RESEARCH STATEMENT

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My main research interests are in number theory, with a focus on the area of automorphic forms and special values of L-functions of elliptic curves. More specifically, I am interested in connecting the classical language of modular forms with powerful tools coming from automorphic representations.

1. Introduction: integers that are sums of rational cubes

In my thesis problem I am interested in finding which integers N can be written as the sum of two rational cubes:

$$N = x^3 + y^3, \;\; x, y \in \mathbb{Q} \tag{1}$$

Despite the simplicity of the problem, an elementary approach to solving the Diophantine equation fails. However, we can restate the problem in the language of elliptic curves. After making the equation homogeneous and making a change of coordinates, the equation becomes:

$$E_N : Y^2 = X^3 - 432N^2,$$

which defines an *elliptic curve* over $\mathbb{Q}$ written in its Weierstrass affine form. Thus the problem reduces to finding if $E_N(\mathbb{Q})$, the set of rational points of the elliptic curve E_N, is non-trivial:

$$N = x^3 + y^3 \text{ has solutions in } \mathbb{Q} \Longleftrightarrow E_N(\mathbb{Q}) \neq \{O\}$$

By the *Mordell-Weil Theorem*, the set of rational points $E_N(\mathbb{Q})$ is a finitely generated abelian group. For simplicity, we will assume that N is cube free, and $N \neq 1, 2$. In these cases, the group has only trivial torsion (see [18]). Thus the existence of nontrivial rational solutions reduces to the problem of finding if the group of rational points $E_N(\mathbb{Q})$ of the elliptic curve E_N has positive rank.

Assuming the *Birch and Swinnerton-Dyer Conjecture (BSD)*, we can find rational non-torsion points on E_N iff the L-function $L(E_N, s)$ of the elliptic curve E_N vanishes at $s = 1$, i.e.:

$$L(E_N, 1) \neq 0 \Longleftrightarrow x^3 + y^3 = N \text{ has no solutions}$$

Without assuming BSD, from the work of Coates-Wiles [4], or more generally Gross-Zagier [7] and Kolyvagin [9], when $L(E_N, 1) \neq 0$, we have $\operatorname{rank} E_N(\mathbb{Q}) = 0$. Note that this implies that we have no rational solutions in (1). We define an invariant S_N of E_N as follows:

$$S_N = \frac{1}{\Omega_{N,\infty} R_{E_N}} L(E_N, 1),$$

where the denominator contains easily computable arithmetic invariants: $\Omega_{N,\infty} = \dfrac{1}{\sqrt[3]{N}} \Gamma\left(\dfrac{1}{3}\right)^3$ is the real period, and R_{E_N} is the regulator of the elliptic curve E_N.

The definition is made such that, when $L(E_N, 1) \neq 0$, we expect to get from the full BSD conjecture:

$$S_N = \#\text{Ш}(E_N) \prod_{p \mid 6N} c_p, \tag{2}$$

where $\#Ш$ the order of the Tate-Shafarevich group and c_p are the Tamagawa numbers corresponding to the elliptic curve E_N:

From the work of Rubin [16], when $L(E_N,1) \neq 0$ we have $\#Ш(E_N)$ is finite. Furthermore, using the Cassels-Tate pairing, Cassels proved in [3] that when Ш is finite, then its order $\#Ш$ is a square. Thus we expect S_N to be an integer square. Current work in Iwasawa theory shows that for semistable elliptic curves at the good primes p we have $\mathrm{ord}_p(Ш[p^\infty]) = \mathrm{ord}_p(S_N)$, