : \alpha \in K \text{ such that } \alpha \equiv a \mod N \text{ for some integer } a \text{ such that } gcd(a, N) = 1\}$. Furthermore, let $O_N := \mathbb{Z} + NO_K$ be the order of $K$ of conductor $N$. Then we can define the ring class field of the order $O_N$ to be

$$Cl(O_N) := I(N)/P_{\mathbb{Z},N}.$$
Note that $K$ has class number one and thus by the Strong Approximation theorem we have:

$$\mathbb{A}_K^\times = K^\times \mathbb{C}^\times \prod_{v|\infty} \mathcal{O}_{K_v}^\times.$$ 

We would like to describe $\text{Cl}(\mathcal{O}_N)$ adelically. We do this below:

**Lemma 2.1.** For $N$ a positive integer, we can think of the ring class group adelically as:

$$\text{Cl}(\mathcal{O}_N) \cong U(N)\backslash \mathbb{A}_{K,f}^\times / K^\times,$$

where $U(N) = \prod_p (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$.

**Proof:** From the Strong approximation theorem, as $K$ is a PID, we have:

$$\mathbb{A}_K^\times \cong K^\times \mathbb{C}^\times \prod_{v|\infty} \mathcal{O}_{K_v}^\times.$$ 

Taking the quotient by $K^\times \mathbb{C}^\times$, we get:

$$\mathbb{A}_{K,f}^\times / K^\times \cong \prod_{v|\infty} \mathcal{O}_{K_v}^\times / \left( K^\times \cap \prod_{v} \mathcal{O}_{K_v}^\times \right) \cong \prod_{v|\infty} \mathcal{O}_{K_v}^\times / \langle -\omega \rangle,$$

where $\langle -\omega \rangle$ is the group of sixth roots of unity.

Furthermore, note that $U(N) \cong \prod_{v|N} \mathcal{O}_{K_v}^\times \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$. Moreover note that $(-\omega) U(N) = U(N)$. Thus we have:

$$\mathbb{A}_{K,f}^\times / K^\times U(N) \cong \prod_{v|\infty} \mathcal{O}_{K_v}^\times / \langle -\omega \rangle U(N) \cong \prod_{v|N} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times \cong \prod_{v|N} \prod_{p|N} \mathcal{O}_{K_v}^\times / (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times.$$

Finally, we need to show an isomorphism between $\text{Cl}(\mathcal{O}_N) = I(N)/P(N)$ and $\prod_{p|N} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$. We construct the map:

$$I(N) \to \prod_{v|N} \mathcal{O}_{K_v}^\times \to \prod_{v|p} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times.$$

Let $(k_0) \in I(N)$ be an ideal. Then we can map $k_0 \to (k_0)_{v|N}$. After taking the projection map, we want to look at the kernel of the composition $I(N) \to \prod_{v|p} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$.

This consists of ideals $(k_0) \in I(N)$ such that $k_0 \equiv a_p \mod N\mathbb{Z}_p[\omega]$, where $a_p \in \mathbb{Z}$ and $(a_p, p) = 1$.

By the Chinese remainder theorem, we can find $a \in \mathbb{Z}$ such that $a \equiv a_p \mod N$ for all $p|N$. Then we have $k_0 \equiv a \mod N\mathbb{Z}_p[\omega]$ for all $a \in \mathbb{Z}$. Thus $(k_0) \in P(N)$ and $P(N)$ is the kernel of the above map. Thus we get:

$$I(N)/P(N) \cong \prod_{v|p} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times,$$

which proves our claim.
Another easy result that we will use is the following straight forward application of the Chinese remainder theorem. This map will be important in our proof:

**Lemma 2.2.** For any \((l_{1,v})_{v|N} \in \prod_{v|N} \mathcal{O}_{K_v}^\times\), we can find \(k_1 \in \mathcal{O}_K\) such that for all \(v|N\) we have:

\[
l_{1,v} \equiv k_1 \mod N\mathcal{O}_{K_v},
\]

**Proof:** For any \(v|N\) we can find \(a_{1,v} \in \mathcal{O}_K\) such that \(l_{1,v} \equiv a_{1,v} \mod N\mathcal{O}_{K_v}\). We will pick for \(N = \prod_{v|N} p_v^{e_v}\), where \(p_v\) is the prime corresponding to the place \(v\):

\[
k_1 = \sum_{v|N} a_{1,v} m_v \frac{N}{p_v},
\]

where \(m_v \in \mathcal{O}_K\), \(m_v \frac{N}{p_v} \equiv 1 \mod p_v^{e_v}\). We can find such an inverse since \(\mathcal{O}_K\) is a PID, thus \(\mathcal{O}_K/N\mathcal{O}_K \cong \prod_{v|N} \mathcal{O}_K/p_v^{e_v}\). \(\) \(\)

2.2.1 Characterization of ideals in ring class fields

Recall that a primitive ideal is an ideal not divisible by any integral ideal. It is easy to prove:

**Lemma 2.3.** Any primitive ideal of \(\mathcal{O}_K\) can be be written in the form \(A = \langle a, \frac{b+\sqrt{-3}}{2} \rangle\) as a \(\mathbb{Z}\)-module, where \(b\) is an integer (determined modulo 2a) such that \(b^2 \equiv -3 \mod 4a\) and \(Nm A = a\). This implies that for \(A = \langle \alpha \rangle\), we have \(\|\alpha\| = a\).

Conversely, given an integer satisfying the above congruence and \(A\) defined as above, we get that \(A\) is an ideal in \(\mathcal{O}_K\) of norm \(a\).

2.3 The cubic character

In the following we will define the cubic character \(\chi_D\) and check that it is well defined on the class group \(\text{Cl}(\mathcal{O}_D)\). Let \(\omega = \frac{-1+\sqrt{-3}}{2}\) be a fixed cube root of unity. Then we can define the cubic residue character following Ireland and Rosen [8].

**Definition 2.1.** For \(\alpha \in \mathbb{Z}[\omega]\) such that \(\alpha\) is prime to 3, we define a cubic residue character \(\chi_\alpha : I(3\alpha) \to \{1, \omega, \omega^2\}\) on the fractional ideals of \(K\) prime to \(3\alpha\). For every prime ideal \(p\) of \(\mathbb{Z}[\omega]\), the character is defined to be:

\[
\chi_\alpha(p) = \omega^j,
\]

for \(j \in \{0, 1, 2\}\) such that \(\omega^j\) is the unique third root of unity for which:

\[
\alpha^{(Nm p-1)/3} \equiv \omega^j \mod p, \text{ for } Nmp \neq 3.
\]

It is further defined multiplicatively on the fractional ideals of \(I(3\alpha)\).

**Notation:** We will also denote \(\chi_D(\cdot) = \langle \frac{\mathcal{D}}{\cdot} \rangle_3\).

First let us check that this definition makes sense. Since \(K\) is a PID, any prime ideal \(p\) has a generator of the form \(\pi = a + b\omega \in \mathbb{Z}_p[\omega]\). Then the norm \(Np = a^2 - ab + b^2\) which is congruent to 0, 1 \mod 3. Then, if \(p\) is prime to 3, we must have \(Np \equiv 1 \mod 3\), implying that 3 divides \(Np - 1\).
Furthermore, the group \( (\mathbb{Z}[\omega]/p\mathbb{Z}[\omega])^\times \) has \( Nmp - 1 \) elements, thus we have \( \alpha^{Nm p^{-1}} \equiv 1 \mod p \). Then since \( Nmp - 1 \) is divisible by 3, we can factor out:

\[
p|\left(\alpha^{(Nm p^{-1})/3} - 1\right)\left(\alpha^{(Nm p^{-1})/3} - \omega\right)\left(\alpha^{(Nm p^{-1})/3} - \omega^2\right)
\]

Finally since \( K = \mathbb{Q}[\sqrt{-3}] \) is an UFD, \( p \) divides exactly one of these terms, say \( \alpha^{(Nm p^{-1})/3} - \omega \). Thus we can take \( \chi_\alpha(p) = \omega^i \) and it is well-defined.

Following Ireland and Rosen, it is natural to look at the primary elements of \( K \):

**Definition 2.2.** For a prime ideal \( p \) of \( K \) we call \( \pi \) primary if \( \pi \) generates \( p \) a prime ideal and \( \pi \equiv 2 \mod 3 \).

**Lemma 2.4.** For any ideal \( A \) prime to 3, we can find a generator \( \alpha \in \mathbb{Z}[\omega] \) such that \( \alpha \equiv 2 \mod 3 \).

**Proof:** Since \( K \) is a PID, we can find a generator \( \alpha_0 = a + b\omega \) be a generator of \( A \). Then note that \( \pm\alpha_0, \pm\alpha_0\omega, \pm\alpha_0\omega^2 \) also generate the ideal \( A \) and exactly one of them is \( \equiv 2 \mod 3 \).

**Remark 2.1.** Note that from the definition of \( \chi_\pi \) we have \( \chi_{\pi_1}(\pi_2) = \chi_{-\pi_1}(\pi_2) \), as \( \pi_1^{(Nm \pi_2 - 1)/3} = (\pi_2)^{(Nm \pi_2 - 1)/3} \) when \( Nm \pi_2 \) is odd and \( \pi_1^{(Nm 2 - 1)/3} = (\pi_1)^{(Nm 2 - 1)/3} \equiv 1 \mod 2 \) when \( \pi_2 = 2 \). Moreover \( \chi_{\pi_1}(\pi_2) = \chi_{\pi_1}(\pi_2) \), as \( \chi_{\pi_1}(1) = 1 \). Then we actually have for any choices of \( \pm \):

\[
\chi_{\pi_1}(\pi_2) = \chi_{\pi_2}(\pi_1)
\]

**Theorem 2.1.** (Cubic reciprocity law). For \( \pi_1, \pi_2 \equiv 2 \mod 3 \) primary generators of primes \( p_1, p_2, N\pi_1 \neq N\pi_2 \) and \( N\pi_1, N\pi_2 \neq 3 \), then:

\[
\left(\frac{\pi_1}{\pi_2}\right)_3 = \left(\frac{\pi_2}{\pi_1}\right)_3
\]

**Corollary 2.1.** For \( \pi_i, \pi_i' \equiv 2 \mod 3 \), we have

\[
\chi_{\pi_1 \ldots \pi_n}(\pm \pi_1' \ldots \pi_n') = \chi_{\pi_1 \ldots \pi_n}(\pm \pi_1 \ldots \pi_n)
\]

**Proof:** We will first show that \( \chi_{\pi_1 \ldots \pi_n}(\pi_i') = \chi_{\pi}(\pi_1 \ldots \pi_n) \). By definition, we have:

\[
\chi_{\pi_1 \ldots \pi_n}(\pi_i') \equiv (\pi_1 \ldots \pi_n)^{(Nm \pi_i - 1)/3} \mod \pi_i'
\]

Thus, we have:

\[
\chi_{\pi_1 \ldots \pi_n}(\prod_{i=1}^{m} \pi_i') = \prod_{i=1}^{m} \chi_{\pi_1 \ldots \pi_n}(\pi_i') = \prod_{i=1}^{m} \prod_{j=1}^{n} \chi_{\pi_j}(\pi_i')
\]

Using the cubic reciprocity, we have \( \chi_{\pi_i}(\pi_i') = \chi_{\pi_i'}(\pi_i) \), thus we get \( \prod_{i=1}^{m} \prod_{j=1}^{n} \chi_{\pi_j}(\pi_i') = \prod_{i=1}^{m} \prod_{j=1}^{n} \chi_{\pi_i'}(\pi_j) \), which furthermore implies:

\[
\chi_{\pi_1 \ldots \pi_n}(\prod_{i=1}^{m} \pi_i') = \chi_{\pi_i' \ldots \pi_m}(\prod_{j=1}^{n} \pi_i).
\]

Note that we can always write the elements of \( \mathbb{Z}[\omega] \) that are congruent to \( \pm 1 \mod 3 \) as a product of primary elements up to sign. Using the above corollary for \( \alpha \) and \( D \), we get:
Corollary 2.2. If \( \alpha \equiv \pm 1 \mod 3 \) and \( D \) an integer prime to 3, then we have:

\[
\chi_D(\alpha) = \chi_\alpha(D)
\]

Proof: Since \( \alpha, D \equiv \pm 1 \mod 3 \), we can write each of them in the form \( \alpha = \pm \prod_{i=1}^{n} \pi_i \) and \( D = \pm \prod_{j=1}^{m} \pi_j \).

Then using the previous Corollary and Remark 2.1, we have
\[
\chi_{\pm \prod_{i=1}^{n} \pi_i} (\pm \prod_{j=1}^{m} \pi_j) = \chi_{\pm \prod_{i=1}^{n} \pi_i} (\pm \prod_{i=1}^{n} \pi_i).
\]

Lemma 2.5. Let \( \alpha \) be prime to 3 and \( p \) a prime ideal prime to 3. Then the cubic residue can also be rewritten as:

\[
\chi_\alpha(p) = \left(\frac{\alpha}{\pi}\right)^{(N\pi - 1)/3} \mod \pi
\]

Proof: We have by definition \( \chi_\alpha(p) \equiv \alpha^{(N\pi - 1)/3} \equiv \omega^i \mod p \). Taking the complex conjugate we have \( \overline{\alpha^{(N\pi - 1)/3}} \equiv \overline{\omega^i} \mod p \). Then by taking the ratio we get:

\[
\left(\frac{\alpha}{\pi}\right)^{(N\pi - 1)/3} \equiv \frac{\overline{\omega^i}}{\omega^i} \mod p
\]

Thus we have \( \chi_\alpha(p) \equiv \alpha^{(N\pi - 1)/3} \equiv \omega^i \equiv \left(\frac{\overline{\alpha}}{\alpha}\right)^{(N\pi - 1)/3} \mod p \) which finishes the proof of the lemma.

Corollary 2.3. Let \( D = \prod_{i=1}^{m} p_i \). For \( \alpha \in \mathcal{P}_{\mathbb{Z}, 3D} \), we have \( \chi_D(\alpha) = 1 \). Thus \( \chi_D \) is well defined on \( \text{Cl}(\mathcal{O}_{3D}) \).

Proof: Recall from the previous Lemma that if \( \alpha \equiv \pm 1 \mod 3 \), then we have:

\[
\chi_\alpha(p) \equiv \left(\frac{\alpha}{\pi}\right)^{(N\pi - 1)/3} \mod p
\]

Let \( p | D \). Since \( \alpha \in \mathcal{P}_{\mathbb{Z}, 3D} \), we have \( \alpha \equiv a \mod 3D \) for some \( a \in \mathbb{Z} \) and \( (a, 3D) = 1 \). Thus \( \alpha \equiv a \mod p \), which also \( \alpha \equiv a \mod p \), which implies:

\[
\chi_\alpha(p) \equiv \left(\frac{\alpha}{a}\right)^{(N\pi - 1)/3} \equiv \left(\frac{\overline{\alpha}}{a}\right)^{(N\pi - 1)/3} \equiv 1 \mod p
\]

Thus we get \( \chi_\alpha(p) = 1 \) for all \( p | D \). Thus we have \( \chi_D(\alpha) = 1 \). Moreover, using Corollary 2.2, we have \( \chi_D(\alpha) = \prod_{i=1}^{m} \chi_{p_i}(\alpha) = \prod_{i=1}^{m} \chi_\alpha(p_i) = 1 \).

Remark 2.2. For any fractional ideal \( \mathcal{A} \) of \( K \), when we write \( \chi_D(\mathcal{A}) \) we will mean:

\[
\chi_D(\mathcal{A}) := \chi_D(\alpha),
\]

where \( \alpha \) is the unique generator of \( \mathcal{A} \) such that \( \alpha \equiv 1 \mod 3 \).
2.3.1 Relating $\chi_D$ to the Galois conjugates of $D^{1/3}$.

There is another way to look at the cubic character using the Galois conjugates of $D^{1/3}$. We have the following lemma:

**Lemma 2.6.** Let $D$ be an integer prime to 3. Then for a prime ideal $p$ of $K$ prime to $3D$, we have:

$$D^{1/3}\chi_D(p) = (D^{1/3})^{\sigma_p},$$

where $\sigma_p \in \text{Gal}(\mathbb{C}/K)$ is the Galois action corresponding to the ideal $p$ in the Artin correspondence.

**Proof:** It is enough to prove the claim for $\sigma_i \in \text{Gal}(F/K)$, where $L = K[D^{1/3}, D^{1/3}\omega, D^{1/3}\omega^2]$. Let $\sigma_p = \left(\frac{L/K}{p}\right)$ the Frobenius element corresponds to $p$ the prime ideal of $\mathcal{O}_K$. Then using the definition of the Frobenius element for $D^{1/3}$, we get:

$$(D^{1/3})^{\sigma_p} = (D^{1/3})^{\text{Nm}_p} \mod p\mathcal{O}_L$$

Furthermore, note that $(D^{1/3})^{\text{Nm}_p} = D^{1/3}D^{(\text{Nm}_p-1)/3} = D^{1/3}\chi_D(p) \mod p\mathcal{O}_L$. Since the Galois conjugates of $D^{1/3}$ are the roots of $x^3 - D$, the Galois conjugate $(D^{1/3})^{\sigma_p} \in \{D^{1/3}, D^{1/3}\omega, D^{1/3}\omega^2\}$ and from the congruences above we get:

$$(D^{1/3})^{\sigma_p} = D^{1/3}\chi_D(p)$$

**Corollary 2.4.** Let $D$ be an integer prime to 3 and $A$ an ideal of $K$ prime to $3D$. Moreover, let $\sigma_A \in \text{Gal}(K^{ab}/K)$ be the Galois action corresponding to the ideal $A$ through the Artin map. Then for the cubic character $\chi_D$, we have:

$$(D^{1/3})^{\sigma_A} = D^{1/3}\chi_D(A).$$

**Proof:** Let $A = \prod_j p_j^{f_j}$ the prime decomposition of $A$ in $K$. Note that $\chi_D(p_i) \in K$, thus it is preserved by the Galois action. Applying the above Lemma we get:

$$((D^{1/3})^{\sigma_{p_i}})^{\sigma_{p_j}} = (D^{1/3}\chi_D(p_i))^{\sigma_{p_j}} = D^{1/3}\chi_D(p_j)^{\sigma(p_j)}$$

Using this step repeatedly, we get $(D^{1/3})^{\sigma_A} = D^{1/3}\chi_D(A) = D^{1/3}\chi_D(A)$.

**Remark 2.3.** Note that for the complex conjugate character $\overline{\chi_D}$ we have a similar result:

$$(D^{2/3})^{\sigma_A} = D^{2/3}\overline{\chi_D(A)}.$$ (7)

2.4 Hecke characters

There are two equivalent ways of defining a Hecke character: classically and adelically. We define the **classical Hecke character** over $K$ to be $\tilde{\chi} : I(f) \to \mathbb{C}^\times$ a character from the set of fractional ideals prime to $f$, where $f$ is a nonzero ideal of $\mathcal{O}_K$. We further say that $\tilde{\chi}$ has
infinity type \( \tilde{\chi}_\infty \) if it is characterized by the condition that on the set of principal ideals \( P(f) \) prime to \( f \) it satisfies the condition:

\[
\tilde{\chi}((\alpha)) = \bar{\epsilon}(\alpha)\tilde{\chi}_\infty^{-1}(\alpha),
\]

where:

- \( \bar{\epsilon} : (\mathcal{O}_K/f\mathcal{O}_K)^\times \to \mathbb{T} \) is called the \((\mathcal{O}_K/f\mathcal{O}_K)^\times\)-type character i.e. \( \bar{\epsilon} \) is a character taking values in a finite group \( \mathbb{T} \).
- \( \tilde{\chi}_\infty \) is an infinity type continuous character i.e. \( \tilde{\chi}_\infty : \mathbb{C}^\times \to \mathbb{C}^\times \) is a continuous character.

We define the **idelic Hecke character** to be a continuous character \( \chi : \mathbb{A}^\times /K^\times \to \mathbb{C}^\times \).

There is a unique correspondence between the idelic and the classical Hecke characters. The correspondence can be explicitly constructed in the following way:

- \( \tilde{\chi}(\mathcal{O}_K^\times \omega_v) := \chi(p_v), v \nmid f \)
- \( \tilde{\chi}_\infty \) is determined by \( \chi_\infty \)
- \( \tilde{\chi}_v \) with \( v \mid f \) is determined by Weak Approximation Theorem.

### 2.4.1 Converting the characters.

We want to compute a formula for \( L(s,\chi) \), where \( \chi : \mathbb{A}_K^\times /K^\times \to \mathbb{C}^\times \) is the Hecke character defined by \( \chi = \chi_3D\varphi \). Here \( \chi_3D\varphi \) are the adelic correspondent Hecke characters of the classical Hecke characters:

1. \( \chi_3D : I(3D) \to \{1,\omega,\omega^2\} \) is the cubic character.

2. \( \varphi : I(3) \to \mathbb{C}^\times \) is the Hecke character defined by \( \chi((\alpha)) = \alpha \) for \( \alpha \equiv 1 \mod 3 \).

By abuse of notation, I will use \( \varphi, \chi_3D \) both for the classical and the adelic Hecke characters. This should be clear from the context. We can rewrite the two characters adelically:

1. \( \varphi : \mathbb{A}_K^\times \to \mathbb{C}^\times \) such that:

\[
\begin{aligned}
\varphi_v(p) &= -p, \text{ if } v|p, p \equiv 2 \mod 3, \\
\varphi_v(\mathcal{O}_K^\times) &= 1, \text{ if } v|p, p \equiv 1 \mod 3, \\
\varphi_v(\omega_v) &= \omega_v, \text{ where } \omega_v \text{ uniformizer of } \mathcal{O}_{K_v}, \omega_v \equiv 1 \mod 3, \\
\varphi_{\infty}(x_{\infty}) &= x_{\infty}^{-1}, \text{ if } v = \infty, \\
\varphi_v(\omega_v) \text{ can be determined from the Weak approximation theorem, } & \text{ if } v = \sqrt{-3}\mathbb{Z}
\end{aligned}
\]

2. Note that \( \chi_3D \) is trivial on \( P_{2,3D} \), thus \( \chi_3D \) is a character on \( Cl(\mathcal{O}_{3D}) \). We will define the character by making it trivial on \( \mathbb{C}^\times, U(3D) \) and \( K^\times \). Then we can define using Lemma 2.2:

\[
\chi_3D(l) = \chi_3D(l_1) = \chi_3D((k_1)).
\]
More precisely, this will be:

\[
\begin{align*}
\chi_{3D,v}(\overline{w}_v) &= \chi_{3D}(\omega_v), \quad \chi_{3D,v}(O_{K_v}^\times) = 1, \\
\chi_{3D,\infty}(\infty) &= 1, \\
\chi_{3D,v}(\overline{w}_v) \text{ can be determined from the Weak approximation theorem}, & \text{ if } v \mid 3D
\end{align*}
\]

We can generally compute \(\chi_f(l_f)\) in the following way:

**Lemma 2.7.** If \(\chi = \chi_{3D}\varphi\), let \(l_f = kl_1\), \(k \in K^\times, l_1 \in \prod_v O_{K_v}^\times\). Note that this decomposition is unique up to a unit of \(O_{K_v}^\times\) and pick \(k\) such that \(l_1 \equiv 1 \mod 3\). Moreover take \(k_1 \in K^\times\) such that \(l_1 \equiv k_1 \mod 3O_{K_v}\). Then:

\[
\chi_f(l_f) = k\chi_{3D}((k_1))
\]

**Proof:** We start by writing:

\[
\chi_f(l_f) = \chi_f(k)\chi_f(l_1) = \chi_{\varphi}(k)^{-1}\chi_{v|3D}(l_1, v)
\]

Moreover, from the Chinese remainder theorem, we can find \(k_1 \in K^\times\) such that \(k_1 \equiv l_1, v \mod 3O_{K_v}\). As we have \(k_1^{-1}l_1 \equiv 1 \mod 3O_{K_v}\) and \(\chi\) is trivial on \((\mathbb{Z} + 3O_{K_v})^\times\) for \(v \mid 3D\), we get \(\chi_v(k_1) = \chi(l_1, v)\). This implies:

\[
\chi_f(l_f) = k\chi_{v|3D}(k_1) = k\chi_{v|3D}(k_1)^{-1}\chi_{\varphi}(k_1)^{-1}
\]

Note that if we write \(k_1 = u \prod_v \omega_v^{r_v}\), where \(u \in O_{K_v}^\times\), we get:

\[
\prod_{v|3D} \chi_v(k_1) = \prod_{v|3D} \chi_v(\omega_v)^{r_v} = \prod_{v|3D} \chi(p_v)^{r_v} = \tilde{\chi}((k_1))
\]

This moreover implies:

\[
\chi_f(l_f) = k\tilde{\chi}((k_1))^{-1}k_1 = kk_1^{-1}k_1\chi_{3D}((k_1)) = k\chi_{3D}((k_1))
\]

### 3 Computing the value \(L(E_D, 1)\) using Tate’s Zeta function

In this section we will compute the value of \(L(E_D, 1) = L(1, \chi_D\varphi)\), working with \(\chi_D, \varphi\) as automorphic Hecke characters. We will show the following result:

**3.0.2 Schwartz-Bruhat functions.**

We take \(V = K\) a quadratic vector space over \(\mathbb{Q}\) and \(V_{\mathbb{Q}} = \mathbb{A}_\mathbb{Q} \otimes \mathbb{Q} K\). Then we can define the Schwartz-Bruhat functions \(\Phi = \prod_v \Phi_v, \Phi_v \in S(V_{\mathbb{Q}})\) to be:
Here \( q(z) = |z|^2 \) the usual absolute value on \( \mathbb{C} \).

**Remark 3.1.** \( \operatorname{char}_{(a+p\mathcal{O}_{K_v})}(m) = \prod_{v|p} \operatorname{char}_{(a+D\mathcal{O}_{K_v})}(m) = \prod_{v|p} \operatorname{char}_{(1+p\mathcal{O}_{K_v})}(a^{-1}m) \) and each \( \operatorname{char}_{(b\mathcal{O}_{K_v})} \) is a locally constant function with compact support. We are taking a linear combination of these Schwartz-Bruhat functions, thus we do get a Schwartz-Bruhat function.

### 3.0.3 Haar measure.

We pick the usual additive Haar measure:

\[
\begin{align*}
  &d^\times x_v = \frac{dx_v}{|x_v|}, \text{ normalized such that vol}(\mathcal{O}^\times_{K_v}) = 1, \quad \text{if } v \nmid \infty \smallsetminus 3D
  \\
  &d^\times z = \frac{dz}{|z|}, \text{ usual Lebesgue measure}, \quad z \in \mathbb{C}, |z|_{\mathbb{C}} = x^2 + y^2, \text{ for } z = x + yi
\end{align*}
\]

We also define the multiplicative Haar measure:

\[
\text{XXX}
\]

### 3.1 Zeta functions

We recall Tate’s zeta function. For a Hecke character \( \chi_v : K_v^\times \to \mathbb{C}^\times \) and a Schwartz-Bruhat function \( \Phi_v \in \mathcal{S}(K_v) \), it is defined locally to be:

\[
Z_v(s, \chi_v, \Phi_v) = \int_{K_v^\times} \chi_v(\alpha_v)|\alpha_v|^s \Phi_v(\alpha_v)d^\times \alpha_v,
\]

where \( d^\times \alpha_v \) is the multiplicative Haar measure defined above.

We define globally \( Z(s, \chi, \Phi) = \prod_v Z_v(s, \chi_v, \Phi_v) \). As a global integral, this is:

\[
Z(s, \chi, \Phi) = \int_{\mathcal{A}_K^\times} \chi(\alpha)|\alpha|^s \Phi(\alpha)d^\times \alpha,
\]

**Lemma 3.1.** For all \( s \) and \( \Phi \) Schwartz-Bruhat functions chosen as above, we have:

\[
L_f(s, \chi_D \varphi) = Z_f(s, \chi_D \varphi, \Phi)V_{3D},
\]

where \( V_{3D} = \operatorname{vol}(1 + 3\mathbb{Z}_3[\omega]) \operatorname{vol}(\mathbb{Z} + D \prod_{p|D} \mathbb{Z}_p[\omega])^\times = \frac{1}{6} \prod_{p|D} \left( \frac{1}{(p - (\frac{3}{2}))} \right) \)

**Proof:** From Tate’s thesis, we have \( L_f(s, \chi_D \varphi) = Z_f(s, \chi_D \varphi) \prod_{p|3D} L_p(s, \chi_D \varphi_p) \).

Since \( \chi_D \varphi \) is ramified at \( 3D \), we have \( L_p(s, \chi_D \varphi_p) = 1 \). We need to compute:
\[ Z_p(s, \chi_D \varphi, \Phi_p) = \int_{\mathbb{Q}^*_p} \chi_D, p(\alpha_p) \varphi_p(\alpha_p) |\alpha_p|_p^s \Phi_p(\alpha_p) d^x \alpha_p \]

From the definition for \( p|D \), we have \( \Phi_p = \text{char}_{\mathbb{Z}+3D\mathbb{Z}_p[\omega]} \), the integral reduces to \( Z_p(s, \chi_D \varphi, \Phi_p) = \int_{\mathbb{Q}^*_p} \chi_D, p(\alpha_p) \varphi_p(\alpha_p) |\alpha_p|_p^s d^x \alpha_p \). Note that for \( p \neq 3 \), all the characters \( \chi_D, \varphi \) and \( \left| \cdot \right|_p \)

are unramified, thus we just get the volume \( \text{vol} \left( \mathbb{Z}+3D\mathbb{Z}_p[\omega] \right)^x \).

For \( p = 3 \), we have \( \Phi_p = \text{char}_{\mathbb{Z}+3\mathbb{Z}_3[\omega]} \). Similarly, we get \( \text{vol} \left( (1+3\mathbb{Z}_3[\omega])^x \right) \).

We compute the volumes. For \( D \) a product of primes, we have

\[ \text{vol} \left( (\mathbb{Z}+3D\mathbb{Z}_p[\omega])^x \right) = \text{vol} \left( (\mathbb{Z}+p\mathbb{Z}_p[\omega])^x \right) = (p-1) \text{vol} \left( 1+p\mathbb{Z}_p[\omega] \right) = \frac{1}{(p-1)^2} \text{vol}(\mathbb{Z}_p^x) \]

Note that \( \text{vol} \left( 1+p\mathbb{Z}_p[\omega] \right) = \frac{1}{(p-1)^2} \text{vol}(\mathbb{Z}_p^x) \) when \( p \) is nonsplit and \( \text{vol} \left( 1+p\mathbb{Z}_p[\omega] \right) = \frac{1}{(p-1)^2} \text{vol}(\mathbb{Z}_p^x) \) when \( p \) is split. This is computed by writing:

- \( p \) nonsplit: \( \text{vol}(\mathbb{Z}_p[\omega]^x) = \sum \text{vol}(a+b\omega+p\mathbb{Z}_p[\omega]) \), where the sum is taken over all \( a+b\omega \) prime to \( p \) and \( 0 \leq a, b \leq p-1 \). We count \( p^2-1 \) of them and we get \( \text{vol}(\mathbb{Z}_p[\omega]^x) = (p^2-1) \text{vol}(1+p\mathbb{Z}_p[\omega]) \).

- \( p \) split: \( \text{vol}(\mathbb{Z}_p[\omega]^x) = \sum \text{vol}(a+b\omega+p\mathbb{Z}_p[\omega]) \). We count similarly \( p^2-2p+1 \) such terms, as \( p \) splits and we have to discard the divisors of \( p \).

For \( p = 3 \), we have \( \text{vol} \left( 1+3\mathbb{Z}_3[\omega] \right) = \frac{1}{3} \).

We compute:

- \( \mathbb{Z}_3[\omega] = \mathbb{Z}_3[\sqrt[3]{-3}] = \{ a_0 + a_1 \sqrt[3]{-3} + a_2(-3) + \ldots, 0 \leq a_i \leq 2 \} \)
- \( \text{vol}(\mathbb{Z}_3[\omega])^x = 1 \)
- \( (\mathbb{Z}_3[\omega])^x = \bigcup (a_0 + a_1 \sqrt[3]{-3})(1+3\mathbb{Z}_3[\omega]), \) where \( a_0 + a_1 \sqrt[3]{-3} \) is prime to 3. Then we have 6 possibilities and thus \( \text{vol}(1+3\mathbb{Z}_3[\omega]) = \frac{1}{6} \).

By plugging in \( s = 1 \) in the above Lemma, we get:

**Corollary 3.1.** The finite part of the L-function at \( s = 1 \) equals:

\[ L_f(1, \chi_D \varphi) = \frac{1}{6} \prod_{p|D} \frac{1}{(p-\left(\frac{3}{p}\right))} Z_f(1, \chi_D \varphi, \Phi), \]

### 3.1.1 Computing the finite part of Tate’s Zeta function \( Z_f(s, \chi_D \varphi, \Phi) \)

In this section we will compute the value of \( Z_f(s, \chi_D \varphi, \Phi) \). We begin by rewriting Tate’s zeta function \( Z_f(s, \chi_D \varphi, \Phi) \) as a linear combination of Hecke characters:
Lemma 3.2. For all \( s \in \mathbb{C} \) and the Schwartz-Bruhat functions \( \Phi_f \in \mathcal{S}(\mathbb{A}_{K,f}) \), we have:

\[
Z_f(s, \chi_D \varphi, \Phi_f) = V_{3D} \sum_{\alpha_f \in U(3D) \setminus \mathbb{A}_{K,f}^*} I(s, \alpha_f, \Phi_f) \chi_D(\alpha) \varphi(\alpha),
\]

where \( I(s, \alpha_f, \Phi_f) = \sum_{k \in K^*} \frac{k}{|k|_\mathbb{C}} \Phi_f(k \alpha_f) \).

Proof: By definition, we have \( Z_f(s, \chi_D \varphi, \Phi_f) = \int_{\mathbb{A}_{K,f}^*} \chi_D(\alpha_f) \varphi(\alpha_f) |\alpha_f|^s \Phi_f(\alpha_f) d^s \alpha_f \). We rewrite the integral by taking a quotient by \( K^* \):

\[
Z_f(s, \chi_D \varphi, \Phi_f) = \int_{\mathbb{A}_{K,f}^* / K^*} \sum_{k \in K^*} \chi_D(\alpha_f) \varphi(\alpha_f) |k \alpha_f|^s \Phi_f(k \alpha_f) d^s \alpha_f
\]

Note that from the definition of Hecke characters, we have \( \chi_D(f(k \alpha_f')) = \chi_{D, \infty}(k) \chi_D(\alpha_f') = \chi_D(f(\alpha_f')) \), \( \varphi_f(k \alpha_f') = \varphi_{\infty}(k) \varphi_f(\alpha_f') = k \varphi_f(\alpha_f') \) and \( |k \alpha_f'|^s = |k|_{\infty}^s |\alpha_f'|^s = |k|_\mathbb{C}^{-2s} |\alpha_f'|^s \), where \(| \cdot |_\mathbb{C} \) is the usual absolute value over \( \mathbb{C} \). Then the integral reduces to:

\[
Z_f(s, \chi_D \varphi, \Phi_f) = \int_{\mathbb{A}_{K,f}^* / K^*} \left( \sum_{k \in K^*} \frac{k}{|k|_\mathbb{C}^2} \chi_D(\alpha_f') \Phi_f(k \alpha_f') \right) \varphi_f(\alpha_f') |\alpha_f'|^s d^s \alpha_f
\]

Furthermore, note that our choice of Schwartz-Bruhat functions \( \Phi_f(k \alpha_f') \) are invariant over \( U(3D) \). Similarly:

- \(| \cdot |_f \) is trivial on units, thus on \( U(3D) \)
- \( \chi_D \) is invariant on \( U(3D) \) by definition
- \( \varphi \) is trivial on all the units at all the unramified places. At \( 3 \) it is invariant under \( 1 + 3 \mathbb{Z}_3 \omega \), thus it is trivial on all of \( U(3D) \)

Thus we can take the quotient by \( U(3D) \) as well. Note that the integral is now a finite sum:

\[
Z_f(s, \chi_D \varphi, \Phi_f) = \text{vol}(U(3D)) \sum_{\alpha_f' \in U(3D) \setminus \mathbb{A}_{K,f}^*/K^*} \left( \sum_{k \in K^*} \frac{k}{|k|_\mathbb{C}^2} \chi_D(\alpha_f') \Phi_f(k \alpha_f') \right) \varphi_f(\alpha_f') |\alpha_f'|^s
\]

Moreover, note that \( \text{vol}(U(3D)) = \text{vol}(1 + 3 \mathbb{Z}_3 \omega) \prod_{p|D} \text{vol}(\mathbb{Z} + D \mathbb{Z}_p[\omega]) = V_{3D} \).

By denoting \( I(s, \alpha_f, \Phi_f) = \sum_{k \in K^*} \frac{k}{|k|_\mathbb{C}^2} \Phi_f(k \alpha_f) \), we get the conclusion of the Lemma.

Combining the Lemma ?? and Lemma ??, we get:

Corollary 3.2. For all \( s \in \mathbb{C} \) and the Schwartz-Bruhat functions \( \Phi_f \in \mathcal{S}(\mathbb{A}_{K,f}) \) chosen above, we have:

\[
L_f(s, \chi_D \varphi) = \sum_{\alpha_f \in U(3D) \setminus \mathbb{A}_{K,f}^*/K^*} I(s, \alpha_f, \Phi_f) \chi_D(\alpha) \varphi(\alpha),
\]
3.1.2 Adelic representatives for $\text{Cl}(O_{3D})$

From the Strong approximation theorem, we can write $\alpha_f \in \mathbb{A}_K^\times = \mathbb{C}^\times K^\times \prod_{v \nmid \infty} O_{K_v}^\times$ in the form $\alpha_f = \gamma_k \kappa \beta_f$, where $\kappa \in K^\times$, $\gamma_k \in \mathbb{C}^\times$ and $\beta_f \in \prod_{v \nmid \infty} O_{K_v}^\times$. Then we can take representatives in $\alpha_f \in U(3D) \backslash \mathbb{A}_K^\times /K^\times$ such that $\alpha_f \in \prod_{v \nmid \infty} O_{K_v}^\times$. Moreover, since we are taking the quotient by the cube roots of six ($\pm 1, \pm \omega, \pm \omega^2$), we can pick $\alpha_f$ such that $\alpha_3 \equiv 1 \mod 3$. This can be done by replacing $\alpha_f$ by $\pm \alpha_f \omega^i$ for some $i, 0 \leq i \leq 2$.

Furthermore, note that representatives $\alpha_f, \alpha'_f$ are in the same class in $U(3D)$ iff $\alpha_f \alpha_f^{-1} \equiv a \mod D\mathbb{Z}_p[\omega]$, for some integer $a$ such that $(a, D) = 1$.

Moreover, we can define an ideal $\mathcal{A}_\alpha$ that is generated by $\kappa \in O_K$ such that

$$\alpha_p \equiv k_\alpha \mod 3D\mathbb{Z}_p[\omega].$$

Note that this ideal is unique only as a class in $\text{Cl}(O_{3D})$.

3.1.3 Connection to the Eisenstein series

Using the above representatives, note that $\varphi_f$ and $|\cdot|_f$ are trivial for the representatives $I_f$ and the Corollary ?? becomes:

$$L_f(s, \chi_D \varphi) = \sum_{\alpha_f \in U(3D) \backslash \mathbb{A}_K^\times /K^\times} I(s, \alpha_f, \Phi_f) \chi_D(\alpha_f).$$

We will now connect $I(s, \alpha_f, \Phi_f)$ to an Eisenstein series. We define the following classical Eisenstein series of weight $1$:

$$E_\varepsilon(s, z) = \sum_{m,n \in \mathbb{Z}} \varepsilon(n) (3mz + n)^3 |3mz + n|^s,$$

where the sum is taken over all $m, n \in \mathbb{Z}$ except for the pair $(0, 0)$, and $\varepsilon = (\frac{-1}{3})$ is the quadratic character associated to the field extension $K/\mathbb{Q}$.

Note that the Eisenstein series does not converge absolutely. However, we can still compute its value at $0$ using the Hecke trick in order for it to converge. We will compute its Fourier expansion in the following section.

Recall that for $\alpha_f \in \prod_{v \nmid \infty} O_{K_v}^\times$, we have the corresponding ideal class $[\mathcal{A}_\alpha]$ in $\text{Cl}(O_{3D})$. Such a representative is $\mathcal{A}_{\alpha_f} = (k_\alpha)$, where $k_\alpha \in O_K$ is chosen such that $k_\alpha \equiv \alpha_p \mod 3D\mathbb{Z}_p[\omega]$ for $p \nmid 3D$. Note that we can pick $\mathcal{A}_\alpha$ to be a primitive ideal.

We can further write $\mathcal{A}_\alpha$ as a $\mathbb{Z}$-lattice $\mathcal{A}_\alpha = [a, \frac{-b+\sqrt{-3}}{2a}]$, where $a = Nm \mathcal{A}_\alpha$ and $b$ is chosen (not uniquely) such that $b^2 \equiv -3 \mod 4a$. Then we can take the corresponding CM point $z_{\mathcal{A}_\alpha} := \frac{-b+\sqrt{-3}}{2a}$.

Using this notation, we have the following equality:

**Lemma 3.3.** For $\alpha_f \in \prod_{v \nmid \infty} O_{K_v}^\times$ and any choice of $z_{\mathcal{A}_\alpha}$ as above, we have:
\[ I(s, \alpha_f, \Phi_f) = \frac{1}{2} \frac{(\mathrm{Nm} A_\alpha)^{1-s}}{k_\alpha} E_v(s, \frac{1}{z\alpha}) \]

**Remark 3.2.** Note that the variable \( z\alpha \) on the left hand side is not uniquely defined. However, the function is going to be invariant on the class \([A_\alpha] \) in \( \text{Cl}(\mathcal{O}_{3D}) \).

**Proof:** Recall that \( I(s, \alpha_f, \Phi_f) = \sum_{k \in \mathcal{O}_K} \frac{k}{|k|_C^2} \Phi_f(k\alpha_f) \). We need to compute \( \Phi_f(k\alpha_f) \). Note that for all finite places \( v \) we have \( \Phi_v(k\alpha_v) \neq 0 \) only for \( k\alpha_v \in \mathcal{O}_{K_v} \), and since \( \alpha_v \in \mathcal{O}_{K_v}^\times \), we must have \( k \in \mathcal{O}_K \) for all \( v \mid \infty \). This implies \( k \in \mathcal{O}_K \) and for all \( v \mid 3D \) we get \( \Phi_v(k\alpha_v) = 1 \) for \( k \in \mathcal{O}_K \). Thus we can rewrite:

\[ I(s, \alpha_f, \Phi_f) = \sum_{k \in \mathcal{O}_K} \frac{k}{|k|_C^2} \Phi_f(k\alpha_f) \]

where \( \Phi_{3D} = \prod_{v \mid 3D} \Phi_v \) and \( \alpha_{3D} = (\alpha_v)_{v \mid 3D} \).

We can further compute \( \Phi_v(k\alpha_v) \) for \( v \mid 3D \). Recall that for \( p|D \) we defined \( \Phi_p = \text{char}(\mathbb{Z} + 3D\mathbb{Z}[\omega])^\times \) and \( \Phi_3 = \text{char}(1 + 3\mathbb{Z}[\omega])^\times \). Then we have \( \Phi_{3D}(k\alpha_{3D}) \neq 0 \) iff \( k\alpha_p \in a + 3D\mathbb{Z}[\omega] \) for some integer \((a, p) = 1 \) and for \( p = 3 \) we need \( k\alpha_3 \in 1 + 3\mathcal{O}_{K_3} \).

Recall that we can define \( k_\alpha \) such that \( k_\alpha \equiv a_p \mod 3\mathbb{Z}[\omega] \) for all \( p|3D \). Then the we have \( kk_\alpha \in a + 3D\mathbb{Z}_p[\omega] \) for \((a, p) = 1 \) and \( kk_\alpha \in 1 + 3\mathbb{Z}[\omega] \) as well. Furthermore, for \( k \in \mathcal{O}_K \) we actually have \( \Phi_{3D}(k\alpha_{3D}) = \Phi_{3D}(kk_\alpha) \). Then we can rewrite \( I(s, \alpha_f, \Phi_f) \) using \( k_\alpha \) in the form:

\[ I(s, \alpha_f, \Phi_f) = \sum_{k \in \mathcal{O}_K} \frac{k}{|k|_C^2} \Phi_f(kk_\alpha) \]

We can rewrite this further:

\[ I(s, \alpha_f, \Phi_f) = \frac{|k_\alpha|_C^2}{k_\alpha} \sum_{k \in \mathcal{O}_K} \frac{kk_\alpha}{|kk_\alpha|_C^2} \Phi_f(kk_\alpha) \]

Finally, we will make this explicit. Note that we must have \( kk_\alpha \in \mathcal{A}_\alpha \), where \( \mathcal{A}_\alpha = (a_\alpha) \), we well as \( kk_\alpha \in a_\alpha + D\mathbb{Z}_p[\omega] \) for some integer \( a_\alpha \), \((a_\alpha, p) = 1 \) as well as \( kk_\alpha \in 1 + 3\mathbb{Z}_3[\omega] \). By the Chinese remainder theorem, we can find an integer \( a \) such that \( a \equiv a_\alpha \mod D \) and \( a \equiv 1 \mod 3 \). Then we have \( kk_\alpha \in a + D \prod_{p|3D} \mathbb{Z}_p[\omega] \cap \mathcal{O}_K \), thus \( kk_\alpha \in P_{2,3D} \cap P_{1,3} \). Here \( P_{2,3D} = \{ k \in K : k \equiv a \mod 3D\mathcal{O}_K \} \) for some integer \( a, \(a, 3D\) = 1 \) and \( P_{1,3} = \{ k \in K : k \equiv 1 \mod 3 \} \). We rewrite:

\[ I(s, \alpha_f, \Phi_f) = \frac{|k_\alpha|_C^2}{k_\alpha} \sum_{k \in \mathcal{A}_\alpha \cap P_{2,3D} \cap P_{1,3}} \frac{k}{|k|_C^2} \]

Finally, we want to write the elements of \( \mathcal{A}_\alpha \cap P_{2,3D} \cap P_{1,3} \) explicitly.

Recall that we can write \( \mathcal{A}_\alpha \) as a \( \mathbb{Z} \)-lattice \( \mathcal{A}_\alpha = [a, \frac{b + \sqrt{3}c}{2}] \). Then all of the elements of \( \mathcal{A} \) are of the form \( ma + n\frac{b + \sqrt{3}c}{2} \) for some integers \( m, n \in \mathbb{Z} \). Moreover, note that the intersection of \( \mathcal{A} \) and \( P_{2,3D} \) is \( \{ k \in \mathcal{O}_K : k \equiv n \mod 3D, \text{ for some integer } n, (n, 3D) = 1 \} \) is \( \{ ma + 3Dn\frac{b + \sqrt{3}c}{2} : m, n \in \mathbb{Z} \} \). Further taking the intersection with \( P_{1,3} \), we must have \( ma \equiv 1 \), thus we must have \( m \equiv 1 \mod 3 \). Thus we can rewrite \( I(s, \alpha_f, \Phi_f) \) in the form:
I(s, αf, Φf) = \frac{a^s}{k_α} \sum_{m,n \in \mathbb{Z}, m \equiv 1 \pmod{3}} \frac{1}{(ma + n\frac{b+\sqrt{-3}}{2a})(ma + n\frac{b+\sqrt{-3}}{2a}-2^s)}.

Here we have also used the fact that \(|k_α| = a\). Note that we can further rewrite this as:

I(s, αf, Φf) = a^{-s-1}k_α \sum_{m,n \in \mathbb{Z}, m \equiv 1 \pmod{3}} \frac{1}{(m + n\frac{b+\sqrt{-3}}{2a})(m + n\frac{b+\sqrt{-3}}{2a}-2^s-2^s)}.

Furthermore, by changing \(n \rightarrow -n\) and taking out a factor of \(a^{1-2s}\), we have:

I(s, αf, Φf) = a^{-s}k_α \sum_{m,n \in \mathbb{Z}, m \equiv 1 \pmod{3}} \frac{1}{(m + n\frac{b+\sqrt{-3}}{2a})(m + n\frac{b+\sqrt{-3}}{2a}-2^s-2^s)}.

Note that for \(Re(s) > 1\) the integral converges absolutely, thus we can rewrite it in the form:

I(s, αf, Φf) = \frac{1}{2} a^{-s}k_α \sum_{m,n \in \mathbb{Z}, m \equiv 1 \pmod{3}} \frac{1}{(m + n\frac{b+\sqrt{-3}}{2a})(m + n\frac{b+\sqrt{-3}}{2a}-2^s-2^s-2^s)} + \frac{1}{2} a^{-s}k_α \sum_{m,n \in \mathbb{Z}, m \equiv 2 \pmod{3}} (-m + n\frac{b+\sqrt{-3}}{2a})(-m + n\frac{b+\sqrt{-3}}{2a}-2^s-2^s).

Changing \(n \rightarrow -n\) in the second sum, we get:

\begin{align*}
I(s, αf, Φf) &= \frac{1}{2} a^{-s}k_α \sum_{m,n \in \mathbb{Z}, m \equiv 1 \pmod{3}} \frac{1}{(m + n\frac{b+\sqrt{-3}}{2a})(m + n\frac{b+\sqrt{-3}}{2a}-2^s-2^s-2^s)} \\
&\quad - \frac{1}{2} a^{-s}k_α \sum_{m,n \in \mathbb{Z}, m \equiv 2 \pmod{3}} \frac{1}{(m + n\frac{b+\sqrt{-3}}{2a})(m + n\frac{b+\sqrt{-3}}{2a}-2^s-2^s)}
\end{align*}

Thus we can write for \(Re(s) > 1\) we can rewrite:

I(s, αf, Φf) = \frac{1}{2} a^{-s}k_α \sum_{m,n \in \mathbb{Z}, m \equiv 1 \pmod{3}} \frac{\varepsilon(m)}{(m + n\frac{b+\sqrt{-3}}{2a})(m + n\frac{b+\sqrt{-3}}{2a}-2^s-2^s-2^s)}.

On the right hand side we can recognize the Eisenstein series \(E_2(2s - 2, \frac{z}{a_\mathcal{A}})\), thus we get:

I(s, αf, Φf) = \frac{1}{2} a^{-s}k_α E_2(2s - 2, z\mathcal{A}) = \frac{1}{2} a^{-s}k_α E_2(2s - 2, z\mathcal{A}) = \frac{1}{2} a^{-s}k_α E_2(2s - 2, z\mathcal{A})\frac{(Nm\mathcal{A})^{1-s}}{k_\mathcal{A}}.

By analytic continuation, we can extend the equality to all \(s \in \mathbb{C}\).

Using this Lemma, we can extend the Corollary \ref{corollary} in the form:

Corollary 3.3. For all \(s\), we have:

\[ L_f(s, \chi D\varphi) = \frac{1}{2} \sum_{\mathcal{A} \in \mathcal{C}(\mathcal{O}_D)} E_2(2s - 2, z\mathcal{A})\frac{(Nm\mathcal{A})^{1-s}}{k_\mathcal{A}}. \]

Proof: Recall that in the Corollary \ref{corollary} we got

\[ L_f(s, \chi D\varphi) = \sum_{\alpha \in U(3\mathcal{D} \setminus K^\times)} I(s, αf, Φf)\chi_D(\alpha)\varphi(\alpha). \]

18
We can rewrite $I(s, \alpha_f, \Phi_f) = \frac{1}{2} a^{1-s}_{\kappa_n} E_\varepsilon(2s - 2, z_{A_n})$ and $\varphi(\alpha) = 1, \chi_D(\alpha) = \chi_D(k_\alpha) = \chi_D(A_\alpha)$. Then we get:

$$L_f(s, \chi_D \varphi) = \sum_{\alpha \in U(3D) \setminus \mathbb{A}_K^K / \mathbb{K}^*} \frac{1}{2} a^{1-s}_{\kappa_\alpha} E_\varepsilon(2s - 2, z_{A_\alpha}) \chi_D(A_\alpha)$$

Finally, consider $\mathcal{A}$ as representatives of $\text{Cl}(\mathcal{O}_{3D})$. Note that by changing $\mathcal{A} \to \overline{\mathcal{A}}$ we just invert the classes of $\text{Cl}(\mathcal{O}_{3D})$. Thus we get the result of the Corollary:

$$L_f(s, \chi_D \varphi) = \sum_{\mathcal{A} \in \text{Cl}(\mathcal{O}_{3D})} \frac{1}{2} a^{1-s}_{\kappa_\mathcal{A}} E_\varepsilon(2s - 2, z_{\mathcal{A}}) \chi_D(\mathcal{A}).$$

### 3.1.4 Fourier expansion of the Eisenstein series $E_\varepsilon(s, z)$ at $s = 0$.

We want to connect the Eisenstein series $E_\varepsilon(s, z)$ to the theta function $\Theta_K(z)$. In order to do this, we will compute the Fourier expansion of $E_\varepsilon(s, z)$ at $s = 0$.

We will use the Hecke trick to compute the Fourier expansion of the Eisenstein series:

$$E_\varepsilon(s, z) = \sum_{d \leq \varepsilon} \frac{\varepsilon(d)}{(3cz + d)^{2s}}$$

We will follow closely the proof of Pacetti [12]. This is also done by Hecke in [7]. We rederive the formula:

$$E_1(z, s) = \sum_{d \leq \varepsilon} \frac{\varepsilon(d)}{d^{1+2s}} + 2 \sum_{c=1}^{\infty} \sum_{r=0}^{2} \sum_{d \in \mathbb{Z}} \frac{\varepsilon(r)}{(3cz + (3d + r))^2}$$

We divide by $3^{2s+1}$ and get:

$$E_1(z, s) = 2L(\varepsilon, 1 + 2s) + \sum_{c=1}^{\infty} \sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} \sum_{d \in \mathbb{Z}} \frac{\varepsilon(r)}{\frac{3cz + r}{3} + d}$$

We define for $z$ in the upper-half plane:

$$H(z, s) = \sum_{m \in \mathbb{Z}} \frac{1}{(z + m)^{2s}}$$

Following Shimura [??], for $z = x + yi$ and $s > 0$ we have the Fourier expansion:

$$H(z, s) = \sum_{n=-\infty}^{\infty} \frac{\tau_n(y, s + 1, s)e^{2\pi i n x}}{\Gamma(2s)(4\pi y)^s},$$

where

$$\tau_n(y, s + 1, s) = \begin{cases} n^{2s}e^{-2\pi n y} \sigma(4\pi n y, s + 1, s), & \text{if } n > 0 \\ n|n|^{2s}e^{-2\pi |n| y} \sigma(4\pi |n| y, s + 1), & \text{if } n < 0 \\ \Gamma(2s)(4\pi y)^{-2s}, & \text{if } n = 0, \end{cases}$$

where $\gamma(Y, \alpha, \beta) = \int_0^\infty (t + 1)^{\alpha-1}t^{\beta-1}e^{-Yt}dt$.
For any $s > 0$, $H(z, s)$ converges, thus we can compute the limits of each of its Fourier coefficients:

- $n = 0$: $\lim_{s \to 0} \frac{(2\pi)^{2s+1} \Gamma(2s)}{\Gamma(s)} \frac{1}{\Gamma(s)} (4\pi y)^{-2s} = -2\pi i \lim_{s \to 0} \frac{\Gamma(2s)}{\Gamma(s)}$

- $n < 0$: $\lim_{s \to 0} \frac{(2\pi)^{2s+1}}{\Gamma(s + 1) \Gamma(s)} |n|^{2s} e^{-2\pi |n| y} \int_{0}^{\infty} (t+1)^{s-1} t^s e^{-4\pi |n| y t} dt = -2\pi i e^{-2\pi |n| y} \lim_{s \to 0} \frac{1}{\Gamma(s)} \int_{0}^{\infty} (t+1)^{s} t^s e^{-4\pi |n| y t} dt$

- $n > 0$: $\lim_{s \to 0} \frac{(2\pi)^{2s+1}}{\Gamma(s + 1) \Gamma(s)} n^{2s} e^{-2\pi n y} \int_{0}^{\infty} (t+1)^{s-1} t^s e^{-4\pi n y t} dt$

(COMPUTATION)

We should get:

$$\lim_{s \to 0} H(s, z) = -\pi i - 2\pi i \sum_{n=1}^{\infty} q^n$$

Finally, note that:

$$E_1(s, z) = 2L(\varepsilon, s) + 2 \sum_{c=1}^{\infty} \sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} H \left( \frac{3dz + r}{3}, s \right)$$

Using the Fourier expansion of $H(z, s)$, we get:

$$E_1(s, z) = 2L(\varepsilon, s) + 2 \sum_{c=1}^{\infty} \sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} \sum_{n \in \mathbb{Z}} \tau_n(yn, s + 1, s) e^{2\pi inz} \omega^{nr}$$

Taking the limit as $s \to 0$, and the Fourier expansion above, we get:

$$E_1(s, z) = 2L(\varepsilon, s) + 2 \sum_{c=1}^{\infty} \sum_{r=0}^{2} \frac{\varepsilon(r)}{3} \left( -\pi i - 2\pi i \sum_{n=1}^{\infty} e^{2\pi inz} \omega^{nr} \right)$$

We compute separately the inner sum:

$$\sum_{r=0}^{2} \frac{\varepsilon(r)}{3} \left( -\pi i + \sum_{n=1}^{\infty} e^{2\pi inz} \omega^{nr} \right) = -2\pi i \sum_{n=1}^{\infty} e^{2\pi inz} \varepsilon(n) 2 \sum_{r=0}^{2} \omega^{nr} \varepsilon(rn) = -2\pi i G(\varepsilon) \sum_{n=1}^{\infty} e^{2\pi inz} \varepsilon(n),$$

where $G(\varepsilon) = \sum_{r=0}^{2} \varepsilon(r) \omega^r = \sqrt{-3}$ is the Gaussian quadratic sum corresponding to $\varepsilon$.

Then we get:

$$E_1(0, z) = 2L(\varepsilon, 1) - \frac{4\pi i \sqrt{-3}}{3} \sum_{c=1}^{\infty} \sum_{n=1}^{\infty} e^{2\pi inz} \varepsilon(n) = 2L(\varepsilon, 1) + \frac{4\pi \sqrt{3}}{3} \sum_{N=1}^{\infty} \left( \sum_{m|N} \varepsilon(m) \right) e^{2\pi inz}$$

20
Since $\varepsilon$ is a quadratic character, we can compute $L(1, \varepsilon) = \frac{\pi \sqrt{3}}{9}$ (see Kowalski [?]). This gives us the Fourier expansion:

$$E_1(0, z) = \frac{2\pi \sqrt{3}}{9} \left( 1 + 6 \sum_{N=1}^{\infty} \left( \sum_{m|N} \varepsilon(m) \right) e^{2\pi i Nz} \right)$$

### 3.1.5 Connection to the theta function $\Theta_K(z)$.

Recall the theta function $\Theta_K$ associated to the number field $K$:

$$\Theta_K(z) = \sum_{a, b \in \mathbb{Z}} e^{2\pi i (a^2 - ab + b^2)z}.$$  

Equivalently, we can rewrite the theta function in the form: $\Theta_K(z) = 1 + 6 \sum_{A} e^{2\pi i NmA z}$, where we sum over all ideals $A$. Thus we have the Fourier expansion for $\Theta_K$:

$$\Theta_K(z) = 1 + 6 \sum_{n \geq 1} c(n)q^n,$$

where $c(n)$ is the number of ideals of norm $n$. We will show the following version of Siegel-Weil theorem:

**Theorem 3.1.** For $E_\varepsilon(s, z)$ defined in the previous section and $\varepsilon$ the quadratic character corresponding to to the extension $K/\mathbb{Q}$, we have:

$$E_\varepsilon(0, z) = 2L(0, \varepsilon)\Theta_K(z)$$

The proof consists of comparing the Fourier expansions of the two sides. This is mainly going to be based on the lemma below:

**Lemma 3.4.** For $n \geq 1$ then for the ideals in $\mathcal{O}_K$ we have:

$$\sum_{d|n} \varepsilon(d) = \# \text{ideals of norm } n$$

**Proof:** We first show the result for powers of primes $p^s$. We consider three cases:

If $p \equiv 1 \mod 3$, then there are two ideals of norm $p$: $(a + b\omega)$ and $(a - b\omega)$ such that $a^2 - ab + b^2 = p$. Then we have $k + 1$ ideals of norm $p^k$: $(a + b\omega)^i(a + b\omega)^{k-i}$ for $0 \leq i \leq k$. Moreover, since $\varepsilon(p) = 1$, we have $(1 + \varepsilon(p) + \ldots + \varepsilon(p^k)) = k + 1$.  

If $p \equiv 2 \mod 3$, then there are no ideals of norm $p$. Thus, if $k$ is even, we have exactly one ideal of norm $p^k$: $A = (p^{k/2})$. In this case $(1 + \varepsilon(p) + \ldots + \varepsilon(p^k)) = 1 - 1 + \ldots + 1 = 1$. If $k$ is odd, we have no ideals of norm $p^{2k+1}$. Moreover $(1 + \varepsilon(p) + \ldots + \varepsilon(p^k)) = 1 - 1 + \ldots - 1 = 0$.  

If $p = 3$, then we have exactly one ideal of norm $3^k$, namely the ideal $(\sqrt{-3})$. Moreover $\varepsilon(3) = 0$, thus $(1 + \varepsilon(3) + \ldots + \varepsilon(3^k)) = 1$.  

It is easy to extend the result to all integers. As $\varepsilon$ is a character, we have:

$$\sum_{d|n} \varepsilon(d) = \prod_{p|n} (1 + \varepsilon(p) + \ldots + \varepsilon(p)^{c_p}),$$

21
where \( n = \prod p_i^{e_i} \), \( e_i \geq 1 \) and \( p_i \) are primes. If we have any ideal \( \mathcal{A} \) of norm \( n \), then
\[
\mathcal{A} = \prod \mathfrak{p}_i^{e_i},
\]
and we must have \( n = \prod N\mathfrak{p}_i^{e_i} \). Moreover, we have \# ideals of norm \( n = \prod \# \text{ideals of norm } (N\mathfrak{p}_i)^{e_i} \), which finishes the proof.

We are ready to state the proof of the theorem. Using the above Lemma we can rewrite the Fourier expansion of \( \Theta_K \) as:
\[
\Theta_K(z) = 1 + 6 \sum_{N=1}^{\infty} \left( \sum_{m|N} \varepsilon(m) \right) e^{2\pi i N z}
\]

Multiplying by a factor of \( \frac{2\sqrt{3}}{9} \), we recognize the Eisenstein series \( E_\varepsilon(0, z) \). Thus it implies \( E_\varepsilon(0, z) = \frac{2\sqrt{3}}{9} \Theta_K(z) \). Note that this is the same as:
\[
E_\varepsilon(0, z) = 2L(1, \varepsilon)\Theta_K(z)
\]

### 3.1.6 Final formula for \( L(1, \chi_D \varphi) \)

Applying Corollary ?? for \( s = 1 \) we get:
\[
L_f(1, \chi_D \varphi) = \frac{1}{2} \sum_{\mathcal{A} \in \text{Cl}(\mathcal{O}_D)} \frac{1}{k_\mathcal{A}} E_\varepsilon(0, Dz_\mathcal{A}) \chi_{3D}(\mathcal{A})
\]

Furthermore, from Theorem 3.1 this is the same as:
\[
L_f(1, \chi_D \varphi) = \frac{\pi \sqrt{3}}{9} \sum_{\mathcal{A} \in \text{Cl}(\mathcal{O}_D)} \frac{1}{k_\mathcal{A}} \Theta_K(Dz_\mathcal{A}) \chi_{3D}(\mathcal{A})
\]

We need one more step before rewriting the formula as a trace. This is going to be the following lemma:

**Lemma 3.5.** For \( \mathcal{A} = \left[ a, \frac{-b + \sqrt{-3}}{2} \right] \) a primitive ideal of norm \( N\mathcal{A} = a \), with generator \( \mathcal{A} = (k_\mathcal{A}) \), where \( k_\mathcal{A} \equiv 1 \mod 3 \), we have:
\[
\Theta_K \left( \frac{-b + \sqrt{-3}}{2a} \right) \frac{k_\mathcal{A}}{a} = \Theta_K \left( \frac{-1 + \sqrt{-3}}{2} \right)
\]

**Proof:** Since \( \mathcal{A} = \left[ a, \frac{-b + \sqrt{-3}}{2} \right] \) as a \( \mathbb{Z} \)-lattice, we can write its generator \( k_\mathcal{A} \) in the form \( k_\mathcal{A} = ma + 3n \frac{-b + \sqrt{-3}}{2} \) for some integers \( m, n \). Moreover, \( k_\mathcal{A} = m - 3n \frac{b + \sqrt{-3}}{2} \) and \( \frac{k_\mathcal{A}}{a} = m - 3n \frac{b + \sqrt{-3}}{2a} \). Moreover, since \( k_\mathcal{A} \) is the generator of a primitive ideal, we have \( \gcd(m, 3n) = 1 \). Then we can find through the Euclidean algorithm integers \( A, B \) such that \( mA + 3nB = 1 \), which makes \( \left( \begin{array}{c} A \\ -3n \end{array} \right) m \) a matrix in \( \Gamma_0(3) \). Since \( \Theta \) is a modular form of weight 1 for \( \Gamma_0(3) \), we have:
\[
\Theta_K \left( \frac{A - \frac{b + \sqrt{-3}}{2a}}{-3n \frac{-b + \sqrt{-3}}{2a} + m} \right) = \left( m - 3n \frac{-b + \sqrt{-3}}{2a} \right) \Theta_K \left( \frac{-b + \sqrt{-3}}{2a} \right)
\]
Noting that \(-3n\frac{-b+\sqrt{-3}}{2a}+m = k_A/a = 1/\kappa_A\), we can compute \(A\frac{-b+\sqrt{-3}}{2a}+B = (A\frac{-b+\sqrt{-3}}{2a}+Ba)\kappa_A\). This is \((aB + A\frac{-b+\sqrt{-3}}{2a})(ma + 3n\frac{b+\sqrt{-3}}{2})/a\). After expanding, we get:
\[-3nA^2 + 3 + abB + aB + \frac{b(-mA + 3nB) + \sqrt{-3}}{2}\]

Note that \(mA + 3nB = 1\) implies that \(mA\) and \(3nB\) have different parities. Also we chose \(b\) odd, since \(b^2 + 3 \equiv 0 \mod 4a\). Then we note that \(-3nA^{b^2+3} + abB + \frac{b(-mA + 3nB + 1)}{2} \in \mathbb{Z}\) and thus using the period 1 of \(\Theta_K\) we get:
\[
\Theta_K \left( A\frac{-b+\sqrt{-3}}{2a} + B \right) = \Theta_K \left( \frac{-1 + \sqrt{-3}}{2} \right)
\]

This finishes the proof.

Note that the Lemma above is equivalent to \(\Theta_K(\tau_A) = \kappa_A \Theta_K(\omega)\), where \(\tau_A = \frac{-b+\sqrt{-3}}{2a}\). Then we can rewrite (9) as:

**Proposition 3.1.**

\[
L_f(E_D, 1) = \frac{\pi\sqrt{3}}{9} \Theta_K(\omega) \sum_{A \in \text{Cl}(\mathcal{O}_{3D})} \frac{\Theta_K(D\tau_A)}{\Theta_K(\tau_A)} \chi_{3D}(A)
\]  

(9)

### 3.1.7 Turning the formula into a trace.

We will rewrite (9) as a trace. First, let \(f(z) = \frac{\Theta_K(Dz)}{\Theta_K(z)}\). This is a modular function for \(\Gamma_0(3D)\).

We will prove in the following section ?? the following proposition:

**Proposition 3.2.** Take \(A\) representative ideals for \(\text{Cl}(\mathcal{O}_{3D})\). We can take all \(A\) to be primitive and we can write them in the form \(A = [a, \frac{-b+\sqrt{-3}}{2a}]\). Then the Galois conjugates of \(f(\omega)\) are:

\[
f(\omega)^{\sigma_A^{-1}} = \frac{\Theta(D\frac{-b+\sqrt{-3}}{2a})}{\Theta(\frac{-b+\sqrt{-3}}{2a})}
\]

We will also rewrite the character \(\chi_D\) to include a trace. In the Introduction we have also showed in Lemma ?? that \((D^{1/3})^{\sigma_A^{-1}} = D^{1/3}\chi_D(A)\).

Then the formula (9) becomes:

\[
L_f(E_D, 1) = \frac{\pi\sqrt{3}}{9} D^{-1/3} \Theta_K(\omega) \sum_{A \in \text{Cl}(\mathcal{O}_{3D})} \left( D^{1/3} \frac{\Theta_K(D\omega)}{\Theta_K(\omega)} \right)^{\sigma_A^{-1}}
\]  

(10)

Moreover, we also have \(D^{1/3} \in H_{3D}\). See Cohn [3] for a proof. Thus we can rewrite the sum on the left hand side as \(\text{Tr}_{H_{3D}/K} \left( D^{1/3} \frac{\Theta_K(D\omega)}{\Theta_K(\omega)} \right)\). We can compute the extra terms as well.
• Rodriguez-Villegas and Zagier in [1] cite $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = -3\Gamma\left(\frac{1}{3}\right)^3/(2\pi)^2$. We will use several properties of $\Theta_K$ proved in section ???. We can rewrite $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right)$ as $\Theta_K\left(\frac{-3+\sqrt{3}}{6} - \frac{1}{3}\right)$ and using formula ??, we get:

$$
\Theta_K\left(\frac{-3+\sqrt{3}}{18} - \frac{1}{3}\right) = (1 - \omega^2)\Theta_K\left(\frac{-3+\sqrt{3}}{6}\right) + \omega^2\Theta_K\left(\frac{-3+\sqrt{3}}{18}\right)
$$

Using $\Theta_K\left(\frac{-3+\sqrt{3}}{6}\right) = 0$, we get $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = \omega^2\Theta_K\left(\frac{-3+\sqrt{3}}{18}\right)$.

Furthermore, the functional equation $\Theta(-1/3z) = -\sqrt{-3z}\Theta(z)$ for $z = \frac{3+\sqrt{3}}{2}$, we get $-\sqrt{-3\frac{3+\sqrt{-3}}{2}}\Theta(\omega) = \Theta_K\left(\frac{-3+\sqrt{3}}{18}\right)$. Note that $-\sqrt{-3\frac{3+\sqrt{-3}}{2}} = 3\omega$, thus we get $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = 3\Theta(\omega)$.

This gives us the value $\Theta(\omega) = \Gamma\left(\frac{1}{3}\right)^3/(2\pi)^2$

• $L_\infty(s, \chi_D \varphi) = L_\infty(s, \varphi_\infty)$, where $\varphi_\infty(z) = z^{-1}$. Then we can compute:

$$
L_\infty(s, \varphi_\infty) = L_\infty(s - 1/2, |\cdot|_{L^2}\varphi_\infty) = 2(2\pi)^s\Gamma(s).
$$

This gives us $L_\infty(1, \chi_D \varphi) = 4\pi$.

XXXX Should have 2 instead here.

• The real period $\Omega_D$ of the elliptic curve $E_D$. The real period of $E_1$ is $\Gamma\left(\frac{1}{3}\right)^3/\sqrt{3}$ (see [1]). Then to compute the real period of $E_D$ we twist by a factor of $D^{-1/3}$ (see [1]) and get:

$$
\Omega_D = D^{-1/3}\Gamma\left(\frac{1}{3}\right)^3
$$

Check this.

Multiplying all the terms, we get:

$$
L(E_D, 1) = 2\frac{\pi\sqrt{3}}{9}D^{-1/3}\frac{\Gamma\left(\frac{1}{3}\right)^3}{(2\pi)^2} \text{Tr}_{H_3/K}\left(D^{1/3}\Theta_K(D\omega)/\Theta_K(\omega)\right)
$$

This gives us the theorem:

**Theorem 3.2.**

$$
L(E_D, 1) = \frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)^3}{18\pi}D^{-1/3}\text{Tr}_{H_3/K}\left(D^{1/3}\Theta_K(D\omega)/\Theta_K(\omega)\right)
$$

**3.1.8 $S_D$ is an integer**

In the previous section we showed that $S_D \in K$. Note that $D^{1/3}\Theta(D\omega)/\Theta(\omega) = D^{1/3}\Theta(-D + D\omega)/\Theta(-1 + \omega) = D^{1/3}\Theta(D\omega)/\Theta(\omega)$. Thus $S_D \in \mathbb{Q}$. We would like to show that $S_D \in \mathbb{Z}$.

First we look at the Fourier expansion of $f(z) = \Theta(Dz)/\Theta(z)$:
\[ \Theta(z) = 1 + 6 \sum_{N \in \mathbb{Z}_{\geq 1}} c(N)q^N, \]

where \( c(N) = \# \text{ ideals with norm } N \text{ in } K \) and, \( q = e^{2\pi iz}. \) Then we also have the Fourier expansion of \( \Theta(Dz) \):

\[ \Theta(Dz) = 1 + 6 \sum_{N \in \mathbb{Z}_{\geq 1}} c(N)q^{DN}, \]

By taking their ratio we get \( \frac{\Theta(Dz)}{\Theta(z)} = \sum_{n \in \mathbb{Z}} a_nq^n, \ a_n \in \mathbb{Z}. \) This is easy to see just by straight computation. The minimal polynomial of \( D^{1/3}f(\omega) \) is:

\[ \prod_{\mathcal{A} \in \text{Cl}(\mathcal{O}_D)} (X - D^{1/3}\chi_D(\mathcal{A})(f(\omega))^{\sigma_A}) \in \mathbb{Z}[\omega, D^{1/3}](X, q) \]

This implies that \( \text{Tr}_{H_D/K} D^{1/3}f(\omega) \in \mathbb{Z}[\omega, D^{1/3}]. \) We already know that \( \text{Tr}_{H_D/K} D^{1/3}f(\omega) \in \mathbb{Q}, \) thus \( \text{Tr}_{H_D/K} D^{1/3}f(\omega) \in \mathbb{Z}. \)

### 4 Shimura reciprocity law in the classical setting.

Let \( \mathcal{F} \) be the field of modular functions over \( \mathbb{Q}. \) From CM theory (see \( \| \), for example), it is known that if \( \tau \in K \cap \mathcal{H} \) and \( f \in \mathcal{F}, \) then we have \( f(\tau) \in K_{ab}, \) where \( K_{ab} \) is the maximal abelian extension of \( K. \) Shimura reciprocity law gives us a way to compute the Galois conjugates of \( f(\tau)^\sigma \) when acting with \( \sigma \in \text{Gal}(K_{ab}/K). \) We will follow the exposition of Stevenhagen \( \| \). For more details see Gee \( \| \).

We recall that \( \mathcal{F} = \bigcup_{N \geq 1} \mathcal{F}_N, \) where \( \mathcal{F}_N \) is the space of modular functions of level \( N. \) Moreover, we can think of \( \mathcal{F}_N \) as the function field of the modular curve \( X(N) = \Gamma(N) \backslash \mathcal{H}^* \) over \( \mathbb{Q}(\zeta_N), \) where \( \zeta_N = e^{2\pi i/N} \) and \( \mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}). \) We can compute explicitly \( \mathcal{F}_N = \mathbb{Q}(j, j_N), \) where \( j \) is the \( j \)-invariant and \( j_N(z) = j(Nz). \) In particular, we have \( \mathcal{F}_1 = \mathbb{Q}(j). \)

When working over \( \mathbb{Q}, \) one has an isomorphism:

\[ \text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \cong \text{GL}_2(\mathbb{Z}/NZ)/\{\pm 1\}. \]

More precisely, if we denote by \( \gamma_\sigma \) the Galois action corresponding to the matrix \( \gamma \in \text{GL}_2(\mathbb{Z}/NZ) \) under the isomorphism above, it is enough to define the Galois action for \( \text{SL}_2(\mathbb{Z}/NZ) \) and for \( G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in (\mathbb{Z}/NZ)^\times \right\}. \) We state explicitly the two actions below.

- **Action of** \( \alpha \in \text{SL}_2(\mathbb{Z}/NZ) \) **on** \( \mathcal{F}_N. \)

  \[ (f(\tau))^{\sigma_\alpha} = f^{\alpha}(\tau) = f(\alpha \tau), \]

  where \( \alpha \) is acting on the upper half plane via fractional linear transformations.

- **Action of** \( \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in (\mathbb{Z}/NZ)^\times \) **on** \( \mathcal{F}_N. \) Note that for \( f \in \mathcal{F}_N \) we have a Fourier expansion
\[
f(z) = \sum_{n \geq 0} a_n q^{n/N} \text{ with coefficients } a_n \in \mathbb{Q}(\zeta_N), \quad q = e^{2\pi iz}.\]
If we denote \( u_d := \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \), then the action of \( \sigma_{u_d} \) is given by
\[
(f(\tau))^{\sigma_{u_d}} = f^{u_d}(\tau) := \sum_{n \geq 0} a_n^{u_d} q^{n/N},
\]
where \( \sigma_d \) is the Galois action in \( \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \) that sends \( \zeta_N \to \zeta_N^d \).

As the restriction maps between the fields \( \mathcal{F}_N \) are in correspondence with the natural maps between the groups \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/(\pm 1) \) we can take the projective limit to get the isomorphism:
\[
\text{Gal}(\mathcal{F}/\mathcal{F}_1) \cong \text{GL}_2(\hat{\mathbb{Z})}/(\pm 1).
\]

To further get all the automorphisms of \( \mathcal{F} \) we need to consider the action of \( \text{GL}_2(\mathbb{A}_{Q,f}) \). We get the exact sequence:
\[
1 \to \{ \pm 1 \} \to \text{GL}_2(\mathbb{A}_{Q,f}) \to \text{Aut}(\mathcal{F}) \to 1.
\]
For this to make sense, we need to extend the action from \( \text{GL}_2(\hat{\mathbb{Z}}) \) to \( \text{GL}_2(\mathbb{A}_{Q,f}) \). We do this by using the action of \( \text{GL}_2(\hat{\mathbb{Q}}]^+\):

- **Action of** \( \alpha \in \text{GL}_2(\hat{\mathbb{Q}}]^+ \) **on** \( \mathcal{F} \).
  \[
  f^\alpha(\tau) = f(\alpha\tau),
  \]
  where \( \alpha \) acts by fractional linear transformations.

We extend the action of \( \text{GL}_2(\hat{\mathbb{Z}}) \) to \( \text{GL}_2(\mathbb{A}_{Q}) \) by writing the elements \( \gamma \in \text{GL}_2(\mathbb{A}_{Q}) \) in the form \( \gamma = u\alpha \), where \( u \in \text{GL}_2(\hat{\mathbb{Z}}) \) and \( \alpha \in \text{GL}_2(\hat{\mathbb{Q}}]^+ \). Note that this decomposition is not uniquely determined. However, by combining the two actions of \( u \) and \( \alpha \), a well defined action is given by:
\[
 f^{u\alpha} = (f^u)^\alpha.
\]

We want to look at the action of \( \text{Gal}(K^{ab}/K) \) inside \( \text{Aut}(\mathcal{F}) \). From class field theory we have the exact sequence:
\[
1 \to K^\times \to \mathbb{A}_{K,f}^\times \xrightarrow{[\cdot,K]} \text{Gal}(K^{ab}/K) \to 1,
\]
where \([\cdot,K]\) is the Artin map.

We are going to embed \( \mathbb{A}_{K,f}^\times \) into \( \text{GL}_2(\mathbb{A}_{Q,f}) \) such that the Galois action of \( \mathbb{A}_{K,f}^\times \) through the Artin map and the action of the matrices in \( \text{GL}_2(\mathbb{A}_{Q,f}) \) are compatible. We do this by constructing a matrix \( g_\tau(x) \) for the idele \( x \in \mathbb{A}_{K,f}^\times \).

Let \( \mathcal{O} \) be the order of \( K \) generated by \( \tau \) i.e. \( \mathcal{O} = \mathbb{Z}[\tau] \). We define the matrix \( g_\tau(x) \) to be the unique matrix in \( \text{GL}_2(\mathbb{A}_{Q}) \) such that \( x \begin{pmatrix} \tau \\ 1 \end{pmatrix} = g_\tau(x) \begin{pmatrix} \tau \\ 1 \end{pmatrix} \). We can compute it explicitly. To do that, consider the minimal polynomial of \( \tau \):
\[
p(X) = X^2 + BX + C
\]
Then if we write \( x_p \in \mathbb{Q}_p^\times \) in the form \( x_p = s_p \tau + t_p \in \mathbb{Q}_p^\times \) with \( s_p, t_p \in \mathbb{Q}_p \), we can compute:

\[
g_\tau(x_p) = \left( \begin{array}{cc} t_p - s_p B & -s_p C \\ s_p & t_p \end{array} \right)
\]

Shimura reciprocity law is going to make the following diagram commute:

\[
\begin{array}{c}
1 \\
\downarrow g_\tau \\
1
\end{array}
\]

\[
\begin{array}{cccc}
K^\times & \xrightarrow{[\cdot,K]} & \mathbb{A}_{K,f}^\times & \xrightarrow{\text{Gal}(K^{ab}/K)} & \text{Gal}(\mathbb{Q}) & \rightarrow 1 \\
\downarrow g_\tau & & & & & \\
\{\pm 1\} & \xrightarrow{\text{GL}_2(\mathbb{A}_{Q,f})} & \text{Aut}(\mathbb{F}) & \rightarrow 1
\end{array}
\]

We make the statement explicit below:

**Theorem 4.1. (Shimura reciprocity law)** For \( f \in \mathcal{F} \) and \( x \in \mathbb{A}_{K,f}^\times \), we have:

\[
(f(\tau))^{[x,K]} = f^{g_\tau(x^{-1})(\tau)},
\]

where \([x, K]\) is the Galois action corresponding to the idele \( x \) via the Artin map, \( g_\tau \) is defined above and the action of \( g_\tau(x) \) is the action in \( \text{GL}_2(\mathbb{A}_{Q,f}) \).

**Remark 4.1.** Note that the elements of \( K^\times \) have trivial action. This can be easily seen by embedding \( K^\times \hookrightarrow \text{GL}_2(\mathbb{Q})^+ \) given by \( k \mapsto g_\tau(k) \). Noting that \( \tau \) is fixed by the action of the torus \( K^\times \), we have:

\[
f^{g_\tau(k^{-1})}(\tau) = f(g_\tau(k^{-1})\tau) = f(\tau)
\]

**Remark 4.2.** We can also rewrite the theorem for ideals in \( K \). Let \( f \in \mathcal{F}_N \) and \( \mathcal{O} = \mathbb{Z}[\tau] \) of conductor \( M \). Going through the Artin map, we can restate Shimura reciprocity in this case in the form:

\[
f(\tau)^{\sigma_A} = f^{g_\tau(A)^{-1}}(\tau),
\]

where \( \sigma_A \) is the Galois action corresponding to the ideal \( A \) through the Artin map and

\[
g_\tau(A) := g_\tau((\alpha)_{p|\text{Nm}(A)}).
\]

Note that \( g_\tau(A) \) is unique up to multiplication by roots of unity in \( K \). However, these have trivial action on \( f \). This can be easily seen by multiplying by an element of \((\pm \omega) \in K^\times \) and noticing that we get trivial action at the unramified places \( p \nmid MN \).

**Remark 4.3.** Note that the action of \( g_\tau(A) \) is the same as the action of \( g_\tau((\alpha)_{p|M^N})^{-1} \).

**Remark 4.4.** Note that the maps above are based on the map between the ideals \( A \) prime to \( MN \) and the ideles:
\[ I(MN) \to \mathbb{A}_{K,f}^x/K^x \]
\[ \mathcal{A} = \prod_v p_v^{c_v} \to (\varpi_v)^{c_v}, \]

where \( \varpi_v \) is the uniformizer of the ideal \( p_v \) at the place \( v \nmid \infty \).

### 4.1 Applying Shimura reciprocity law to \( K = \mathbb{Q}[\sqrt{-3}] \).

**Lemma 4.1.** The function \( f(z) = \frac{\Theta_K(Dz)}{\Theta_K(z)} \) is a modular function of level \( 3D \) with integer Fourier coefficients at the cusp \( \infty \).

**Proof:** Since \( \Theta_K(z) \) is a modular form of weight 1 for \( \Gamma_0(3) \), it can be easily seen that \( \Theta(Dz) \) is a modular form of weight 1 for \( \Gamma(3D) \). Furthermore, their ratio is modular function for \( \Gamma_0(3D) \). We check this below. For \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(3D) \), we have:

\[
 f(\gamma z) = \frac{\Theta\left( \begin{array}{cc} D & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) z}{\Theta\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) z} = \frac{\Theta\left( \begin{array}{cc} a & b D \\ c & d \end{array} \right)(Dz)}{\Theta\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) z} = \frac{(cz+d)\Theta(Dz)}{(cz+d)\Theta(z)} = f(z)
\]

To find the Fourier expansion of \( f(z) \) at \( \infty \), it is enough to write the Fourier expansions of \( \Theta(Dz) \) and \( \Theta(z) \):

\[
 \frac{\Theta(Dz)}{\Theta(z)} = 1 + \sum_{N \geq 1} c(N)q^{ND} = \sum_{M \geq 0} a_M q^M
\]

We can compute the Fourier coefficients explicitly from the equality:

\[
 1 + \sum_{N \geq 1} c(N)q^{ND} = (1 + \sum_{N \geq 1} c(N)q^N)(\sum_{M \geq 0} a_M q^M)
\]

Note that we have \( a_0 = 1 \) and \( a_M = -a_{M-1}c(1) - a_{M-2}c(2) - \cdots - a_1c(M-1) - a_0c(M)\) if \( D \nmid M \) and \( a_M = c(M/D) - a_{M-1}c(1) - a_{M-2}c(2) - \cdots - a_1c(M-1) - a_0c(M) \) if \( D \mid M \). By induction, since \( c(N) \in \mathbb{Z} \), we get all the coefficients \( a_M \in \mathbb{Z} \).

#### 4.1.1 \( f(\omega) \) is in the ring class field \( H_{3D} \).

From CM-theory, we have that if \( f \in F_{3D} \) and \( \tau \) generating \( \mathcal{O}_K \), we have \( f(\tau) \in H_{3D,\mathcal{O}_K} \) the ray class field of conductor 3D. We claim that \( f(\omega) \in H_{3D} \). Recall that we have \( \text{Gal}(K^{ab}/H_{3D}) \cong U(3D)\backslash \mathbb{A}_{K,f}^x/K^x \). Thus in order to show that \( f(\omega) \in H_{3D} \), we need to check that \( f(\omega) \) is invariant under the action of \( U(3D) \).

**Lemma 4.2.** For \( \omega = \frac{-1 + \sqrt{-3}}{2} \) and \( f(z) = \frac{\Theta_K(Dz)}{\Theta_K(z)} \) we have \( f(\omega) \in H_{3D} \).
Proof: In order for \( f(\omega) \in H_{3D} \), we need to show that it is invariant under \( \text{Gal}(K^{ab}/H_{3D}) \).

Using Shimura reciprocity law, we need to show:

\[ f(\omega) = f^{r_\omega(s)}(\omega), \]

for all \( s \in K^x U(3D) \). From Remark 4.1, the action of \( K^x \) is trivial. Thus it is enough to show the result for all elements \( l = (A_p + B_p \omega)_p \in U(3D) \). By the definition of \( U(3D) \), this implies that \( A_p + B_p \omega \in (\mathbb{Z}_p[\omega])^x \) for all \( p \) and \( A_3 \equiv 1 \mod 3, B_3 \equiv 1 \mod 3, B_p = 0 \mod D \) for all \( p|D \). Since the action for \( p \nmid 3D \) is trivial, \( s \) has the same action \( l_D = (A_p + B_p \omega)_{p|3D} \in U(3D) \). Moreover, this has the same action as \( l_0 = (A + B \omega)_{p|3D} \), where \( A + B \omega \in \mathcal{O}_K \) and \( A \equiv A_p \mod 3DZ_p \) and \( B \equiv B_p \mod 3DZ_p \) for all \( p|3D \).

Note further that we can pick \( A, B \) such that \( (A + B \omega) \) generates a primitive ideal \( \mathcal{A} \) in \( \mathcal{O}_K \). Moreover, from above we have \( 3D|B \) and \( A \equiv 1 \mod 3 \). Recall that we can rewrite any primitive ideal in the form \( \mathcal{A} = [a, \frac{-b + \sqrt{-3}}{2}]_\mathbb{Z} \), where \( a = N \mathcal{A} \) and \( b^2 \equiv -3 \mod 4a \). Then the generator is \( A + B \omega = ta + s\frac{-b + \sqrt{-3}}{2} \), where \( t, s \in \mathbb{Z}, 3D|s \).

Now observe that \( f(\omega) = f(\tau) \), where \( \tau = \frac{-b + \sqrt{-3}}{2} \), thus from Shimura reciprocity law, we have:

\[ (f(\tau))^{\sigma_{l^{-1}}} = f^{r_\tau(l)}(\omega). \]

Here \( r_\tau(l) = \begin{pmatrix} A_p - bB_p - B_p c \\ B_p \end{pmatrix} \) and \( r_\tau(l) \) has the same action as \( r_\tau(l_0) \), where \( l_0 = (A + B \omega)_{p|3D} \) and \( A + B \omega = ta + s\frac{-b + \sqrt{-3}}{2} \). Then we need to compute the action of:

\[ (f(\tau))^{\sigma_{l_0}} = f^{r_\tau(l_0)}(\tau). \]

Note that \( r_\tau(l_0) = \begin{pmatrix} ta - sb - sc/a \\ ta \end{pmatrix}_{p|3D} \), where \( c = \frac{b^2 + 3}{4} \). Then we can rewrite the action of \( r_\tau(l_0) \):

\[ f^{r_\tau(l_0)}(\tau) = f^{\begin{pmatrix} ta - sb - sc/a \\ ta \end{pmatrix}_{3D}}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = f^{\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{p|3D}}(\begin{pmatrix} ta - sb - sc/a \\ ta \end{pmatrix}) \]

Since \( a|c \), the matrix \( \begin{pmatrix} ta - sb - sc/a \\ ta \end{pmatrix} \in SL_2(\mathbb{Z}) \) and we can rewrite:

\[ f((ta - sb - sc/a \ t)z) = \frac{\Theta_K \left( \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} (ta - sb - sc/a \ t)z \right)}{\Theta_K \left( \begin{pmatrix} ta - sb - sc/a \\ t \end{pmatrix}z \right)} = \frac{\Theta_K \left( \begin{pmatrix} ta - sb - scD/a \ t \\ s \end{pmatrix}z \right)}{\Theta_K \left( \begin{pmatrix} ta - sb - sc/a \\ t \end{pmatrix}z \right)} \]

Note that since \( 3D|s \), we actually have \( \begin{pmatrix} ta - sb - scD/a \\ s \end{pmatrix}, \begin{pmatrix} ta - sb - sc/a \\ t \end{pmatrix} \in \Gamma_0(3) \) and we can apply the properties of the modular form \( \Theta_K \):

\[ \frac{\Theta_K \left( \begin{pmatrix} ta - sb - scD/a \\ t \end{pmatrix}z \right)}{\Theta_K \left( \begin{pmatrix} ta - sb - sc/a \\ t \end{pmatrix}z \right)} = (sz + t)^{-1} \Theta_K (Dz) = (sz + t)^{-1} \Theta_K (z) = f(z). \]

Finally, note that since \( (a, 3D) = 1 \) and \( f \) has rational coefficients, the action of \( (\frac{1}{a}, 0)_{p|3D} \) is trivial. This finishes the proof that \( f(\omega) \) is invariant under the Galois action coming from \( U(3D) \), thus \( f(\omega) \in H_{3D} \).
Lemma 4.3. Let its Fourier expansion. Let $A$ be a primitive ideal prime to $3D$. For $\tau_1 = \frac{-b+\sqrt{-3}}{2}$, let $\mathcal{O}_D = \mathbb{Z} + D\tau\mathbb{Z}$.

4.1.2 Galois conjugates of $f(\omega)$.

Let $A = \left[ a, \frac{-b+\sqrt{-3}}{2} \right]_\mathbb{Z}$ be a primitive ideal prime to $3D$. For $\tau_1 = \frac{-b+\sqrt{-3}}{2}$ be a CM point and let $\mathcal{A} = \left[ a, \frac{-b+\sqrt{-3}}{2} \right]$ be a primitive ideal prime to $N$. Then we have the Galois action:

$$f(\tau)^{\sigma_A^{-1}} = f(\tau/a)$$

Proof: From Shimura reciprocity (11), we have:

$$f(\tau)^{\sigma_A^{-1}} = f^{g_{\tau}(A)}(\tau).$$

Note that the minimum polynomial of $\tau$ is $p_{\tau}(X) = X^2 + bX + \frac{b^2 + 3}{4}$. Now let $\alpha = ta + \frac{s-b+\sqrt{-3}}{2} = ta + s\tau$ be a generator of $A$. Then we have $g_{\tau}(A) = \left( \frac{ta-sb - \frac{b^2+3}{4} ta}{s} \right)_p$. We can rewrite the matrix in the form:

$$g_{\tau}(A) = \left( \frac{ta-sb - \frac{b^2+3}{4} ta}{s} \right)_p \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{p|a}$$

As $\left( \frac{ta-sb - \frac{b^2+3}{4} ta}{s} \right)_p \in \text{SL}_2(\mathbb{Z}_p)$ for $p \nmid ND$, it has a trivial action. Then:

$$f^{g_{\tau}(A)}(\tau) = f^{\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{p|a}}(\tau)$$

We rewrite the matrix $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{p|a} = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{p|a} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{Q}$, where $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{p|a} \in \text{GL}_2(\mathbb{Z})$ and $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{Q} \in \text{GL}_2(\mathbb{Q})^\times$.

Note that the action of $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{p|a}$ is only given by $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{p|NM}$. However, since $f$ has rational Fourier coefficients in its Fourier expansion, this action is trivial. Thus we are left with:

$$f^{\sigma_{\tau}(A)}(\tau) = f^{\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{a}}(\tau)$$

This is just $f^{g_{\tau}(A)}(\tau) = f(\tau/a)$.

Lemma 4.4. Take the primitive ideals $A = \left[ a, \frac{-b+\sqrt{-3}}{2} \right]_\mathbb{Z}$ to be the representatives of the ring class field $H_3$ such that all norms $N_{\mathbb{Q}}A$ are relatively prime to each other and $b^2 \equiv -3 \mod 4a$ for all the $a = N_{\mathbb{Q}}A$ chosen.

Then the only Galois conjugates of $f(\omega) = \frac{\Theta_K(D\omega)}{\Theta_K(\omega)}$ are the following:

$$\left( \frac{\Theta_K(D\omega)}{\Theta_K(\omega)} \right)^{\sigma_A^{-1}} = \frac{\Theta_K \left( \frac{D}{-b+\sqrt{-3}} \right)}{\Theta_K \left( \frac{-b+\sqrt{-3}}{2a} \right)}$$

30
Proof: Note that \( \Theta_K(D\omega) = \Theta_K(D^{\frac{b+\sqrt{-3}}{2}}) \) and apply lemma 4.3 to \( \tau = \frac{b+\sqrt{-3}}{2} \) and 
\( f(z) = \frac{\Theta_K(Dz)}{\Theta_K(z)} \). These are the only Galois conjugates from Lemma ??.

5 Writing \( S_D \) as a square.

In this section we will show the following result:

**Theorem 5.1.** For \( D = \prod_{p_i \equiv 1 \mod 3} p_i^{e_i} \), let \( \tau \equiv -3 \mod 12D^2 \). Moreover, let \( b^* \equiv b^{-1} \mod D \).

Let \( H_0 \) be the ray class field of conductor \( 3D \) and let \( \mathcal{H}_0 \subset H_0 \) be the subfield of \( H_0 \) that is the fixed field of \( G_0 = \{ r \in (\mathbb{Z}/D\mathbb{Z})^\times, r \equiv 1 \mod 6 : A_r^c = (1 + b^*(1-r)\frac{b+\sqrt{-3}}{2}) \} \). Then we have

\[ S_D = |\text{Tr}_{H_0/H_0}(f_1(\tau)D^{2/3})|^2 \]

and \( S_D \in \mathbb{Z} \).

The main tool in proving Theorem 5.1 is a Factorization Formula of Rodriguez-Villegas and Zagier [13]. We will apply the Factorization Formula 12 to the formula for the L-function \( L(E_D, 1) \) in Theorem ??.

5.1 Factorization Formula

We recall the version of Factorization Formula ([13], Theorem, page 7) simplified to the case of \( \alpha = p = 0 \):

**Theorem 5.2.** (Factorization formula.) For \( a \in \mathbb{Z}_{>0}, \mu, \nu \in \mathbb{Q}, z = x + yi \in \mathbb{C} \) and \( Q_z(m, n) = \frac{|mz-n|^2}{2y} \), we have:

\[ \sum_{m, n \in \mathbb{Z}} e^{2\pi i (\mu m + \nu n)} e^{\pi i mn - \frac{|mz-n|^2}{2y}} = \sqrt{2ay} \theta \left[ \begin{array}{c} a \mu \\ \nu \end{array} \right] (a^{-1}z) \cdot \theta \left[ \begin{array}{c} \mu \\ -a\nu \end{array} \right] (-az), \]

where \( \theta \left[ \begin{array}{c} \mu \\ \nu \end{array} \right] (z) = \sum_{n \in \mathbb{Z}+\mu} e^{\pi n^2 z + 2\pi in} v \) is a theta function of half integral weight.

Using the formula above, we will prove the following Proposition:

**Proposition 5.1.** For \( a \equiv 1 \mod 6, D \equiv 1 \mod 6 \) and \( b^2 \equiv -3 \mod 4D^2a^2a_1, b \equiv 1 \mod 16 \), we have:

\[ \frac{3}{2} \Theta \left( D^{\frac{b+\sqrt{-3}}{2a}} \right) - \frac{1}{2} \Theta \left( D^{\frac{b+\sqrt{-3}}{6a}} \right) = \sum_{r \in \mathbb{Z}/D\mathbb{Z}} \frac{\sqrt[4]{3}}{D\sqrt{a_1}} e^{-\frac{\pi i(a-1)r}{6}} \theta_{ar} \left( \frac{b+\sqrt{-3}}{2a^2a_1} \right) \theta_r \left( \frac{b+\sqrt{-3}}{2a_1} \right), \]

\[ (13) \]
where \( \theta_s(z) = \sum_{n \in \mathbb{Z}} e^{\pi(n+s/D-1/6)z}(-1)^n \) is a theta function of weight 1/2 for s non-negative integer. Here we use the notation \( r \in \mathbb{Z}/D \mathbb{Z} \) to mean any representatives \( r \) for the residues mod \( D \).

We start by applying the Factorization Formula (12) several times for \( \mu := \frac{\nu + \theta}{D} \), \( 0 \leq r \leq D-1 \) and \( z := z/D \). Summing up the formulas, we are going to get:

**Lemma 5.1.** We have the following factorization lemma:

\[
\sum_{r \in \mathbb{Z}/D \mathbb{Z}} \frac{\sqrt{2 \nu y}}{\sqrt{D}} \theta \left[ a(\mu + r)/D \nu \right] \left( D \frac{z}{a} \right) \theta \left[ (\mu + r)/D - a \nu \right] (-aDz) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\nu + n\mu)} e^{\pi(mni - \frac{\nu m^2}{2Y})\frac{D}{n^2}}
\]

**Proof:** Plugging in \( \mu := \frac{\nu + \theta}{D} \), \( z := z/D \) in (12), we get:

\[
\sqrt{2 \nu y} \theta \left[ a(\mu + r)/D \nu \right] \left( D \frac{z}{a} \right) \theta \left[ (\mu + r)/D - a \nu \right] (-aDz) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\nu + n(\mu + r)/D)} e^{\pi(mni - \frac{\nu m^2}{2Y})\frac{1}{n}}
\]

We sum from \( r \) in \( \mathbb{Z}/D \mathbb{Z} \):

\[
\sum_{r \in \mathbb{Z}/D \mathbb{Z}} \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\nu + n(\mu + r)/D)} e^{\pi(mni - \frac{\nu m^2}{2Y})\frac{1}{n}} = \sum_{m,n \in \mathbb{Z}} \sum_{r \in \mathbb{Z}/D \mathbb{Z}} e^{2\pi i (m\nu + n(\mu + r)/D)} e^{\pi(mni - \frac{\nu m^2}{2Y})\frac{1}{n}}
\]

Note that the LHS can be rewritten as \( \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\nu + n(\mu)/D)} e^{\pi(mni - \frac{\nu m^2}{2Y})\frac{1}{n}} \sum_{r \in \mathbb{Z}/D \mathbb{Z}} e^{2\pi i rD/D} \)
and note further that:

\[
\sum_{r \in \mathbb{Z}/D \mathbb{Z}} e^{2\pi i rD/D} = \sum_{r=0}^{D-1} e^{2\pi i rD/D} = \begin{cases} 0, & \text{for } D \nmid n \\ D, & \text{for } D \mid n \end{cases}
\]

Thus we are only summing over the \( n \)'s that are multiples of \( D \):

\[
\sum_{r \in \mathbb{Z}/D \mathbb{Z}} \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m\nu + n(\mu + r)/D)} e^{\pi(mni - \frac{\nu m^2}{2Y})\frac{1}{n}} = D \sum_{m,n' \in \mathbb{Z}} e^{2\pi i (mn' + m(\nu + r)/D')} e^{\pi(mn'i - \frac{\nu m'^2}{2Y/D'})\frac{1}{n'}}
\]

Going back to our initial equality, we can replace \( z' = z/D \) and get:

\[
\sum_{r \in \mathbb{Z}/D \mathbb{Z}} \theta \left[ a(\mu + r)/D \nu \right] \left( D \frac{z'}{a} \right) \theta \left[ (\mu + r)/D - a \nu \right] (-aDz') = D \sum_{m,n \in \mathbb{Z}} e^{2\pi i (mn + m(\mu + r))} e^{\pi(mni - \frac{\nu m'^2}{2Y/D'})\frac{1}{n'}}
\]

32
Lemma 5.2. For $b \equiv 1 \mod 16$, $b \equiv 0 \mod 3$, $b^2 \equiv -3 \mod 4a^2a_1D$, we have:

$$e^{2\pi i (m/2+n/2) \frac{nD_{aa+1-m+b+b\sqrt{-3}}}{D_{aa+1} \sqrt{3}}} = e^{2\pi i \frac{\left| nD_{aa+1-m+b+b\sqrt{-3}} \right|^2}{D_{aa+1} \sqrt{3}} D_{\frac{b+b\sqrt{-3}}{6}}}$$
Proof: We only need to show that:
\[ 2\pi i \left( \frac{m}{2} + \frac{n}{2} + \frac{Dmn}{2a} \right) = -2\pi i \frac{|naa_1D - m \frac{b+\sqrt{b^2-4}}{2}|^2}{aa_1D} \frac{D}{b} \frac{b}{6a} \mod 2\pi i \mathbb{Z}. \]
After dividing by \(2\pi i\), we compute the RHS of the identity:
\[ \frac{|naa_1D - m \frac{b+\sqrt{b^2-4}}{2}|^2}{aa_1D} \frac{D}{b} \frac{b}{6a} = \left( Dm^2 b(b^2 + 3) \frac{2b^2mn}{24a^2a_1} - D \frac{b^2mn}{6a} + \frac{Dba_1n^2}{6} \right) \]
Thus our claim turns into:
\[ \left( \frac{m}{2} + \frac{n}{2} + \frac{Dmn}{2a} \right) = \left( Dm^2 b(b^2 + 3) \frac{2b^2mn}{24a^2a_1} - D \frac{b^2mn}{6a} + \frac{Dba_1n^2}{6} \right) \mod \mathbb{Z} \]
Equivalently:
\[ \frac{m}{2} + \frac{n}{2} = \left( D \frac{m^2 b (b^2 + 3)}{2} \frac{3}{4a^2a_1} - D \frac{(b^2 + 3)mn}{6a} + \frac{n^2 b}{2} \frac{3}{Da_1} \right) \mod \mathbb{Z} \]
We have \( b^2 \equiv -3 \mod 4ana_1^2 \), \( b \equiv 1 \mod 16 \), \( b \equiv 0 \mod 3 \). Note that this implies that \( b \) is odd and that \( b^2 + 3 \equiv 4 \mod 8 \), as well as \( b^2 + 3 \equiv 0 \mod 3 \). Then, since \( a, a_1, D \) are odd, we get:
- \( m/2 \equiv m^2/2 \equiv D \frac{m^2 b (b^2 + 3)}{2} \frac{3}{4a^2a_1} \mod \mathbb{Z} \)
- \( n/2 \equiv n^2/2 \equiv \frac{n^2 b}{2} \frac{3}{Da_1} \mod \mathbb{Z} \)
- \( -D \frac{(b^2 + 3)mn}{6a} \in \mathbb{Z} \)
This finishes the proof.

Lemma 5.3. Under the same conditions as above we have:
\[ \sum_{m,n \in \mathbb{Z}} e^{2\pi i/3} e^{2\pi i \frac{|m-b+\sqrt{b^2-4}}{2a_1} + naa_1|^2} \frac{z}{3} = \frac{3}{2} \Theta(3z) - \frac{1}{2} \Theta(z), \]
where \( z \in \mathbb{H} \), \( A_1 = [aa_1, \frac{-b+\sqrt{b^2-4}}{2}] \) and \( b \equiv 0 \mod 3 \), \( b^2 \equiv -3 \mod 4ana_1^2 \).

Proof: Note first that by changing \( m \rightarrow -m \) and \( -m \cdot \frac{-b+\sqrt{b^2-4}}{2} + naa_1 \) to its conjugate, we have:
\[ \sum_{m,n \in \mathbb{Z}} e^{2\pi i/3} e^{2\pi i \frac{|-m-b+\sqrt{b^2-4}}{2a_1} + naa_1|^2} \frac{z}{3} = \sum_{m,n \in \mathbb{Z}} e^{2\pi i/3} e^{2\pi i \frac{|b+\sqrt{b^2-4}}{2a_1} + naa_1|^2} \frac{z}{3}. \]
We can split the sum in three terms, depending on \( n \mod 3 \):
\[ \sum_{m,3|n \in \mathbb{Z}} e^{2\pi i \frac{|b+\sqrt{b^2-4}}{2a_1} + naa_1|^2} + \omega \sum_{m,n \in \mathbb{Z}, n \equiv 1 \mod 3} e^{2\pi i \frac{|b+\sqrt{b^2-4}}{2a_1} + naa_1|^2} z + \omega^2 \sum_{m,n \in \mathbb{Z}, n \equiv 2 \mod 3} e^{2\pi i \frac{|b+\sqrt{b^2-4}}{2a_1} + naa_1|^2} z \]
Note that the first term equals:
\[ \sum_{m,n \in \mathbb{Z}} e^{2\pi i \frac{|b+\sqrt{b^2-4}}{2a_1} + naa_1|^2} \frac{z}{3} = \Theta_K(3z). \]
Also note that the two terms $\sum_{m,n \in \mathbb{Z}, n \equiv 1 \mod 3} e^{2\pi i \frac{|m b + \sqrt{-3} + n a_1|}{a_1}} z$ and $\sum_{m,n \in \mathbb{Z}, n \equiv 2 \mod 3} e^{2\pi i \frac{|m b + \sqrt{-3} + n a_1|}{a_1}} z$ equal each other, by changing in the latter $n \rightarrow -n$ and $m \rightarrow -m$. Thus we got so far:

$$\Theta(3z) + (\omega + \omega^2) \sum_{m,n \in \mathbb{Z}, n \equiv 1 \mod 3} e^{2\pi i \frac{|m b + \sqrt{-3} + n a_1|}{a_1}} z$$

Furthermore, we have:

$$\sum_{m,n \in \mathbb{Z}, n \equiv 1 \mod 3} e^{2\pi i \frac{|m b + \sqrt{-3} + n a_1|}{a_1}} z = \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (n,3) = 1} e^{2\pi i \frac{|m b + \sqrt{-3} + n a_1|}{a_1}} z$$

Finally, this is just:

$$\frac{1}{2} \sum_{m,n \in \mathbb{Z}} e^{2\pi i \frac{|m b + \sqrt{-3} + n a_1|}{a_1}} z = 2 \sum_{m,n \in \mathbb{Z}} e^{2\pi i \frac{|m b + \sqrt{-3} + 3n a_1|}{3a_1}} 3z = \frac{1}{2} (\Theta(z) - \Theta(3z))$$

Finally, we get:

$$\sum_{m,n \in \mathbb{Z}} e^{2\pi i / 3} e^{2\pi i \frac{|m b + \sqrt{-3} + n a_1|}{a_1}} z = \Theta(3z) - \frac{1}{2} (\Theta(z) - \Theta(3z)) = \frac{3}{2} \Theta(3z) - \frac{1}{2} \Theta(z).$$

From the previous two lemmas, we get the following corollary:

**Corollary 5.2.** Under the above conditions, we have:

$$\sum_{m,n \in \mathbb{Z}} e^{2\pi i (m/2 - n/6)} e^{\pi (m n i - \frac{|na_1 D - m b + \sqrt{-3}|^2}{a_1 D \sqrt{3}})} \frac{D}{a_1} = \frac{3}{2} \Theta \left( D \frac{-b + \sqrt{-3}}{2a} \right) - \frac{1}{2} \Theta \left( \frac{D - b + \sqrt{-3}}{6a} \right)$$

**Proof:** Note that we can rewrite the LHS in the form:

$$\sum_{m,n \in \mathbb{Z}} e^{2\pi i (m/2 - n/6)} e^{\pi (m n i - \frac{|na_1 D - m b + \sqrt{-3}|^2}{a_1 D \sqrt{3}})} \frac{D}{a_1} = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m/2 - n/2 + n/3)} e^{\pi (m n i - \frac{|na_1 D - m b + \sqrt{-3}|^2}{a_1 D \sqrt{3}})} \frac{D}{a_1}$$

Then, from Lemma 5.2, we have:

$$\sum_{m,n \in \mathbb{Z}} e^{2\pi i (m/2 - n/6)} e^{\pi (m n i - \frac{|na_1 D - m b + \sqrt{-3}|^2}{a_1 D \sqrt{3}})} \frac{D}{a_1} = \sum_{m,n \in \mathbb{Z}} e^{2\pi i / 3} e^{\pi (m n i - \frac{|na_1 D - m b + \sqrt{-3}|^2}{a_1 D \sqrt{3}}) D - b + \sqrt{-3}}$$

Now apply Lemma 5.3 for $z = D \frac{-b + \sqrt{-3}}{6a}$, we get:

$$\sum_{m,n \in \mathbb{Z}} e^{2\pi i / 3} e^{\frac{|na_1 D - m b + \sqrt{-3}|^2}{a_1 D - b + \sqrt{-3}}} D - b + \sqrt{-3} = \frac{3}{2} \Theta \left( D \frac{-b + \sqrt{-3}}{2a} \right) - \frac{1}{2} \Theta \left( \frac{D - b + \sqrt{-3}}{6a} \right)$$

Finally, from (15) and Corollary 5.2 we get the result of Proposition 5.1:
\[
\frac{3}{2} \Theta \left( D - \frac{b + \sqrt{-3}}{2a} \right) - \frac{1}{2} \Theta \left( D - \frac{b + \sqrt{-3}}{6a} \right) = \frac{\sqrt{3}}{\sqrt{a_1}} \sum_{r \in \mathbb{Z} / D \mathbb{Z}} e^{\pi i (a-1)/6} \theta_{ar} \left( -\frac{b + \sqrt{-3}}{2a^2 a_1} \right) \theta_r \left( \frac{b + \sqrt{-3}}{2a_1} \right).
\]

A particular case of Proposition 5.1 is going to be the following result:

**Corollary 5.3.** For \( b^2 \equiv -3 \mod 2a^2a_1, \ b \equiv 1 \mod 16, \) we have:

\[
\frac{3}{2} \Theta \left( -\frac{b + \sqrt{-3}}{2a} \right) - \frac{1}{2} \Theta \left( -\frac{b + \sqrt{-3}}{6a} \right) = \frac{\sqrt{3}}{\sqrt{a_1}} e^{\pi i (a-1)/6} \theta_0 \left( -\frac{b + \sqrt{-3}}{2a^2 a_1} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2a_1} \right),
\]

where \( \theta_0(z) = \sum_{n \in \mathbb{Z}} e^{\pi i (n-1/6)^2 z (-1)^n}. \)

**Proof:** Applying the Proposition 5.1 for \( D = 1 \) we get:

\[
\frac{3}{2} \Theta \left( -\frac{b + \sqrt{-3}}{2a} \right) - \frac{1}{2} \Theta \left( -\frac{b + \sqrt{-3}}{6a} \right) = \frac{\sqrt{3}}{\sqrt{a_1}} e^{\pi i (a-1)/6} \theta_0 \left( -\frac{b + \sqrt{-3}}{2a^2 a_1} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2a_1} \right).
\]

Furthermore, using the result from Appendix A, Lemma 6.3 that \( \Theta \left( -\frac{b + \sqrt{-3}}{6a} \right) = 0, \) we get the result of the Corollary.

We can rewrite further the ratios:

**Corollary 5.4. Under the same conditions as above, we have:**

\[
\frac{\Theta \left( D - \frac{b + \sqrt{-3}}{2a} \right) \Theta \left( D - \frac{b - \sqrt{-3}}{2a} \right)}{\Theta \left( -\frac{b + \sqrt{-3}}{2a} \right) \Theta \left( -\frac{b - \sqrt{-3}}{2a} \right)} = \frac{1}{3} \sum_{r \in \mathbb{Z} / D \mathbb{Z}} \theta_{ar} \left( -\frac{b + \sqrt{-3}}{2a^2 a_1} \right) \theta_r \left( \frac{b + \sqrt{-3}}{2a_1} \right).
\]

**Proof:** We begin by writing the ratio of the formulas in Proposition 5.1 and Corollary 5.3:

\[
\frac{\Theta \left( D - \frac{b + \sqrt{-3}}{2a} \right) \Theta \left( D - \frac{b - \sqrt{-3}}{2a} \right)}{\Theta \left( -\frac{b + \sqrt{-3}}{2a} \right) \Theta \left( -\frac{b - \sqrt{-3}}{2a} \right)} = \frac{\frac{\sqrt{3}}{\sqrt{a_1}} e^{\pi i (a-1)/6}}{\frac{\sqrt{3}}{\sqrt{a_1}} e^{\pi i (a-1)/6}} \sum_{r \in \mathbb{Z} / D \mathbb{Z}} \theta_{ar} \left( -\frac{b + \sqrt{-3}}{2a^2 a_1} \right) \theta_r \left( \frac{b + \sqrt{-3}}{2a_1} \right).
\]

Simplifying, we get the result of the Corollary.

**Remark 5.1.** Note that \( \theta_0(z) = \eta(z/3), \) where \( \eta \) is the Dedekind eta function.

### 5.2 Ratios of \( \theta_r \) and \( \theta_0 \)

Now we will apply the Factorization Lemma once more to connect the theta functions \( \theta_r \) to the theta function \( \theta_0. \) We do this by applying the Factorization Formula (12) twice and comparing the results.

Note first that any primitive ideal \( A \) in \( \mathcal{O}_K \) prime to 6 has a generator \( (n_a a + m_a - \frac{b + \sqrt{-3}}{2}) \) such that \( a = \text{Nm}(A), \ b^2 \equiv -3 \mod 12a \) and \( n_a \equiv 1 \mod 3. \) Moreover, note that \( a = n_a^2 a^2 + m_a^2 \frac{b + \sqrt{-3}}{2} - m_a n_a b, \) thus \( m_a n_a b \equiv 1 \mod a, \) as \( a | (b^2 + 3)/4. \)

Using this notation, we have:
Lemma 5.4. For $b \equiv 0 \mod 3$, $b^2 \equiv -3 \mod 4D^2a'$, $n_{a'} \equiv 1 \mod 3$, we have:

$$\theta_r \left(-\frac{b + \sqrt{-3}}{2aa'}\right) \theta_0 \left(\frac{b + \sqrt{-3}}{2D^2aa'}\right) = \frac{1}{\sqrt{a'}} \theta_{n_{a'}r} \left(-\frac{b + \sqrt{-3}}{2a}\right) \theta_0 \left(\frac{b + \sqrt{-3}}{2D^2a}\right)$$

Proof: We write the generator of $\mathcal{A}'$ in the form $(n_{a'}a' + m_{a'} - \frac{b + \sqrt{-3}}{2})$, where $b^2 \equiv -3 \mod 4a'D^2$. Moreover, we can pick $n_{a'} \equiv 1 \mod 3$. Then, using the Factorization Formula (12) for $\mu = -\frac{1}{6} + \frac{r_D}{D}$, $\nu = \frac{1}{2}$, $a := D$ and $z = \frac{-b + \sqrt{-3}}{2a'D}$, we have:

$$\sqrt[3]{\frac{\pi}{2}} \theta \left[\frac{-\frac{1}{6} + \frac{r_D}{D}}{D/2}\right] \left(D \frac{b + \sqrt{-3}}{2aa'D}'\right) \theta \left[-\frac{D}{6} - \frac{1}{2}\right] \left(\frac{b + \sqrt{-3}}{2D^2aa'}\right) = \sum_{m,n} e^{2\pi i \frac{6m}{a'} e^{2\pi i \frac{2aD + \frac{1}{2}}{2}} e^{2\pi i \left[\frac{m \frac{-b + \sqrt{-3} + m + naD}{D}^2 - \frac{b + \sqrt{-3}}{6D}}{2}\right]}}.$$ 

Thus we got:

$$\sqrt[3]{\frac{\pi}{2}} \theta_r \left(D \frac{b + \sqrt{-3}}{2aa'D}'\right) \theta_0 \left(\frac{b + \sqrt{-3}}{2D^2aa'}\right) = \sum_{m,n} e^{2\pi i \frac{6m}{a'} e^{2\pi i \frac{2aD + \frac{1}{2}}{2}} e^{2\pi i \left[\frac{m \frac{-b + \sqrt{-3} + m + naD}{D}^2 - \frac{b + \sqrt{-3}}{6D}}{2}\right]}}.$$ (16)

Note that if we write any element of $\mathcal{A}'$, we can write it as an element of $\mathcal{A}D$ multiplied by the generator of $\mathcal{A}'$. Thus if we write an element of $\mathcal{A}D$, in the form $m \frac{-b + \sqrt{-3} + naaD}{2} + naaD$, it is going to equal an element $m_0 \frac{-b + \sqrt{-3}}{2} + naaD \in \mathcal{A}D$ times the generator $m_{a'} \frac{-b + \sqrt{-3} + naaD}{2} + naa'D$ of $\mathcal{A}'$:

$$m \frac{-b + \sqrt{-3}}{2} + naa'D = (m_0 \frac{-b + \sqrt{-3}}{2} + naaD)(m_{a'} \frac{-b + \sqrt{-3}}{2} + naa'D).$$

This gives us:

$$\begin{cases} m = m_0m_{a'} + na_0a' - m_0m_{a'}b \\ n = na_0a' - m_0m_{a'} \frac{b^2 + 3}{4aaD} \end{cases}$$

Since $b^2 + 3 \equiv 0 \mod 4D^2$, it implies that $n \equiv na_0 \mod D$. Then we have:

$$\sum_{m,n} e^{2\pi i \frac{6m}{a'} e^{2\pi i \frac{2aD + \frac{1}{2}}{2}} e^{2\pi i \left[\frac{m \frac{-b + \sqrt{-3} + m + naD}{D}^2 - \frac{b + \sqrt{-3}}{6D}}{2}\right]}} = \sum_{m_0,n_0} e^{2\pi i \frac{n_0a_{a'} r \theta_{n_{a'}r} \left(-\frac{b + \sqrt{-3}}{2a}\right) \theta_0 \left(\frac{b + \sqrt{-3}}{2D^2a}\right)}} e^{2\pi i \frac{m_0 \frac{-b + \sqrt{-3} + m_{a'}D}{D}^2 - \frac{b + \sqrt{-3}}{6D}}{2}}.$$ 

Since we picked $n_{a'} \equiv 1 \mod 3$, this is the same as:

Then applying the Factorization Formula (12) again for $\mu := -\frac{1}{6} + \frac{r_{a'} r}{D}$, $\nu := \frac{1}{2}$, $a := D$ and
Thus:

\[ z := \frac{-b + \sqrt{-3}}{2aD}, \]

we get:

\[
\sum_{m,n} e^{2\pi i m \frac{n_0}{D}} e^{2\pi i \frac{m-b+n_0 D^2}{2\sqrt{a}}} = \left( \frac{-b + \sqrt{-3}}{2\sqrt{a}} \right) \left( \frac{-b + \sqrt{-3}}{2D^2a} \right).
\]

Moreover, on the RHS we have the theta functions \( \theta \left[ \frac{1}{b} + \frac{n r}{D} \right] (z) = e^{-\pi i / 6} \theta_0 (z) \) and \( \theta \left[ \frac{1}{b} + \frac{n r}{D} \right] (z) = e^{\pi i / 6} \theta_0 (z) \). Thus we can rewrite the equality as:

\[
\sum_{m,n} e^{2\pi i m \frac{n_0}{D}} e^{2\pi i \frac{m-b+n_0 D^2}{2\sqrt{a}}} = \left( \frac{-b + \sqrt{-3}}{2\sqrt{a}} \right) \left( \frac{-b + \sqrt{-3}}{2D^2a} \right).
\]

Comparing the two relations (16) and (17), we get:

\[
\frac{1}{\sqrt{a}} e^{\pi i r \theta_r \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)} \theta_0 \left( \frac{b + \sqrt{-3}}{2D^2} \right) = e^{\pi i r \theta_{n_0} \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)} \theta_0 \left( \frac{b + \sqrt{-3}}{2D^2} \right)
\]

**Lemma 5.5.** Under the same conditions as above, we have:

\[
\frac{e^{\pi i r \theta_r \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)}}{\theta_0 \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)} = \frac{e^{\pi i r \theta_{n_0} \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)}}{\theta_0 \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)}
\]

**Proof:** Note that from Corollary 5.3, we have \( \frac{3}{2} \Theta \left( \frac{-b + \sqrt{-3}}{2aD^2} \right) = \frac{3\pi}{D\sqrt{a}} \theta_0 \left( \frac{-b + \sqrt{-3}}{2aD^2} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2aD^2} \right) \).

Moreover, we also have from the same corollary that \( \frac{3}{2} \Theta \left( \frac{-b + \sqrt{-3}}{2aD^2} \right) = \frac{3\pi}{D\sqrt{a}} \theta_0 \left( \frac{b + \sqrt{-3}}{2aD^2} \right) \theta_0 \left( \frac{-b + \sqrt{-3}}{2aD^2} \right) \),

thus:

\[
\frac{1}{\sqrt{a}} \theta_0 \left( \frac{-b + \sqrt{-3}}{2aD^2} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2aD^2} \right) = \theta_0 \left( \frac{-b + \sqrt{-3}}{2aD^2} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2aD^2} \right)
\]

Recall from the previous Lemma that we also have:

\[
\frac{1}{\sqrt{a}} e^{\pi i r \theta_r \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)} \theta_0 \left( \frac{b + \sqrt{-3}}{2aD^2} \right) = e^{\pi i r \theta_{n_0} \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)} \theta_0 \left( \frac{b + \sqrt{-3}}{2aD^2} \right)
\]

Dividing the two relations, we get exactly:

\[
\frac{e^{\pi i r \theta_r \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)}}{\theta_0 \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)} = \frac{e^{\pi i r \theta_{n_0} \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)}}{\theta_0 \left( \frac{-b + \sqrt{-3}}{2aD^2} \right)}
\]

38
5.3 Applying the factorization lemma to get a square

We would like to apply the factorization lemma for the formula in Theorem ?? for certain ideals that are representatives of the ring class field $\text{Cl}(\mathcal{O}_{3D})$. We will pick this ideals below.

5.3.1 Representatives of $\text{Cl}(\mathcal{O}_{3D})$

Recall that, using Cox [4], for the ideal class group of conductor $3D$, we have:

$$\text{Cl}(\mathcal{O}_{3D}) = (\mathcal{O}_{3D}/3D\mathcal{O}_K)^\times/(\mathbb{Z}/3D\mathbb{Z})^\times(\mathcal{O}_K^\times/\{\pm 1\})$$

Moreover, we can compute explicitly that for $D = \prod_{p_i \equiv 1 \mod 3} p_i$ we have $\text{Cl}(\mathcal{O}_{3D}) \cong (\mathbb{Z}/D\mathbb{Z})^\times$ which also gives us $\# \text{Cl}(\mathcal{O}_{3D}) = \phi(D)$, where $\phi$ is Euler’s totient function.

Furthermore, we are claiming that we can take as representatives of $\text{Cl}(\mathcal{O}_{3D})$ ideals with norm $\text{Nm} \mathcal{A}_k \equiv k \mod D$ for $k \in (\mathbb{Z}/D\mathbb{Z})^\times$. We construct these ideals in the following lemma:

**Lemma 5.6.** We can take as representatives of $\text{Cl}(\mathcal{O}_{3D})$ the ideals:

$$\mathcal{A}_k = \left( n_k a_k + m_k \frac{-b + \sqrt{-3}}{2} \right),$$

where $\text{Nm} \mathcal{A}_k = a_k \equiv k \mod D$ for $k \in (\mathbb{Z}/D\mathbb{Z})^\times$, $a_k \equiv 1 \mod 6$ and $n_k \equiv 1 \mod D$. We can pick such an ideal if we take $m_k \equiv b^{-1}(k + 1) \mod D$. We can further put the conditions $n_k, m_k \equiv 1 \mod 3$ to determine the ideal uniquely modulo $3D$.

**Proof:** Note first that two ideals $\mathcal{A}, \mathcal{B}$ are in the same class in $\text{Cl}(\mathcal{O}_{3D})$ if we can find generators $\alpha, \beta$ for $\mathcal{A}$ and $\mathcal{B}$, respectively, such that $\alpha \beta^{-1} \equiv m \mod 3D$, where $m$ is an integer prime to $3D$. Note that this implies $\alpha \beta^{-1} \equiv \pm 1 \mod 3$.

Let us assume that $\mathcal{A}_k$ and $\mathcal{A}_l$ are in the same class in $\text{Cl}(\mathcal{O}_{3D})$. Then we must have

$$\pm \omega^i \left( n_k a_k + m_k \frac{-b + \sqrt{-3}}{2} \right) \equiv \pm \omega^j R \left( n_l a_l + m_l \frac{-b + \sqrt{-3}}{2} \right) \mod 3D$$

for some $i, j$. Since we chose $n_k, m_k, n_l, m_l \equiv 1 \mod 3$ and $b$ is odd we actually have

$$n_k a_k + m_k \frac{-b + \sqrt{-3}}{2} \equiv n_l a_l + m_l \frac{-b + \sqrt{-3}}{2} \equiv \omega \mod 3,$$

which determines the choice of $\pm \omega^i = \pm \omega^j$ on both sides. We further need the condition:

$$n_k a_k + m_k \frac{-b + \sqrt{-3}}{2} \equiv R(n_l a_l + m_l \frac{-b + \sqrt{-3}}{2}) \mod D$$

Note that this is equivalent to:

$$k + b^{-1}(k + 1) \frac{-b + \sqrt{-3}}{2} \equiv R(l + b^{-1}(l + 1) \frac{-b + \sqrt{-3}}{2}) \mod D$$

Furthermore, this can be rewritten as:

$$\frac{kb + (k + 1)\sqrt{-3}}{2} \equiv R lb + (l + 1)\sqrt{-3} \mod D$$

This implies $k \equiv lR \mod D$ and $k + 1 \equiv lR + R \mod D$, thus $R \equiv 1 \mod D$ and $k \equiv l \mod D$.

Finally, we have $\#(\mathbb{Z}/D\mathbb{Z})^\times$ such ideals, all in different classes of $\text{Cl}(\mathcal{O}_{3D})$, thus we have representatives in every class of $\text{Cl}(\mathcal{O}_{3D})$.  

39
5.3.2 Using the factorization formula

We will pick representatives as in the above Lemma to rewrite the Proposition 5.1 and apply Corollary 5.4. We will denote by \( \{ s \in (\mathbb{Z}/D\mathbb{Z})^\times, s \equiv 1 \mod 6 \} \) the norms of the ideals chosen in Lemma 5.6. Furthermore, we are going to choose in Proposition 5.1 all \( r \) to be even. We will use the notation \( \{ r \in \mathbb{Z}/D\mathbb{Z}, r \text{ even} \} \) to express this.

**Lemma 5.7.** Picking representatives of \( s \in (\mathbb{Z}/D\mathbb{Z})^\times \) such that \( s \equiv 1 \mod 6 \) and \( r \in \mathbb{Z}/D\mathbb{Z} \) also such that \( r \equiv 0 \mod 2 \), we have

\[
\sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times} \sum_{s \equiv 1 \mod 6} \frac{\Theta \left( \frac{D-b+\sqrt{-3}}{2a_s} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{2a_s} \right)} \chi(A_s) = \sum_{r \in \mathbb{Z}/D\mathbb{Z}, r \text{ even}} \sum_{s \equiv 1 \mod 6} \frac{\theta_{sr} \left( \frac{-b+\sqrt{-3}}{2} \right)}{\theta_0 \left( \frac{-b+\sqrt{-3}}{2D^2} \right)} \chi(A_r) \frac{\theta_r \left( \frac{b+\sqrt{-3}}{2} \right)}{\theta_0 \left( \frac{b+\sqrt{-3}}{2D^2} \right)} \chi(A_r)
\]

**Proof:** We fix \( \phi(D) \) ideals \( A_k \) as in Lemma 5.6. Recall that we pick \( A_k \) such that \( \text{Nm} A_k = a_k \equiv k \mod D \) for \( k \in (\mathbb{Z}/D\mathbb{Z})^\times \), \( a_k \equiv 1 \mod 6 \) and \( n_k \equiv 1 \mod D \). We can pick such an ideal if we take \( A_k = (n_k a_k + m_k \frac{-b+\sqrt{-3}}{2}) \) with \( m_k \equiv b^{-1}(k+1) \mod D \). We will try to compute:

\[
S = \sum_{k \in (\mathbb{Z}/D\mathbb{Z})^\times} \frac{\Theta \left( \frac{D-b+\sqrt{-3}}{2a_k} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{2a_k} \right)} \chi(A_k)
\]

Recall that from Corollary 5.4, we have:

\[
\frac{\Theta \left( \frac{D-b+\sqrt{-3}}{2a_s} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{2a_s} \right)} = \frac{1}{3} \frac{\Theta \left( \frac{D-b+\sqrt{-3}}{2a_s} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{6a_s} \right)} = \sum_{r \in \mathbb{Z}/D\mathbb{Z}, r \text{ even}} \frac{\theta_{sr} \left( \frac{-b+\sqrt{-3}}{2D^2} \right)}{\theta_0 \left( \frac{-b+\sqrt{-3}}{2D^2} \right)} \frac{\theta_r \left( \frac{b+\sqrt{-3}}{2} \right)}{\theta_0 \left( \frac{b+\sqrt{-3}}{2D^2} \right)}
\]

Moreover since \( r, a_s \) are both even, we have \( e^{\pi i a_s^2 b} = e^{\pi i r} = 1 \) and thus in Lemma 5.5 we have:

\[
\theta_{sr} \left( \frac{-b+\sqrt{-3}}{2D^2} \right) = \theta_0 \left( \frac{-b+\sqrt{-3}}{2D^2} \right)
\]

Then our sum can be written in the form:

\[
\frac{\Theta \left( \frac{D-b+\sqrt{-3}}{2a_s} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{2a_s} \right)} = \frac{1}{3} \frac{\Theta \left( \frac{D-b+\sqrt{-3}}{6a_s} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{6a_s} \right)} = \sum_{r \in \mathbb{Z}/D\mathbb{Z}, r \text{ even}} \frac{\theta_{sr} \left( \frac{-b+\sqrt{-3}}{2D^2} \right)}{\theta_0 \left( \frac{-b+\sqrt{-3}}{2D^2} \right)} \frac{\theta_r \left( \frac{b+\sqrt{-3}}{2} \right)}{\theta_0 \left( \frac{b+\sqrt{-3}}{2D^2} \right)}
\]

Now summing up for all \( s \in (\mathbb{Z}/D\mathbb{Z})^\times \), we get the result of the lemma:
From Lemma 6.7 in Appendix A, we have
\[ \sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times \ mod \ 6 \atop s \equiv 1 \ mod \ 6} \frac{\Theta \left( D \frac{-b+\sqrt{-3}}{2a_s} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{2a_s} \right)} \chi(A_s) = 0. \] This gives us the result of the Lemma.

**Proposition 5.2.** Under the conditions above, we have:
\[ \sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times \ mod \ 6 \atop s \equiv 1 \ mod \ 6} \frac{\Theta \left( D \frac{-b+\sqrt{-3}}{2a_s} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{2a_s} \right)} \chi(A_s) = \left| \sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times \ mod \ 6 \atop s \equiv 1 \ mod \ 6} \frac{\theta_s \left( \frac{-b+\sqrt{-3}}{2D^2} \right)}{\theta_0 \left( \frac{-b+\sqrt{-3}}{2D^2} \right)} \chi(A_s) \right|^2. \]

**Proof:** Only for the purpose of this proposition we will use the following notation for \( \theta_r \), to emphasize how it depends on \( D \):
\[ \theta_{r/D}(z) = \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{r}{D} - b)^2 z} (-1)^n \]

Using the new notation, in the previous Lemma we have proved:
\[ \sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times \ mod \ 6 \atop s \equiv 1 \ mod \ 6} \Theta \left( D \frac{-b+\sqrt{-3}}{2a_s} \right) \chi(A_s) = \sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times \ mod \ 6 \atop s \equiv 1 \ mod \ 6} \sum_{s_{\text{mod } 2D} \ even} \theta_{sr/D}( \frac{-b+\sqrt{-3}}{2D^2} ) \theta_{r/D}( \frac{b+\sqrt{-3}}{2D^2} ) \chi(A_s). \]

Note that using Corollary 5.3 for \( a = D^2 \) can rewrite:
\[ \theta_0 \left( \frac{-b + \sqrt{-3}}{2D^2} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2D^2} \right) = \frac{D}{\sqrt{3}} \Theta \left( \frac{b + \sqrt{-3}}{2} \right) \]

Thus the equation becomes:
\[ \sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times \ mod \ 6 \atop s \equiv 1 \ mod \ 6} \Theta \left( D \frac{-b+\sqrt{-3}}{2a_s} \right) \chi(A_s) = \frac{1}{\sqrt{3}} \frac{D}{\sqrt{3}} \Theta \left( \frac{b + \sqrt{-3}}{2} \right) \sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times \ mod \ 6 \atop s \equiv 1 \ mod \ 6} \sum_{s_{\text{mod } 2D} \ even} \theta_{sr/D}( \frac{-b+\sqrt{-3}}{2D^2} ) \theta_{r/D}( \frac{b+\sqrt{-3}}{2D^2} ) \chi(A_s) \]

Let \( R \equiv R' \ mod \ D, \ R \ even \ and \ S \equiv 1 \ mod \ 6 \). Then we have by definition:
\[ \theta_{RS}(z_1) \Theta_R(z_2) = \sum_{n \in \mathbb{Z}} e^{\pi i (n + R/S + D - 1/6)^2 z_1} e^{\pi i n} \sum_{m \in \mathbb{Z}} e^{\pi i (m + R/D - 1/6)^2 z_2} e^{\pi i m} \]

By changing \( n \to n + S \) and \( m \to m + 1 \), we change \( R \to D + R \) and \( R + D \equiv R' \ mod \ 2D \). We get
\[ \theta_{RS}(z_1) \Theta_R(z_2) = \sum_{n \in \mathbb{Z}} e^{\pi i (n + R'S/D - 1/6)^2 z_1} e^{\pi i n} (-1)^S \sum_{m \in \mathbb{Z}} e^{\pi i (m + R'/D - 1/6)^2 z_2} e^{\pi i m} (-1) = \theta_{R'S}(z_1) \Theta_R'(z_2) \]

41
Thus we can choose in the formulas above all $r$ to be actually odd. Furthermore, by making a change of $r \pm 2D$ we can also choose $r \equiv 1 \mod 3$. Then we can rewrite the equation as:

\[
\sum_{s \in \mathbb{Z}/DZ} \sum_{r \equiv 6} \frac{\Theta \left( D \frac{b + \sqrt{-3}}{2a} \right)}{\Theta \left( \frac{b + \sqrt{-3}}{2a} \right)} \chi(\mathcal{A}_s) = \frac{1}{D} \sum_{s \in \mathbb{Z}/DZ} \sum_{r \equiv 6} \theta_{s/D} \left( \frac{b + \sqrt{-3}}{2} \right) \theta_{r/D} \left( \frac{b + \sqrt{-3}}{2} \right) \chi(\mathcal{A}_s)
\]

(20)

Denote $\tau_D = \frac{b + \sqrt{-3}}{2}$. Note that we are summing over all residues $r \mod D$. We can separate the terms, depending on whether a prime divisor $p_i$ divides both $D$ and $r$. We do this by using the Inclusion-Exclusion principle and note that the sum gets rewritten as:

\[
\sum_{s \in \mathbb{Z}/DZ} \sum_{r \equiv 6} \theta_{s/D}(\tau_D)\theta_{r/D}(\tau_D)\chi(\mathcal{A}_s) = \sum_{s \in \mathbb{Z}/DZ} \sum_{r \equiv 6} \theta_{s/D}(\tau_D)\theta_{r/D}(\tau_D)\chi(\mathcal{A}_s) - \sum_{p_i | D} \sum_{r \equiv 6} \frac{\Theta \left( \frac{b + \sqrt{-3}}{2} \right)}{\Theta \left( \frac{b + \sqrt{-3}}{2} \right)} \chi(\mathcal{A}_s)
\]

\[
\sum_{p_i | D} \sum_{r \equiv 6} \theta_{s/D}(\tau_D)\theta_{r/D}(\tau_D)\chi(\mathcal{A}_s) - \sum_{p_i | D} \sum_{r \equiv 6} \frac{\Theta \left( \frac{b + \sqrt{-3}}{2} \right)}{\Theta \left( \frac{b + \sqrt{-3}}{2} \right)} \chi(\mathcal{A}_s) - \sum_{p_i, p_j | D} \sum_{r \equiv 6} \frac{\Theta \left( \frac{b + \sqrt{-3}}{2} \right)}{\Theta \left( \frac{b + \sqrt{-3}}{2} \right)} \chi(\mathcal{A}_s)
\]

\[\vdots\]

\[
+ (-1)^{n-1} \sum_{p_1 \cdots p_n | D} \sum_{r \equiv 6} \frac{\Theta \left( \frac{b + \sqrt{-3}}{2} \right)}{\Theta \left( \frac{b + \sqrt{-3}}{2} \right)} \chi(\mathcal{A}_s)
\]

Using Lemma 5.8 proved below, all of the terms except for the first one equal 0. Thus getting back to the equation (19), we get:

\[
\sum_{s \in \mathbb{Z}/DZ} \sum_{s \equiv r \equiv 6} \frac{\Theta \left( D \frac{b + \sqrt{-3}}{2a} \right)}{\Theta \left( \frac{b + \sqrt{-3}}{2a} \right)} \chi(\mathcal{A}_s) = \frac{1}{D} \sum_{s \equiv r \equiv 6} \theta_{s/D} \left( \frac{b + \sqrt{-3}}{2} \right) \theta_{r/D} \left( \frac{b + \sqrt{-3}}{2} \right) \chi(\mathcal{A}_r)
\]

\[
= \frac{1}{D} \sum_{s \equiv r \equiv 6} \theta_{s/D} \left( \frac{b + \sqrt{-3}}{2} \right) \theta_{r/D} \left( \frac{b + \sqrt{-3}}{2} \right) \chi(\mathcal{A}_s)
\]

Below we prove Lemma 5.8 used in the proof of Proposition 20:
Lemma 5.8. If \( D = p_1 \ldots p_n \) and \( D' = D/(p_1 \ldots p_k) \), then:

\[
\sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times} \sum_{s \equiv 1 \mod 6, r \equiv 1 \mod 6} \theta_{sr/D'} \left( \frac{-b + \sqrt{-3}}{2} \right) \theta_{r/D'} \left( \frac{b + \sqrt{-3}}{2} \right) \chi(A_s) = 0
\]

**Proof:** Note that first that we can rewrite the sum in the form:

\[
\sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times} \sum_{s \equiv 1 \mod 6, r \equiv 1 \mod 6} \theta_{sr/D'} \left( \frac{-b + \sqrt{-3}}{2} \right) \theta_{r/D'} \left( \frac{b + \sqrt{-3}}{2} \right) \chi(A_s) = \theta_0 \left( \frac{-b + \sqrt{-3}}{2D^2} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2D^2} \right) \sum_{s \equiv 1 \mod 6, r \equiv 1 \mod 6} \theta_{sr/D'} \left( \frac{-b + \sqrt{-3}}{2} \right) \theta_{r/D'} \left( \frac{b + \sqrt{-3}}{2D^2} \right) \chi(A_s)
\]

Using (18) for \( D := D' \), we recognize the sum on the LHS to be:

\[
\sum_{s \equiv 1 \mod 6} \frac{\Theta \left( \frac{-b + \sqrt{-3}}{2D} \right)}{\Theta \left( \frac{-b + \sqrt{-3}}{2a_s} \right)} \chi(D_A) \frac{\Theta \left( \frac{b + \sqrt{-3}}{2D} \right)}{\Theta \left( \frac{b + \sqrt{-3}}{2a_s} \right)} \chi(D_A) = \sum_{s \equiv 1 \mod 6} \frac{\chi(D_A)}{\chi(D_A)} m^{1/3} \chi_m(A_s)
\]

Denote \( m = D/D' = p_1 \ldots p_k \). Moreover, recall that from the definition of the cubic character we have:

\[
D^{1/3} \chi_D(A_s) = (D^{1/3})^\sigma A_s = (D^{1/3})^\sigma A_s (m^{1/3})^\sigma A_s = D^{1/3} \chi_{D'}(A_s) \chi(D_A) m^{1/3} \chi_m(A_s)
\]

Then we can rewrite the sum as:

\[
\sum_{s \in (\mathbb{Z}/D\mathbb{Z})^\times} \frac{\Theta \left( \frac{-b + \sqrt{-3}}{2a_s} \right)}{\Theta \left( \frac{-b + \sqrt{-3}}{2a_s} \right)} \chi(D_A) = \sum_{s' \equiv 1 \mod 6} \frac{\Theta \left( \frac{-b + \sqrt{-3}}{2a_s} \right)}{\Theta \left( \frac{-b + \sqrt{-3}}{2a_s} \right)} \chi(D_A) \sum_{s \equiv 1 \mod 6, s' \equiv 1 \mod 6} \frac{\chi(D_A)}{\chi(D_A)} m^{1/3} \chi_m(A_s)
\]

Note that as \( D = p_1 \ldots p_n \), we have \( \{ s \in (\mathbb{Z}/D\mathbb{Z})^\times, s \equiv 1 \mod D' \} \cong (\mathbb{Z}/m\mathbb{Z})^\times \). Moreover, note that \( \chi_m(A_s) \) depends only on \( s \mod m \). Thus we are summing the character \( \chi_m(A_s) = \chi_m(A_{s''}) \) over \( s'' \in (\mathbb{Z}/m\mathbb{Z})^\times \).

Moreover, \( \chi_m(A_s) \) is a nontrivial character as a function of \( s \), as \( m^{1/3} \chi_m(A_s) = (m^{1/3})^\sigma A_s = m^{1/3} \) for all \( A_s \). As we are summing a non-trivial character over a group, the sum is just 0:

\[
\sum_{s'' \in (\mathbb{Z}/m\mathbb{Z})^\times} \chi_m(A_{s''}) = 0,
\]

thus the whole sum is zero.

We left out the case \( r \equiv 0 \mod D \). In this case we have:
\[
\sum_{s \in \mathbb{Z}/D\mathbb{Z}} \frac{\theta_0 \left( \frac{b + \sqrt{-3}}{2} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2} \right)}{\chi(A_s)} = \sum_{s \in \mathbb{Z}/D\mathbb{Z}} \frac{1}{D} \chi(A_s) = 0
\]

### 5.4 Shimura reciprocity applied to \( \theta_r \)

We define:

\[
f_r(z) = \frac{\theta_r(z)}{\theta_0(z)} = \frac{\sum\limits_{n \in \mathbb{Z}} e^{\pi i (n + \frac{3}{2} - \frac{1}{2})^2 z e^{\pi i n}}}{\sum\limits_{n \in \mathbb{Z}} e^{\pi i (n - \frac{1}{2})^2 z e^{\pi i n}}}
\]

Then we can rewrite Proposition 5.2 as:

\[
\sum_{s \in \mathbb{Z}/D\mathbb{Z}} \frac{\Theta \left( D \frac{b + \sqrt{-3}}{2a_s} \right) \chi(A_s)}{\Theta \left( \frac{b + \sqrt{-3}}{2a_s} \right)} = \left| \sum_{s \in \mathbb{Z}/D\mathbb{Z}} f_s(\tau) \chi(A_s) \right|^2 \left| \frac{\theta_0 \left( \frac{b + \sqrt{-3}}{2} \right)}{\theta_0 \left( \frac{-b + \sqrt{-3}}{2D^2} \right)} \right|^2
\]

Note that from the Corollary 5.3, we can compute \( \frac{3}{2} \Theta \left( \frac{-b + \sqrt{-3}}{2} \right) = \frac{3\sqrt{3}}{4} \theta_0 \left( \frac{-b + \sqrt{-3}}{2} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2} \right) \) as well as \( \frac{3}{2} \Theta \left( \frac{-b + \sqrt{-3}}{2} \right) = \frac{3\sqrt{3}}{4} \theta_0 \left( \frac{-b + \sqrt{-3}}{2D^2} \right) \theta_0 \left( \frac{b + \sqrt{-3}}{2D^2} \right) \). Taking the ratio of the two relations, gives us:

\[
\left| \frac{\theta_0 \left( \frac{-b + \sqrt{-3}}{2} \right)}{\theta_0 \left( \frac{-b + \sqrt{-3}}{2D^2} \right)} \right|^2 = D
\]

Thus we get:

\[
\sum_{s \in \mathbb{Z}/D\mathbb{Z}} \frac{\Theta \left( D \frac{b + \sqrt{-3}}{2a_s} \right) \chi(A_s)}{\Theta \left( \frac{b + \sqrt{-3}}{2a_s} \right)} = D \left| \sum_{s \in \mathbb{Z}/D\mathbb{Z}} f_s(\tau) \chi(A_s) \right|^2.
\]

By further multiplying by \( D^{1/3} \), we have:

\[
\sum_{s \in \mathbb{Z}/D\mathbb{Z}} \frac{\Theta \left( D \frac{-b + \sqrt{-3}}{2a_s} \right) D^{1/3} \chi(A_s)}{\Theta \left( \frac{-b + \sqrt{-3}}{2a_s} \right)} = \left| \sum_{s \in \mathbb{Z}/D\mathbb{Z}} f_s(\tau) \chi(A_s) D^{2/3} \right|^2.
\]  \hspace{1cm} (21)

Our goal in this section is to show that all the terms \( f_s(\tau) \chi(A_s) D^{2/3} \) are Galois conjugates of each other.

#### 5.4.1 \( \theta_r \) as an automorphic form

We will first look closer at the function \( \theta_r \). We will rewrite \( \theta_r \) as an automorphic theta function \( \Theta : \text{SL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C} \):
\[ \Theta(g) = \sum_{m \in \mathbb{Q}} r(g) \Phi(m), \]

where \( \Phi \in (\mathbb{A}_\mathbb{Q}) \) is a Schwartz-Bruhat function and \( r \) is the Weil representation defined by:

- \( r \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) (x) = \chi_0(a)|a|^{1/2} \Phi(ax) \)
- \( r \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) (x) = \psi(bx^2) \Phi(x) \)
- \( r \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) (x) = \gamma \hat{\Phi}(x), \)

where \( \psi_p(x) = e^{-2\pi i \text{Frac}_p(x)} \) and \( \psi_{x_0}(x) = e^{2\pi i x} \), \( \gamma \) is an 8th root of unity, and \( \chi_0 \) is a character.

**XXX define \( \chi_0 \) and \( \gamma \)**

We define the following Schwartz-Bruhat functions for \( \theta \). Let \( f_p = \prod_{v_p(\mu) < 0} \Phi_{\mu, v} \), where:

\[
\begin{align*}
\Phi^{(r)}_p &= \text{char}_{\mathbb{Z}_p}, & \text{if } p \nmid D \\
\Phi^{(r)}_p &= \text{char}_{\mathbb{Z}_p - \frac{1}{p}}, & \text{if } p | D, p \nmid 2, 3 \\
\Phi^{(r)}_q &= \text{char}_{\mathbb{Z}_p + \frac{1}{p}}, \\
\Phi^{(r)}_2(n) &= e^{\pi i \text{Frac}_2(n)} \text{char}_{\mathbb{Z}_p + \frac{1}{p}}(n), \\
\Phi^{(r)}_\infty(x) &= e^{-2\pi q(x)}. 
\end{align*}
\]

We define the theta function:

\[ \Theta_{\Phi^{(r)}}(g) = \sum_{n \in \mathbb{Q}} r(g) \Phi^{(r)}(n) \]

Note that \( \Phi^{(r)}_f(n) \neq 0 \) for \( n \in \mathbb{Q} \) implies \( n - \frac{r}{2} + \frac{1}{6} \in \mathbb{Z}_p \) for all \( p \). This implies \( n - \frac{r}{2} + \frac{1}{6} \in \mathbb{Z} \), thus \( n \in \mathbb{Z} + \frac{r}{2} - \frac{1}{6} \). Also note that for \( g_z = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \), we have \( r(g_z) \Phi_{\infty}(n) = r \left( \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & xy^{-1} \\ 0 & 1 \end{pmatrix} \right) (n) = y^{1/2} e^{2\pi iz(x+y)n^2} \). Then we can compute:

\[ \Theta_{\Phi^{(r)}}(g_z, 1f) = \sum_{n \in \mathbb{Z} + \frac{r}{2} - \frac{1}{6}} e^{2\pi izn^2} e^{\pi i \text{Frac}_2(n)} = y^{1/2} \theta_r(2z) \]

Note that: \( \theta_r(2z) = y^{-1/2} \Theta_{\Phi^{(r)}}(g_z, 1f) \) and \( \theta_0(2z) = y^{-1/2} \Theta_{\Phi^{(0)}}(g_z, 1f) \), which implies:

\[ \frac{\theta_r(z)}{\theta_0(z)} = \frac{\Theta_{\Phi^{(r)}}(g_z/2, 1f)}{\Theta_{\Phi^{(0)}}(g_z/2, 1f)} \]
5.4.2 Galois action on modular functions (Shimura reciprocity)

Recall the function $f_r$: 

$$f_r(z) = \frac{\theta_r(z)}{\theta_0(z)} = \frac{\Theta_{\Phi^{(r)}(g_{z/2}, 1f)}}{\Theta_{\Phi^{(0)}(g_{z/2}, 1f)}}$$

Lemma 5.9. The theta function $\theta_r(z)$ is modular form of weight $1/2$ for $\Gamma(72D^2)$.

**Proof:** Recall that $\theta_r(z) = \Theta_{\Phi^{(r)}(g_{z/2})}$. We will compute $\theta_r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}z\right)$, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(72D^2)$. Note first that:

$$\theta_r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}z\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}g_z\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\right)$$

As $\Theta_{\Phi^{(r)}}$ is invariant under $\text{SL}_2(\mathbb{Q})$, we can rewrite $\Theta_{\Phi^{(r)}}$ as:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\right)$$

We will compute separately the two terms, using the Weil representations. For the RHS, note that we have to compute $r\left(\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\right)^{-1}\Phi_f^{(r)} = r\left(\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}\right)\Phi_f^{(r)}$. We will show:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\right) = \prod_{p|6D} \gamma_p^2 \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\right)$$

We rewrite the matrix as:

$$\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix} = \begin{pmatrix} 1/a & -b/2a \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2c/a & 1 \end{pmatrix}$$

At $p \nmid 6D$, the action of $\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}$ is trivial, as it belongs to $\text{SL}_2(\mathbb{Z}_p)$ and $\Phi_p^{(r)}$ is the characteristic function of $\mathbb{Z}_p$. For $p|6D$, we compute:

First we compute $r\left(\begin{pmatrix} 1 & 0 \\ -2c/d & 1 \end{pmatrix}\right)\Phi_p^{(r)}(x) = \gamma_p^2 \Phi_p^{(r)}(x)$. We rewrite the matrix as:

$$\begin{pmatrix} 1 & 0 \\ -2c/d & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2c/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and compute the Weil representation action:

- $r\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\Phi_p^{(r)}(x) = \gamma_p \Phi_p^{(r)}(x)$. Note that we can compute:

$$\Phi_p^{(r)}(x) = \int_{\mathbb{Q}_p} e^{-2\pi i \text{Frac}_p(2xy)} \text{char}_{\mathbb{Z}_p + \frac{r}{p}}(y) dy = \int_{\mathbb{Z}_p} e^{-2\pi i \text{Frac}_p(2x(y+r/D))} dy = e^{-2\pi i \text{Frac}_p(2x/D)} \text{char}_{\mathbb{Z}_p}(x)$$

46
- \( r \begin{pmatrix} 1 & 2c/d \\ 0 & 1 \end{pmatrix} \Phi^{(r)}_p(x) = e^{-2\pi i \text{Frac}_p(2c/d) x^2} e^{-2\pi i \text{Frac}_p(2x/D)} \text{char}_p(x). \) As \( v_p(c/d) \geq 0, \) we have \( e^{-2\pi i \text{Frac}_p(2c/d) x^2} = 1, \) thus the action is trivial on \( \Phi^{(r)}_p(x) \)

- \( r \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi^{(r)}_p(x) = \gamma_p \Phi^{(r)}_p(x). \) By the choice of the self-dual Haar measure, this equals \( \gamma_p \Phi^{(r)}_p(-x). \)

- \( r \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi^{(r)}_p(-x) = \Phi^{(r)}_p(x) \)

Now we also want to compute the action of \( r \begin{pmatrix} 1/a & -b/2 \\ 0 & a \end{pmatrix} \Phi^{(r)}_p(x). \) We rewrite the matrix as:

\[
\begin{pmatrix} 1/a & -b/2 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & -ba/2 \\ 0 & 1 \end{pmatrix}
\]

and compute the action:

- \( r \begin{pmatrix} 1/a & -ba/2 \\ 0 & 1 \end{pmatrix} \Phi^{(r)}_p(x) = e^{2\pi i \text{Frac}_p(ba/2x^2)} \text{char}_{x+r/D}(x). \) As \( D^2(ba/2), \) we have \( e^{2\pi i \text{Frac}_p(ba/2x^2)}, \) thus we have trivial action.

- \( r \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix} \Phi^{(r)}_p(x) = \chi_0(a)|a|^{1/2} \Phi^{(r)}_p(x/a). \) As \( a \equiv 1 \mod D, \) we get \( \Phi^{(r)}_p(x/a) = \Phi^{(r)}_p(x), \)

as well as \( \chi_0(a) = |a|^{1/2} = 1. \)

For \( p = 3 \) the computation is similar. For \( p = 2, \) we compute again the action of \( r \begin{pmatrix} 1 & 0 \\ -2c/d & 1 \end{pmatrix} \Phi^{(r)}_p(x) = \gamma_p^{2} \Phi^{(r)}_p(x): \)

- \( r \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi^{(r)}_2(x) = \gamma_2 \Phi^{(r)}_2(x). \) Note that we can compute:

\[
\Phi^{(r)}_2(x) = \int_{\mathbb{Q}_2} e^{-2\pi i \text{Frac}_2(2xy)} \text{char}_{x+1/2}(y) e^{\pi i \text{Frac}_2(y)} dy = e^{\pi i/2} \int_{\mathbb{Q}_2} e^{-2\pi i \text{Frac}_2(2xy+1/2)} e^{\pi i \text{Frac}_2(y)} dy =
\]

\[
e^{\pi i/2} e^{-2\pi i \text{Frac}_2(x)} \int_{\mathbb{Z}_2} e^{-2\pi i \text{Frac}_2((2x-1/2)y)} dy = e^{\pi i/2} e^{-2\pi i \text{Frac}_2(x)} \text{char}_{1/2}(x+1/2)(x)
\]

- \( r \begin{pmatrix} 1 & 2c/d \\ 0 & 1 \end{pmatrix} \Phi^{(r)}_2(x) = e^{-2\pi i \text{Frac}_2(2c/dx^2) \Phi^{(r)}_2(x). \) As \( v_p(2c/d) \geq 4, \) we have \( e^{-2\pi i \text{Frac}_p(2c/dx^2) = 1, \) thus the action is trivial on \( \Phi^{(r)}_p(x) \)

- \( r \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi^{(2)}_2(x) = \gamma_2 \Phi^{(2)}_2(x). \) By the choice of the self-dual Haar measure, this equals \( \gamma_2 \Phi^{(2)}_2(-x). \)

- \( r \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi^{(r)}_2(-x) = \Phi^{(r)}_2(x) \)
We compute similarly the action of \( r \begin{pmatrix} 1/a & -b/2 \\ 0 & a \end{pmatrix} \Phi_2^{(r)}(x): \)

- \( r \begin{pmatrix} 1 & -ba/2 \\ 0 & 1 \end{pmatrix} \Phi_2^{(r)}(x) = e^{2\pi i \text{Frac}_2(ba/2\pi^2)} e^{2\pi i \text{Frac}_2(x)} \text{char}_{\mathbb{Z} + 1/2}(x). \) As \( 4|ba/2, \) we have \( e^{2\pi i \text{Frac}_2(ba/2\pi^2)} = 1, \) thus we have trivial action.

- \( r \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix} \Phi_2^{(r)}(x) = \chi_0(a)|a|^{1/2} \Phi_2^{(r)}(x/a). \) As \( a \equiv 1 \mod 8, \) we get \( \Phi_2^{(r)}(x/a) = \Phi_2^{(r)}(x), \)
  as well as \( \chi_0(a) = |a|^{1/2} = 1. \)

This finishes the computation of the finite part. We get:

\[
\Theta_{\Phi^{(r)}} \left( \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix} g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}^{-1} \right) = 2^{-1/4} y^{1/2} \sum_{m \in \mathbb{Z} + \frac{a}{2}} e^{2\pi im^2z} (-1)^m = 2^{-1/4} y^{1/2} \theta_r(z) \tag{22}
\]

We will compute now the infinite part. Note first that \( r(g_z) \Phi_\infty(m) = y^{1/4} e^{2\pi iz|m|^2} \) We rewrite the matrix:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/d & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -c/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

We compute the Weil representation action:

- \( F_1(m) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{2\pi i z m^2} = \gamma_\infty \frac{1}{\sqrt{8}} e^{-2\pi i \frac{z}{8}} \)

- \( F_2(m) := r \begin{pmatrix} 1 & -c/d \\ 0 & 1 \end{pmatrix} F_1(m) = e^{-2\pi i \frac{z}{4} m^2} F_1(m) = \gamma_\infty \frac{1}{\sqrt{2}} e^{-2\pi i \frac{z}{4} m^2} \)

- \( F_3(m) := r \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F_3(m) = \gamma_\infty F_3(m) = \gamma_\infty \frac{1}{\sqrt{8}} e^{2\pi i \frac{(d \chi_2 + d \chi_4)}{(8 \pi i \chi_2 \chi_4)}} (\gamma_\infty \frac{1}{\sqrt{8}}) \sqrt{\frac{dz}{cz+d}} = \gamma_\infty \frac{1}{\sqrt{8}} e^{2\pi i \frac{d \chi_2 + d \chi_4}{cz+d} m^2} \)

- \( F_4(m) := r \begin{pmatrix} 1/d & 0 \\ 0 & d \end{pmatrix} F_4(m) = \text{sgn}(d) d^{1/2} F_3(m/d) = \text{sgn}(d) d^{1/2} F_3(m/d) \gamma_\infty^2 (\gamma_\infty \frac{1}{\sqrt{8}}) \sqrt{\frac{dz}{cz+d}} = \text{sgn}(d) F_3(m/d) \gamma_\infty (\gamma_\infty \frac{1}{\sqrt{8}}) \sqrt{\frac{dz}{cz+d}} e^{2\pi i \frac{d \chi_2 + d \chi_4}{cz+d} m^2} \)

- \( F_5(m) := r \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} F_5(m) = e^{2\pi i \frac{b}{4} m^2} F_4(m) = \text{sgn}(d) \gamma_\infty^2 (\gamma_\infty \frac{1}{\sqrt{8}}) \sqrt{\frac{dz}{cz+d}} e^{2\pi i \frac{(b \chi_2 + b \chi_4)}{(c \chi_2 + d \chi_4)}} m^2 \)

- \( r \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} F_5(m) = -F_5(-m) = -\text{sgn}(d) \gamma_\infty^2 (\gamma_\infty \frac{1}{\sqrt{8}}) \sqrt{\frac{dz}{cz+d}} e^{2\pi i \frac{a+b}{c \chi_2 + d \chi_4} m^2} \)

We still have to compute the action of \( r \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix} \) on \( -y^{1/2} \text{sgn}(d) \gamma_\infty^2 (\gamma_\infty \frac{1}{\sqrt{8}}) \sqrt{\frac{dz}{cz+d}} e^{2\pi i \frac{a+b}{c \chi_2 + d \chi_4} m^2}. \)

This gives us just:
\[ 2^{-1/4}y^{1/2} \text{sgn}(d)\gamma_{Z, \gamma}^2(*) = 2^{-1/4}y^{1/2} \text{sgn}(d)\gamma_{Z, \gamma}^2(*) \sqrt{\frac{1}{cz + d}} e^{\pi i \left( \frac{az + b}{cz + d} \right)^2} \]

Thus we have:

\[ \Theta \left( \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} z, 1 \right) = 2^{-1/4}y^{1/2} \text{sgn}(d)\gamma_{Z, \gamma}^2(*) \sqrt{\frac{1}{cz + d}} \sum_{m \in \mathbb{Z}^+) \frac{1}{cz + d} e^{\pi i \left( \frac{az + b}{cz + d} \right)^2} (-1)^m \]

Note that this is exactly:

\[ 2^{-1/4}y^{1/2} \text{sgn}(d)\gamma_{Z, \gamma}^2(*) \sqrt{\frac{1}{cz + d}} \theta_r \left( \frac{az + b}{cz + d} \right) = \gamma_0 \theta_r(z) \]

(23)

From (21) and (22) we get that:

\[ \text{sgn}(d)\gamma_{Z, \gamma}^2(*) \sqrt{\frac{1}{cz + d}} \theta_r \left( \frac{az + b}{cz + d} \right) = \gamma_0 \theta_r(z) \]

Lemma 5.11. The modular function \( f_\tau \) has rational Fourier coefficients.

Proof: Note that \( \theta_r(z) = q^{(D-r)^2/72} \left( 1 + \sum_{M \geq 1} a_M q^{M/(72D^2)} \right) \), where \( a_m \in \mathbb{Z} \) and \( \theta_0(z) = q^{1/72} \). Then we can compute \( f_\tau(z) = q^{(D-r)^2/72} \left( 1 + \sum_m a_m q^{m/72D^2} \right) \) with \( a_m \in \mathbb{Z} \).
for $x \in \mathbb{A}_{K,f}^\times$, $g_r(x) = \left( \frac{t - sB}{s} - sC \right)$.

In our case, we want to compute the Galois conjugates of $f_r(\tau)$, where $\tau = \frac{-k + \sqrt{-3}}{2}$. Note that it has the minimum polynomial $X^2 + bX + \frac{b^2 + 3}{4}$. Thus we have to compute the action of all $g_r((x_p)_p) = \prod_p \left( \frac{t_p - s_p b}{s_p} - s_p \frac{\sqrt{b^2 + 3}}{4} \right)$.

We will compute all these actions. However, we claim that it is enough to compute the action of the ideals $\mathcal{A}$ through the correspondence:

$$I(3) \to \mathbb{A}_{K,f}^\times / K^\times$$

$$\mathcal{A} = (A + B\omega) \to (A + B\omega)_{p \mid 6D},$$

where $A + B\omega \equiv 1 \mod 3$ is the generator of the ideal $\mathcal{A}$.

More precisely, in order to find the Galois conjugates over $K$, we will compute the action of all Galois actions corresponding to $(A_p + B_p\omega)_p \in \mathbb{A}_{K}^\times$ and we will prove that the Galois action from Shimura reciprocity law is:

**Proposition 5.3.** For $A = (n_a a + m_a \frac{-b + \sqrt{-3}}{2})$, where $b^2 \equiv -3 \mod 4Db^2$ is an ideal prime to $6D$, we have:

$$f_1(\tau)^{\mathcal{A}} = f_{n_a}(\tau)$$

and $f_r(\tau)$ are all the Galois conjugates of $f(\tau)$, where $r \in (\mathbb{Z}/D\mathbb{Z})^\times$. Moreover, this implies that $f_1(\tau) \in H_{6D}$.

**Proof:** First we note that we do not have to consider the action of all $(x_p)_p \in \mathbb{A}_{K}^\times$. By applying the Strong Approximation Theorem for $GL_1$ and the number field $K$ that is a PID, we have:

$$\mathbb{A}_{K}^\times = K^\times \times \prod_{v \mid \mathcal{O}} \mathcal{O}_{K_v}^\times \times \mathbb{C}^\times$$

This implies:

$$\mathbb{A}_{K,f}^\times = K^\times \times \prod_{v \mid \mathcal{O}} \mathcal{O}_{K_v}^\times$$

Then any $x = (x_v) \in \mathbb{A}_{K,f}^\times$ can be written as $x = k(l_v)$, where $k \in K^\times$, $(l_v)_v \in \prod_{v \mid \mathcal{O}} \mathcal{O}_{K_v}^\times$.

Since $\text{Nm} k > 0$, we have the embedding:

$$k \in K^\times \hookrightarrow \text{GL}_2(\mathbb{Q})^+$$

50
We also have the embedding:

\[(l_v)_v \in \prod_{v \mid \infty} \mathcal{O}_{K_v}^\times \to \prod_p \text{GL}_2(\mathbb{Z}_p)\]

Thus if we know the Galois action of \(K^\times\) and of \(\hat{\mathcal{O}}_{K_v}^\times\), we will know the Galois action of \(\hat{\mathbb{A}}_{K,f}^\times\).

We recall the way the action of \(g_r(x)\) is defined for. For \(\alpha \in \text{GL}_2(\mathbb{Q})^+\), \(f^\alpha\) is defined by \(f^\alpha(\tau) = f(\alpha \tau)\). In our case we only need to look at the action of \(K^\times\). Recall that \(k \in K^\times\) embeds into \(\text{GL}_2(\mathbb{Q})^+\) under the map:

\[k = t + s \frac{-b + \sqrt{-3}}{2} \mapsto g_r(k) = \begin{pmatrix} t - sb & -sc \\ s & t \end{pmatrix}\]

Then the Galois action from Shimura reciprocity is:

\[f(\tau)^{k^{-1}} = f^{g_r(k)}(\tau) = f(g_r(k)\tau)\]

Note that \(t + s\tau \mapsto (\frac{t - sb}{s}, \frac{-sc}{t})\) is the torus that preserves \(\tau\), thus we have:

\[f(\tau)^{k^{-1}} = f(g_r(k)\tau) = f(\tau)\]

Now all we have left is to compute the action of \(\prod_v \mathcal{O}_{K_v}^\times\). Note that for all \(v \nmid 6D\) the action is trivial. For \(v \mid 6D\) we project the action of \((g_r(x_v))_v \to g_r(x') \in \text{GL}_2(\mathbb{Z}/6D^2\mathbb{Z})\).

**Remark 5.2.** Note that we have for \((\pm \omega^i)_p \mapsto \hat{\mathbb{A}}_{K,j}^\times\) acting trivially. Thus we have for \(x \in \hat{\mathbb{A}}_{K,j}^\times\):

\[(f_r(\tau))^{\sigma_{\pm \omega^i}} = ((f_r(\tau))^{\sigma_{\pm \omega^i}})^{\sigma_x} = (f_r^{g_r(\pm \omega^i)}(\tau))^{\sigma_x} = f_r(\tau)^{\sigma_x}\]

**Lemma 5.12.** For \(x \in \prod_v \mathcal{O}_{K_v}^\times\) we can find \(\omega^i, i = 0, \pm 1\) such that:

\[(x_2 \pm \omega^i)_2 = (t_2 + s_2 \omega)\]

with \(v_2(t_2) = 0, v_2(s_2) \geq 1\) and

\[(x_3 \pm \omega^i)_3 = (t_3 + s_3 \omega)\]

with \(t_3 + s_3 \equiv 1 \pmod{3}\).

**Proof:** Note first that if \(v_2(s) \geq 1\), then we must have \(v_2(t_2) = 0\), as we need \(x_2 \omega^i \in (\mathbb{Z}_2[\omega])^\times\). Thus we must find \(x \omega^i\) such that \(v_2(s) \geq 1\). We write \(x = t'_2 + s'_2 \omega\). Then:

\[x_2 \omega = t'_2 \omega + s'_2 \omega^2 = (t'_2 - s'_2) \omega + s'_2 \]

\[x_2 \omega^2 = t'_2 \omega^2 + s'_2 = (-t'_2) \omega + (s'_2 - t'_2)\]

One of \(t'_2, s'_2, t'_2 - s'_2\) must have positive valuation. Assume this is not true: \(v_2(t'_2) = v_2(s'_2) = 0\). Then \(s'_2, t'_2 \equiv 1 \pmod{2}\), thus \(s'_2 - t'_2 \equiv 0 \pmod{2}\) and has positive valuation. Thus we can always pick \(x \omega^i\) as claimed above at the place 2.
Now since take \( x_3 \omega^i = s'_3 \omega + t'_3 = s'_3 - \frac{3+\sqrt{3}}{2} + (t'_3 + s'_3) \). Then, since \( x_3 \) is a unit in \( \mathbb{Z}_3[\omega] \), we must have \( t_3(s'_3 + t'_3) = 0 \), thus \( s'_3 + t'_3 \equiv \pm 1 \mod 3 \). We pick \( x_3 \omega \) or \(-x_3 \omega\) to get the condition \( s'_3 + t'_3 \equiv 1 \mod 3 \).

Since from the remark above \( x \) and \( \pm \omega^i x \) act the same, we can consider the Galois action of \( \sigma_{x\omega^i} \) as in the lemma above. We compute it below.

Let \( x_p \in \prod_v O^\times_{K_v} \) chosen as above. Then:

\[
x_p = t_p + s_p \frac{-b + \sqrt{3}}{2} \quad \iff \quad g_r(x_p) = \begin{pmatrix} t_p - s_p b & s_p c \\ s_p & t_p \end{pmatrix}
\]

Elements of \( \prod_v \text{GL}_2(\mathbb{Z}_p) \) project to \( \text{GL}_2(\mathbb{Z}/6D^2\mathbb{Z}) \), which is the action we care about. From Chinese remainder theorem, we can find \( k_0 \in K \) such that \( k_0 \equiv x_p \mod 6D^2\mathbb{Z}_p \) for all \( p \mid 6D \).

Note that \( k_0 \) is independent of the choice of \( \tau \).

Then we only need to compute the action of:

\[
f_r(\tau)^{\sigma_{x\omega^i}^{-1}} = f_{g_r(x)}(\tau) = f_r^{g_r(x), \psi|6D}(\tau) = f_r^{g_r(t + \sigma \tau), \psi|6D}(\tau)
\]

We will now compute \( f_r^{g_r(x), \psi|6D}(\tau) \). Note that, for \( c' = \frac{b^2 + 3}{4} \), we have the map:

\[
k_0 = s\tau + t \rightarrow g_r(k_0) = \begin{pmatrix} t - sb' & -sc \\ s & t \end{pmatrix}
\]

Let \( \text{Nm}(k_0) = a \). We write the action:

\[
f(\tau)^{\sigma_{x\omega^i}} = f \begin{pmatrix} t - sb & -sc/a \\ s & t/a \end{pmatrix}_{\psi|6D} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{\psi|6D}
\]

Note that \( \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{\psi|6D} \) acts trivially on \( f_r \) as both functions \( \theta \left[ \begin{smallmatrix} \frac{-1}{6} + \frac{\tau}{2} \\ \frac{1}{2} \end{smallmatrix} \right] e^{-\pi i (\tau/6 - 1/6)} \) and \( \theta \left[ \begin{smallmatrix} -\frac{1}{6} \\ \frac{1}{2} \end{smallmatrix} \right] e^{\pi i /6} \) have rational Fourier coefficients.

Thus we need to compute the action:

\[
f_r \begin{pmatrix} t - sb & -sc/a \\ s & t/a \end{pmatrix}_{\psi|6D}(\tau)
\]

Note that \( \begin{pmatrix} t - sb & -sc/a \\ s & ta* \end{pmatrix} \begin{pmatrix} t - sb & -sca* \\ s & ta* \end{pmatrix} \in \text{SL}_2(\mathbb{Z}/6D^2\mathbb{Z}) \) and we can lift it to an element of \( \text{SL}_2(\mathbb{Z}) \).

**Lift from** \( \text{SL}_2(\mathbb{Z}/6D^2) \) to \( \text{SL}_2(\mathbb{Z}) \).

**Lemma 5.13.** We can always lift a matrix in \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}_2(\mathbb{Z}/NZ) \) to \( \text{SL}_2(\mathbb{Z}) \).

**Proof:** Take \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}_2(\mathbb{Z}/NZ) \), \( A, B, C, D \in \mathbb{Z} \). We can further assume \( (C, D) = 1 \). Let \( AD - BC = k \in \mathbb{Z} \). Then we can take:
$A_0 = A + NA_1$
$B_0 = B + NB_1$
$C_0 = C + NC_1$
$D_0 = D + ND_1$

We want to have the condition:

$$1 = A_0D_0 - B_0C_0 = AB - CD + N(AD_1 + A_1D - BC_1 - B_1C) + N^2(A_1D_1 - B_1C_1) = 1 + Nk + N(AD_1 + A_1D - BC_1 - B_1C) + N^2(A_1D_1 - B_1C_1)$$

For example, pick $D_1 = C_1 = 0$. Then we only need:

$$(A_1D - B_1C) = -k$$

Note that since $(C, D) = 1$, we can find $mC + nD = 1$. Then $(-kn)D - kmC = -k$, thus pick $A_1 = -kn$ and $B_1 = km$.

We look at such a matrix

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\in \text{SL}_2(\mathbb{Z})
$$

such that:

$$
\begin{pmatrix}
  a_0 & b_0 \\
  c_0 & d_0
\end{pmatrix}
\equiv
\begin{pmatrix}
  s - tb & -s \frac{b^2 + 3}{2} a^* \\
  s & \frac{t}{2}
\end{pmatrix}
\pmod{6D^2}
$$

Conditions obtained:

- $v_2(s) \geq 0$ and $v_2(t) = 0$ imply $b_0, c_0 \equiv 0 \pmod{2}$, $a_0, d_0 \equiv 1 \pmod{2}$.
- From the choice $3|b$ we also have $a_0 \equiv d_0 \pmod{3}$ and $b_0 \equiv 0 \pmod{3}$. Since we picked $k_0 = t_0 + s_0 \omega \equiv s \frac{b^2 + 3}{2} + t$ with $s_0 + t_0 \equiv 1 \pmod{3}$, we must have $t \equiv t_0 + s_0 \pmod{3}$, thus $d_0 \equiv t_0 \pmod{3}$.
- From the choice of $t + s \frac{b^2 + 3}{2}$ unit in $\prod_{v|6D} \mathcal{O}^\times_{K_v}$, we have $(t, D) = 1$. Otherwise note that the norm is $t^2 - tsb + s^2 \frac{b^2 + 3}{4}$ is divisible by $p|D$, a contradiction.

We will find the action using the following lemma:

**Lemma 5.14.** For

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\in \text{SL}_2(\mathbb{Z})
$$

such that $v_p(d) = 0$ and $d \equiv 1 \pmod{6}$, we have:

$$
\Theta_{\Phi, r} \left( \begin{pmatrix}
  1 & 0 \\
  0 & 2
\end{pmatrix}
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
z
\right) = \Theta_{\Phi, (d^{-1}, r)}(z/2)
$$

Here by $d^{-1}$ we mean $d^{-1} \pmod{D}$.

**Proof:** We compute:

$$
\Theta_{\Phi, r} \left( \begin{pmatrix}
  1 & 0 \\
  0 & 2
\end{pmatrix}
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\right) = \Theta_{\Phi, (c, 2, r)} \left( \begin{pmatrix}
  a & b/2 \\
  2c & d
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  0 & 2
\end{pmatrix}
\right)
$$

Moreover, it equals:

$$
\Theta_{\Phi, r} \left[ z/2, \left( \begin{array}{cc}
  d & -b/2 \\
  -2c & a
\end{array} \right) \right]
$$

Note that for $p \nmid 6D$ we have $\left( \begin{array}{cc}
  d & -b/2 \\
  -2c & a
\end{array} \right)_p$ in $\text{SL}_2(\mathbb{Z}_p)$, thus acts trivially.
For $p|3D$, we have $\Phi_p = \text{char}_{\mathbb{Z}_p - \frac{1}{6} + \frac{r}{6}}$. For now, we will call $\mu_r := -\frac{1}{6} + \frac{r}{6}$.

If $v_p(d) = 0$, we rewrite:

$$\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2c/d & 1 \end{pmatrix} \begin{pmatrix} d & -b/2 \\ 0 & d^{-1} \end{pmatrix}$$

We can further write it in the form:

$$\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2c/d & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b/(2d) \\ 0 & 1 \end{pmatrix}$$

- $r \begin{pmatrix} 1 & -b/(2d) \\ 0 & 1 \end{pmatrix} \Phi_p(x) = e^{-2\pi i \text{Frac}(b/(2d)x^2)} \Phi_p(x) = \Phi_p(x)$

- $r \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \Phi_p(x) = |d|_p \chi_p(d) \Phi_p(dx) = \Phi_p^{(d^{-1})}(x)$

Note that $\Phi_p(dx) \neq 0$ iff $dx \in \mathbb{Z}_p + \mu_r$ iff $x \in d^{-1}\mathbb{Z}_p + d^{-1}\mu_r = \mathbb{Z}_p + d^{-1}\mu_r$. Note that $d^{-1}\mu_r = d^{-1}r/D - d^{-1}/6$. Since we picked $d \equiv 1 \mod 6$, this is the same as $\mu_{d^{-1}}$.

Note: We need to check that the character corresponding to $Q$ is trivial on units ($\chi_p(d) = 1$).

- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_p^{d^{-1}}(x) = e^{2\pi i \text{Frac}(2d^{-1}xr/D)} \text{char}_{\mathbb{Z}_p + 1/2}(x)$

- $r \begin{pmatrix} 1 & 2c/d \\ 0 & 1 \end{pmatrix} (e^{2\pi i \text{Frac}(2xd^{-1}r/D)} \text{char}_{\mathbb{Z}_p + 1/2}(x)) = e^{2\pi i \text{Frac}(2c/dx^2)} (e^{2\pi i \text{Frac}(2xd^{-1}r/D)} \text{char}_{\mathbb{Z}_p}(x))$

- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (e^{2\pi i \text{Frac}(2xd^{-1}r/D)} \text{char}_{\mathbb{Z}_p}(x)) = \Phi_p^{(d^{-1})}(-x)$

- $r \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_p^{(d^{-1})}(-x) = \Phi_p^{(d^{-1})}(x)$

In here we have used the Fourier transform:

$$\hat{\Phi}_3^{(r)}(x) = \int_{\mathbb{Q}_p} \Phi_p^{(r)}(y) e^{-2\pi i \text{Frac}(2xy)} dy = \int_{\mathbb{Z}_p + \frac{r}{6}} \Phi_p^{(r)}(y) e^{2\pi i 2xy} dy = \int_{\mathbb{Z}_p} e^{-2\pi i \text{Frac}(2x(y+r/D))} dy$$

$$= \int_{\mathbb{Z}_p} e^{-2\pi i \text{Frac}(2xy)} e^{-2\pi i \text{Frac}(xrt/D)} dy = e^{-2\pi i \text{Frac}(2xr/D)} \int_{\mathbb{Z}_p} e^{-2\pi i \text{Frac}(2xy)} dy$$

$$= e^{-2\pi i \text{Frac}(2xr/D)} \text{char}_{\mathbb{Z}_p-1/2}(x) = e^{-2\pi i \text{Frac}(2xr/D)} \text{char}_{\mathbb{Z}_p}(x)$$

Similarly we get $\hat{\Phi}_3^{(r)}(x) = e^{-2\pi i \text{Frac}(x/3)} \text{char}_{\mathbb{Z}_p}(x)$
Note that the only difference for \( p = 3 \) in the action of \( \begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix} \) is that it does not modify \( r/D \). Instead, it leaves \( \Phi^{(r)} \) unchanged.

At the place \( p = 2 \), we have \( \Phi_2 = e^{\pi i \text{Frac}(x)} \text{char}_{Z_2-1/2}(x) \). We can compute:

\[ r \begin{pmatrix} 1 & -b/(2d) \\ 0 & 1 \end{pmatrix} \Phi_p(x) = e^{-\pi i \text{Frac}(-b/(2d)x^2)} e^{\pi i x} \text{char}_{Z_2-1/2}(x) = e^{2\pi ib/8d} e^{\pi i x} \text{char}_{Z_2-1/2}(x) \Phi_p(x) \]

Note that we picked \( 2|b \). Then we have \( x \in Z_2 - 1/2 \) iff \( x = n - 1/2 \) for \( n \in Z_2 \). Then \( -b/2d(n - 1/2)^2 = -b/(2d)n^2 + b/(2d)n - b/(8d) \in Z_2 - b/8d \).

\[ r \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} e^{\pi i b/8d} e^{\pi i x} \text{char}_{Z_2-1/2}(x) = e^{2\pi ib/8d} e^{\pi i dx} \text{char}_{Z_2-1/2}(dx) = e^{2\pi ib/8d} e^{\pi i x} \text{char}_{Z_2-1/2}(x) \]

Note that we have used above \( \nu_2(d) = 0 \).

\[ r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_2(r)(x) = e^{2\pi ib/8d} e^{\pi i x} \text{char}_{Z_2-1/2}(x) = e^{2\pi ib/8d} e^{\pi i \text{Frac}(x^2+1/4)} \text{char}_{Z_2-1/2}(x) \]

Below we compute the Fourier transform:

\[ \int_{Z_2} e^{\pi i \text{Frac}(y)} \text{char}_{Z_2-1/2}(y) e^{-\pi i \text{Frac}(2xy)} dy = \int_{Z_2} e^{2\pi i \text{Frac}(y^2 + 2xy + x)} dy = \int_{Z_2} e^{2\pi i \text{Frac}(y/2 + 1/4 + 2xy + x)} dy = e^{2\pi i \text{Frac}(x+1/4)} \text{char}_{Z_2-1/4}(x) \]

\[ r \begin{pmatrix} 1 & 0 \\ 2c/d & 1 \end{pmatrix} e^{2\pi i \text{Frac}(x+1/4)} \text{char}_{Z_2-1/4}(x) = e^{2\pi i \text{Frac}(2c/dx)} e^{2\pi i \text{Frac}(x+1/4)} \text{char}_{Z_2-1/4}(x) \]

Note that we have the assumptions \( 2|c \) and \( 2 \nmid d \). We have \( x + 1/4 = 1/2n, n \in Z_2 \). Note \( x^2 = (n - 1/2)^2 = (n^2 - n + 1/4)/4 \) and then \( e^{2\pi i \text{Frac}(2c/dx^2)} = e^{2\pi i \text{Frac}(c/2d(n^2 - n + 1/4))} = e^{2\pi i \text{Frac}(c/8d)} \). Here we have used the fact that \( c/2d(n^2 - n) \in Z_2 \). Thus we get:

\[ e^{2\pi i \text{Frac}(c+b)/8d} e^{2\pi i \text{Frac}(x+1/4)} \text{char}_{Z_2-1/4}(x) \]

\[ r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{2\pi i \text{Frac}((c+b)/8d)} e^{2\pi i \text{Frac}(x+1/4)} \text{char}_{Z_2-1/4}(x) = e^{2\pi i \text{Frac}((c+b)/8d)} e^{2\pi i \text{Frac}(-x)} \text{char}_{Z_2-1/2}(-x) \]

\[ r \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} e^{2\pi i \text{Frac}((c+b)/8d)} e^{2\pi i \text{Frac}(-x)} \text{char}_{Z_2-1/2}(-x) = e^{2\pi i \text{Frac}((c+b)/8d)} e^{2\pi i \text{Frac}(x)} \text{char}_{Z_2-1/2}(x) = e^{2\pi i \text{Frac}((c+b)/8d)} \Phi_2(x) \]

Finally we are ready to prove the proposition. We have showed so far that:

\[ f_r(\tau) \gamma_x^{-1} = f_r^{g_r(x)}(\tau) = f_r^{(g_r(\text{Frac}))_r} \text{char}_{D^{1/2}}(\tau) = \begin{pmatrix} t - sb & -sc \\ s & t \end{pmatrix} \rho^{(\sigma)}(\tau) = f_r(\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}) \rho^{(\sigma)}(\tau) \]

From the above lemma we get immediately:

\[ \left( \frac{\Theta^{(\tau)}(\tau/2)}{\Theta^{(0)}(\tau/2)} \right)^{\gamma_x} = f_r \left( \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \right)^{-1} = \frac{\Theta^{(\tau+\pi)}(\tau/2)}{\Theta^{(\tau)}(\tau/2)} = f_r^{(g_r(\phi))}(\tau) \]

55
For $\mathcal{A} \in Cl(O_{3D})$, $\mathcal{A} = (k, \mathcal{A}) = (na + m_n\frac{b + \sqrt{3}}{2})$, where $a = Nm\mathcal{A}$, we take the map:

$$x = (k, \mathcal{A})_{p|D} \leftrightarrow \mathcal{A}$$

This gives us:

$$x^{-1} \leftrightarrow \mathcal{A}^{-1}$$

Then we have:

$$f_r(\tau)^{\sigma_{\mathcal{A}^{-1}}} = f_r(\tau)^{\sigma_{\mathcal{A}}} = f_r^{g_r(x_p)}(\tau) = f_r^{g_r(k, \mathcal{A})_{p|D}}(\tau) = f_r^{n_{\mathcal{A}}}^{-1}(\tau)$$

This implies for $r \equiv n_{\mathcal{A}} \ mod \ D$ that we have $f_{n_{\mathcal{A}}}(\tau)^{\sigma_{\mathcal{A}}} = f_1(\tau)$, or equivalently:

$$f_1(\tau)^{\sigma_{\mathcal{A}}} = f_{n_{\mathcal{A}}}(\tau)$$

**Remark.** This implies that for $\mathcal{A}_r = (1 + b^*(r - 1) \frac{b + \sqrt{3}}{2})$ we have:

$$f_1(\tau)^{\sigma_{\mathcal{A}_r}} = f_1(\tau)$$

Also it implies that $a_r = (r^{-1}) = (r \cdot r^{-2} + \frac{b + \sqrt{3}}{2})$ we have:

$$f_1(\tau)^{\sigma_{a_r}} = f_r(\tau)$$

### 5.5 The square is invariant under Galois action

We are finally ready to prove Theorem 5.1.

We define $\mathcal{A}_r^c = \left(1 + b^*(1 - r) \frac{b + \sqrt{3}}{2}\right)$. Note $n_r = r^{-1}$. Note that $\mathcal{A}_r^c = \mathcal{A}_r(r^{-1})$, thus $\mathcal{A}_r$ and $\mathcal{A}_r^c$ are in the same class in $Cl(O_{3D})$. This implies:

$$\chi_D(\mathcal{A}_r) = \chi_D(\mathcal{A}_r^c)$$

Moreover, from the definition of $\chi_D$ we have: $(D^{2/3})^{\sigma_{\mathcal{A}_r}} = D^{2/3}\chi_D(\mathcal{A}_r)$

Moreover, from Proposition 5.3:

$$f_1(\tau)^{\sigma_{\mathcal{A}_r}} = f_{n_{\mathcal{A}_r}}(\tau) = f_r(\tau)$$

Then we can rewrite the term in Proposition 20:

$$\kappa := \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^*} f_r(\tau)D^{2/3}\chi(\mathcal{A}_r) = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^*} f_r(\tau)D^{2/3}\chi(\mathcal{A}_r^c) = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^*} f_1(\tau)^{\sigma_{\mathcal{A}_r}}(D^{2/3})^{\sigma_{\mathcal{A}_r}}$$

$$= \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^*} (f_1(\tau)D^{2/3})^{\sigma_{\mathcal{A}_r}}$$

We want to write $\kappa$ as a Galois trace of a modular function at a CM-point. Note that the ideals $\{\mathcal{A}_{r,r}^c, (r\mathbb{Z}/D\mathbb{Z})^*\}$ for a group, as we have $\mathcal{A}_r^c, \mathcal{A}_{r,r}^c = \mathcal{A}_{r,r}^c$. Then take $G_0 = \{r \in (\mathbb{Z}/D\mathbb{Z})^* :$
that is a subgroup of $\text{Gal}(H_\mathcal{O}/K)$, where $H_\mathcal{O}$ is the ray class field of conductor $3D$.

We define fixed field of $G_0$ in $H$:

$$H_0 = \{ h \in H_\mathcal{O} : \sigma(h) = h, \forall \sigma \in G_0 \}$$

From abelian Galois theory this implies $\text{Gal}(H_\mathcal{O}/H_0) \cong G_0$. Then we got:

$$\kappa = \text{Tr}_{H_\mathcal{O}/H_0}(f_1(\tau)D^{2/3})$$

Thus we have proved so far that:

$$S_D = |\kappa|^2,$$

where $\kappa \in H_0$. We claim that actually $|\kappa|^2 \in \mathbb{Q}$. To prove this, it is enough to show that $|\kappa|^2 \in K^\times$, as

**Lemma 5.15.** We have $\kappa^3 \in K$.

**Proof:** We will show that the Galois conjugates of $\kappa$ over $K$ are $\kappa \omega$ and $\kappa \omega^2$.

Take $\mathcal{A} \in \text{Cl}(\mathcal{O})$. Then we have:

$$\kappa^{\mathcal{A}} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau)D^{2/3})^{\mathcal{A}_r^{\mathcal{A}} \mathcal{A}}$$

We can write $\mathcal{A} = A_5^\circ(m)$. Then we have:

$$\kappa^{\mathcal{A}} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau)D^{2/3})^{\mathcal{A}_r^{\mathcal{A}} \mathcal{A}}$$

Note that $(m)$ acts trivially on $D^{2/3}$, but acts as $A_m^\circ$ on $f_1(\tau)$. Then we have:

$$\kappa^{\mathcal{A}} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau))^{\mathcal{A}_r \mathcal{A}} D^{2/3}\chi(A_r^\circ) = \chi(A_m^\circ) \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau))^{\mathcal{A}_r \mathcal{A}} D^{2/3}\chi(A_r^\circ) = \chi(A_m^\circ)\kappa$$

**Remark 5.3.** Recall that $|\kappa|^2 \in \mathbb{Q}$. Let $\kappa^3 = a + b\sqrt{-3} \in K$. Then $|\kappa|^6 = a^2 + 3b^2$ and we must have $a^2 + 3b^2 = m^3$ for some $m \in \mathbb{Q}$. With this notation we have $|\kappa|^2 = m = \sqrt{a^2 + 3b^2}$.

**Remark 5.4.** If we try to apply $\pi$, this implies:

$$\pi^{\mathcal{A}_r} = (\kappa^{\mathcal{A}_r \mathcal{A}})^{-1} = \pi$$

$$\pi^{\mathcal{A}_r \mathcal{A}} = (\kappa^{\mathcal{A}_r \mathcal{A}}) = \chi(A_r^{\mathcal{A}_r})$$

$$\pi^{\mathcal{A}_r \mathcal{A}} = \kappa \chi(A_r) = \kappa \chi(A_r)$$

6 Appendix A: properties of $\Theta_K$

In this appendix we would like to present a few properties of $\Theta_K$. First, we have a functional equation for the theta function (see [9]):
\[ \Theta_K(-1/3z) = \frac{3}{\sqrt{-3}} z \Theta_K(z) \] (25)

Furthermore, we can compute the transformation of \( \Theta_K(z \pm 1/3) \) in the lemma below:

**Lemma 6.1.** We have the following relations:

(i) \( \Theta \left( z + \frac{1}{3} \right) = (1 - \omega) \Theta(3z) + \omega \Theta(z) \)

(ii) \( \Theta \left( z - \frac{1}{3} \right) = (1 - \omega^2) \Theta(3z) + \omega^2 \Theta(z) \)

**Proof:** We will rewrite the Fourier expansion of \( \Theta(z) \) for \( z := z + 1/3 \):

\[ \Theta \left( z + \frac{1}{3} \right) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2 + n^2 - mn)} \left( z + \frac{1}{3} \right). \]

We split the sum in two parts, depending on whether the ideal \((m + n \omega)\) is prime to \((\sqrt{-3})\). Then we have:

\[ \Theta \left( z + \frac{1}{3} \right) = \sum_{m,n \in \mathbb{Z}, (\sqrt{-3})(m+n\omega)} e^{2\pi i (m^2 + n^2 - mn)} (z + \frac{1}{3}) + \sum_{m,n \in \mathbb{Z}, (\sqrt{-3})(m+n\omega)} e^{2\pi i (m^2 + n^2 - mn)} (z + \frac{1}{3}). \]

Note that on the RHS we can rewrite the first term as:

\[ \sum_{m,n \in \mathbb{Z}, (\sqrt{-3})(m+n\omega)} e^{2\pi i (m^2 + n^2 - mn)} (z + \frac{1}{3}) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2 + n^2 - mn)(3z+1)} = \Theta(3z + 1) = \Theta(3z) \]

Also note that when \( 3 \nmid m^2 + n^2 - mn \), then we have \( m^2 + n^2 - mn \equiv 1 \mod 3 \). Then the second term on the RHS can be rewritten as:

\[ \sum_{m,n \in \mathbb{Z}, (\sqrt{-3})(m+n\omega)} e^{2\pi i (m^2 + n^2 - mn)} \left( z + \frac{1}{3} \right) = \sum_{m,n \in \mathbb{Z}, (\sqrt{-3})(m+n\omega)} e^{2\pi i (m^2 + n^2 - mn)z} \cdot \overline{\omega}. \]

We rewrite this:

\[ \sum_{m,n \in \mathbb{Z}, (\sqrt{-3})(m+n\omega)} e^{2\pi i (m^2 + n^2 - mn)} (z + \frac{1}{3}) = \omega \sum_{m,n \in \mathbb{Z}} e^{2\pi i (m^2 + n^2 - mn)z} - \omega \sum_{m,n \in \mathbb{Z}, (\sqrt{-3})(m+n\omega)} e^{2\pi i (m^2 + n^2 - mn)z} \]

Finally we recognize the two terms as theta functions \( \Theta_K \):

\[ \sum_{m,n \in \mathbb{Z}, (\sqrt{-3})(m+n\omega)} e^{2\pi i (m^2 + n^2 - mn)} (z + \frac{1}{3}) = \omega \Theta(z) - \omega \Theta(3z) \]

Now going back to our initial computation, we get:
\[ \Theta \left( z + \frac{1}{3} \right) = \Theta(3z) + \omega \Theta(z) - \omega \Theta(3z) = (1 - \omega) \Theta(3z) + \omega \Theta(z) \]

This finishes the proof of the first formula. We get the second formula by applying the first formula for \( z := z - 1/3 \). We get \( \Theta(z) = (1 - \omega) \Theta(3z - 1) + \omega \Theta(z - 1/3) \) and this is easily rewritten to give us the second formula.

### 6.1 Properties of \( \Theta_K((-b + \sqrt{3})/6) \).

**Lemma 6.2.** \( \Theta_K \left( \frac{-3 + \sqrt{3}}{6} \right) = 0 \)

**Proof:** We apply the functional equation 24 for \( z = \frac{-3 + \sqrt{3}}{6} \):

\[ \Theta \left( \frac{-3 + \sqrt{3}}{6} \right) = (-\sqrt{-3}) \frac{-3 + \sqrt{3}}{6} \Theta \left( \frac{3 + \sqrt{3}}{6} \right). \]

Since \( \Theta \left( \frac{-3 + \sqrt{3}}{6} \right) = \Theta \left( \frac{3 + \sqrt{3}}{6} \right) \), we get the result of the lemma.

**Lemma 6.3.** For the primitive ideal \( \mathcal{A} = [a, \frac{-b + \sqrt{3}}{2}] \) prime to 3, where \( a = \text{Nm} \mathcal{A}, b \equiv 0 \mod 3 \) and \( b^2 \equiv -3 \mod 4a \), we have:

\[ \Theta_K \left( \frac{-b + \sqrt{3}}{6a} \right) = 0 \]

**Proof:** The proof is similar to that of Lemma 3.5. We can write the generator of primitive ideal \( \mathcal{A} = [a, \frac{-b + \sqrt{3}}{2}] \) in the form \( k_A = ma + n \frac{-b + \sqrt{3}}{2} \) for some integers \( m, n \). Note that \( (m, 3) = 1 \), thus we can find through the Euclidean algorithm integers \( A, B \) such that \( mA + 3nB = 1 \), which makes \( \begin{pmatrix} A & B \\ -3n & m \end{pmatrix} \) a matrix in \( \Gamma_0(3) \). Since \( \Theta \) is a modular form of weight 1 for \( \Gamma_0(3) \), we have:

\[ \Theta_K \left( \frac{A \frac{-b + \sqrt{3}}{6a} + B}{-3n \frac{-b + \sqrt{3}}{6a} + m} \right) = \left( m - n \frac{-b + \sqrt{3}}{2a} \right) \Theta_K \left( \frac{-b + \sqrt{3}}{6a} \right) \]

Noting that \( -3n \frac{-b + \sqrt{3}}{6a} + m = k_A/a = 1/\mathcal{F}_A \), we can compute \( \frac{A \frac{-b + \sqrt{3}}{6a} + B}{-n \frac{-b + \sqrt{3}}{6a} + m} = \frac{(A \frac{-b + \sqrt{3}}{6a} + B)}{3a} \). This is \( (3aB + A \frac{-b + \sqrt{3}}{2})(ma + n \frac{-b + \sqrt{3}}{2})/(3a) \). After expanding, we get:

\[ \frac{-nA}{4a} b^2 + 3 + abB/3 + \frac{b(-mA + 3nB)}{6} + \frac{\sqrt{3}}{6} \]

Note that \( mA + 3nB = 1 \) implies that \( mA \) and \( 3nB \) have different parities. Also we chose \( b \) odd, since \( b^2 + 3 \equiv 0 \mod 4a \). Finally, recall \( 3|b \) and thus using the period 1 of \( \Theta_K \) we get:

\[ \Theta_K \left( \frac{A \frac{-b + \sqrt{3}}{6a} + B}{-3n \frac{-b + \sqrt{3}}{2a} + m} \right) = \Theta_K \left( \frac{-3 + \sqrt{3}}{6} \right) \]
From the previous Lemma, we have \( \Theta_K \left( \frac{-3+\sqrt{-3}}{6} \right) \), thus \( \Theta_K \left( \frac{-b+\sqrt{-3}}{6a} \right) = 0 \) which finishes the proof.

6.2 About \( \Theta_K(D(-3 + \sqrt{-3})/6) \).

In this section we will show that for \( D \) a product of split primes \( p \equiv 1 \mod 3 \) and for the representative ideals \( A = [a, \frac{-b+\sqrt{-3}}{2}] \) of \( \text{Cl}(\mathcal{O}_D) \) with \( b \equiv 0 \mod 3 \), we have:

\[
\sum_{A \in \text{Cl}(\mathcal{O}_D)} \frac{\Theta \left( \frac{-b+\sqrt{-3}}{6a} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{2a} \right)} \chi_D(A) D^{1/3} = 0
\]

We will first show that the LHS is equal to the trace of \( \frac{\Theta_K(D \frac{-b+\sqrt{-3}}{3})}{\Theta(z)} D^{1/3} \) with \( b \equiv 0 \mod 3 \). We will show this by using Shimura reciprocity law. Note first that:

**Lemma 6.4.** The modular function \( f_0(z) = \frac{\Theta(Dz/3)}{\Theta(z)} \) is a modular function for \( \Gamma(3D) \) and \( f_0(z) \) has rational Fourier coefficients at the cusp \( \infty \).

**Proof:** The proof that \( f_0 \) is invariant under \( \Gamma(3D) \) is straightforward. The proof that the Fourier coefficients are rational is also similar to the proof of Lemma ??.

**Lemma 6.5.** For \( f_0 \) as above and \( \tau = \frac{-b_0+\sqrt{-3}}{2} \), we have \( f_0(\tau) \in H_{3D} \).

**Proof:** To show that \( f(\tau) \in H_{3D} \), we need to look at action of \( U(3D) \). We follow closely the proof of Lemma ???. We rewrite the primitive ideal \( A = (A + B \omega) \) as \( A = [a, \frac{-b+\sqrt{-3}}{2}] \) with \( b \equiv b_0 \mod 3 \). The only difference is computing:

\[
f_0 \left( \frac{ta-sb-sc/a}{t} \right) = \frac{\Theta_K \left( \left( \frac{D}{3} \right) \left( \frac{ta-sb-sc/a}{t} \right) \right)}{\Theta_K \left( \left( \frac{ta-sb-sc/a}{t} \right) \right)} = \frac{\Theta_K \left( \left( \frac{ta-sb-scD/(3a)}{3s/D} \right) \left( \frac{Dz}{3} \right) \right)}{\Theta_K \left( \left( \frac{ta-sb-sc/a}{t} \right) \right)}.
\]

Note that we still have \( \left( \frac{ta-sb-scD/(3a)}{3s/D} \right), \left( \frac{ta-sb-sc/a}{t} \right) \in \Gamma_0(3) \), thus we simply get \( f_0(z) \) and all the arguments from Lemma ?? follow.

**Lemma 6.6.** For \( A = \left[ a, \frac{-b+\sqrt{-3}}{2} \right] \) a primitive ideal ideal with \( a = \text{Nm} A \) and \( b^2 \equiv -3 \mod 4a \), we have:

\[
\frac{\Theta \left( \frac{-b+\sqrt{-3}}{6a} \right)}{\Theta \left( \frac{-b+\sqrt{-3}}{2a} \right)} = \left( \frac{\Theta \left( \frac{-b+\sqrt{-3}}{6} \right)}{\Theta(\omega)} \right)^{\sigma_A^{-1}}
\]

**Proof:** Note that \( f_0(z) \) satisfies the properties of Lemma ??, thus applying its result for \( f_0 \left( \frac{-b+\sqrt{-3}}{2} \right) \) gives us the result.

From the previous two lemmas, we immediately get the following Corollary:
Corollary 6.1. For $A = \left[ a, \frac{-b + \sqrt{-3}}{2} \right]$ primitive ideals that are representatives of $\text{Cl}(O_{3D})$ as in Lemma ??, we have:

$$\text{Tr}_{H_{3D}/K} \frac{\Theta(D - \frac{b + \sqrt{-3}}{6a})}{\Theta\left(\frac{-b + \sqrt{-3}}{2a}\right)} D^{1/3} = \sum_{A \in \text{Cl}(O_{3D})} \frac{\Theta\left(\frac{-b + \sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b + \sqrt{-3}}{2a}\right)} \chi_D(A) D^{1/3}$$

6.2.1 Traces of theta functions

We will show the following lemma:

Lemma 6.7. For $D \equiv 1 \mod 3$, $b_0 \equiv 0 \mod 3$ as before, we have:

$$\text{Tr}_{H_{3D}/K} \frac{\Theta\left(D - \frac{3}{6D}\right)}{\Theta(\omega)} D^{1/3} = \sum_{A \in \text{Cl}(O_{3D})} \frac{\Theta_K\left(D - \frac{b_0 + \sqrt{-3}}{6a}\right)}{\Theta_K\left(D - \frac{b_0 + \sqrt{-3}}{2a}\right)} \chi_D(A) D^{1/3} = 0.$$  

Proof: The method will be to apply Lemma 6.1 two times. We first apply Lemma ?? (i) for $z = \frac{1 - 2D}{6D}$ to get:

$$\Theta\left(\frac{1 + \sqrt{-3}}{6D}\right) = (1 - \omega)\Theta\left(\frac{1 + \sqrt{-3}}{2D}\right) + \omega\Theta\left(\frac{1 - 2D + \sqrt{-3}}{6D}\right)$$

This can be rewritten as:

$$\frac{\Theta\left(\frac{1 + \sqrt{-3}}{6D}\right)}{\Theta(\omega)} = (1 - \omega)\frac{\Theta\left(\frac{1 + \sqrt{-3}}{2D}\right)}{\Theta(\omega)} + \omega\frac{\Theta\left(\frac{1 - 2D + \sqrt{-3}}{6D}\right)}{\Theta(\omega)}$$

By taking the inverses and denoting $B_1 := -1 + 2D$, $a_1 := (B_1^2 + 3)/4$, we have:

$$3D \frac{\Theta\left(D - \frac{1 + \sqrt{-3}}{2a_1}\right)}{\Theta(\omega/3)} = (1 - \omega)\frac{\Theta\left(\frac{1 + \sqrt{-3}}{2D}\right)}{\Theta(\omega)} + 3D\omega\frac{\Theta\left(D - \frac{B_1 + \sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1 + \sqrt{-3}}{6a}\right)}$$

Note that $B_1 \equiv 1 - 2D \equiv 1 \mod 3$. Furthermore, noting that $\Theta(\omega/3) = (1 - \omega)\Theta(\omega)$ and $\Theta\left(\frac{B_1 + \sqrt{-3}}{6a}\right) = (1 - \omega^2)\Theta\left(\frac{B_1 + \sqrt{-3}}{2a_1}\right)$, we get:

$$3D \frac{\Theta\left(D - \frac{1 + \sqrt{-3}}{2a_1}\right)}{\Theta(\omega)} = (1 - \omega)\frac{\Theta\left(\frac{1 + \sqrt{-3}}{2D}\right)}{\Theta(\omega)} + 3D\omega\frac{\Theta\left(D - \frac{B_1 + \sqrt{-3}}{2a_1}\right)}{1 - \omega^2}$$

Multiplying by $D^{1/3}$ and rewriting the first term on the RHS, we have:

$$\frac{3D}{1 - \omega} \frac{\Theta\left(D - \frac{1 + \sqrt{-3}}{2a_1}\right)}{\Theta(\omega)} D^{1/3} = (1 - \omega)(1 - \omega^2) \frac{\Theta\left(\frac{1 + \sqrt{-3}}{2D}\right)}{(1 - \omega^2)\Theta(\omega)} D^{1/3} + 3D\omega \frac{\Theta\left(D - \frac{B_1 + \sqrt{-3}}{2a_1}\right)}{1 - \omega^2} \chi_D(A_1)^{-1} \frac{\Theta\left(D - \frac{B_1 + \sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1 + \sqrt{-3}}{2a_1}\right)} D^{1/3} \chi_D(A_1)$$

61
By taking the trace from $H_{3D}$ to $K$ and denoting by $A_1 := \left( \frac{B_1 + \sqrt{-3}}{2} \right)$, we have:

\[
\frac{3D}{1 - \omega} \text{Tr}_{H_{3D}/K} \frac{\Theta(D\omega)}{\Theta(\omega)} D^{1/3} = 3 \text{Tr}_{H_{3D}/K} \frac{\Theta(-D\omega^2)}{\Theta(-\omega^2/3)} D^{1/3} + 3D\omega \frac{\Theta(\omega)}{1 - \omega^2} \text{Tr}_{H_{3D}/K} \frac{\Theta(D\omega)}{\Theta(\omega)} D^{1/3} \chi_D(A_1).
\]

Note that by definition we have $\chi_D(A_1) = \chi_D \left( \frac{B_1 + \sqrt{-3}}{2} \right)$. We can compute the value of the character using Lemma ?? . For each $p | D$, we have:

\[
\chi_p \left( \frac{B_1 + \sqrt{-3}}{2} \omega \right) = \left( \frac{(1 - 2D - \sqrt{-3})\omega^2}{1 - 2D + \sqrt{-3}} \right)^{N\text{m}p^{-1}/3} = \left( \frac{1}{1} \right)^{N\text{m}p^{-1}/3} = 1.
\]

Thus we get $\chi_D(A_1) = 1$, and we can rewrite the equation above as:

\[
\frac{3D}{1 - \omega} \text{Tr}_{H_{3D}/K} \frac{\Theta(D\omega)}{\Theta(\omega)} D^{1/3} = 3 \text{Tr}_{H_{3D}/K} \frac{\Theta(-D\omega^2)}{\Theta(-\omega^2/3)} D^{1/3} + 3D\omega \text{Tr}_{H_{3D}/K} \frac{\Theta(D\omega)}{\Theta(\omega)} D^{1/3} \chi_D(A_1).
\]

Furthermore, from Lemma ??, we have $\frac{\Theta(D\omega)}{\Theta(\omega)} D^{1/3} \chi_D(A_1) = \left( \frac{\Theta(D\omega)}{\Theta(\omega)} D^{1/3} \right)^\sigma_{\chi_D(A_1)}$, thus:

\[
\text{Tr}_{H_{3D}/K} \frac{\Theta(D\omega)}{\Theta(\omega)} D^{1/3} \chi_D(A_1) = \text{Tr}_{H_{3D}/K} \frac{\Theta(D\omega)}{\Theta(\omega)} D^{1/3}.
\]

Denoting $S := \text{Tr}_{H_{3D}/K} \frac{\Theta(D\omega)}{\Theta(\omega)} D^{1/3}$, we get the relation:

\[
\frac{3D}{1 - \omega} S = 3 \text{Tr}_{H_{3D}/K} \frac{\Theta(1 + \sqrt{-3})}{2D} D^{1/3} + 3D\omega \frac{\Theta(\omega)}{1 - \omega^2} S
\]

This implies:

\[
\frac{3D}{1 - \omega^2} S = 3 \text{Tr}_{H_{3D}/K} \frac{\Theta(1 + \sqrt{-3})}{2D} D^{1/3}
\]

This is equivalent to:

\[
\frac{D}{1 - \omega^2} S = \text{Tr}_{H_{3D}/K} \frac{\Theta(1 + \sqrt{-3})}{2D} D^{1/3}
\]

Note that if we apply the transformation $z \rightarrow -1/3z$ given by the functional equation (24) to both theta functions on the RHS we get:

\[
\frac{1}{1 - \omega^2} S = \frac{1}{3} \text{Tr}_{H_{3D}/K} \frac{\Theta(D - 1 + \sqrt{-3})}{6\Theta(\omega)} D^{1/3}
\]
This is equivalent to:

\[(1 - \omega)S = \text{Tr}_{H^D/K} \frac{\Theta \left(D \frac{-1 + \sqrt{-3}}{6}\right)}{\Theta(\omega)} D^{1/3}.\]  

(26)

We will apply now Lemma 6.1 (ii) for \(z = D \frac{-b_1 + \sqrt{-3}}{2a}\), where \(b_1 \equiv 1 \mod 3\). We denote by \(b_0\) an integer \(b_0 \equiv 0 \mod 3\) such that \(b_0 \equiv b_1 \mod 4a\). Then we have:

\[\Theta \left(D \frac{-b_0 + \sqrt{-3}}{6a}\right) = (1 - \omega^2)\Theta \left(D \frac{-1 + \sqrt{-3}}{2a}\right) + \omega^2 \Theta \left(D \frac{-b_1 + \sqrt{-3}}{6a}\right)\]

This can be rewritten as:

\[\frac{\Theta \left(D \frac{-b_0 + \sqrt{-3}}{6a}\right)}{\Theta \left(D \frac{-b_0 + \sqrt{-3}}{2a}\right)} D^{1/3} \chi_D(A) = (1 - \omega^2) \frac{\Theta \left(D \frac{-b_0 + \sqrt{-3}}{2a}\right)}{\Theta \left(D \frac{-b_0 + \sqrt{-3}}{2a}\right)} D^{1/3} \chi_D(A) + \omega^2 \frac{\Theta \left(D \frac{-b_1 + \sqrt{-3}}{6a}\right)}{\Theta \left(D \frac{-b_1 + \sqrt{-3}}{2a}\right)} D^{1/3} \chi_D(A)\]

By taking the sums, we get:

\[M = (1 - \omega^2)S + \omega^2(1 - \omega)S = 0\]

References


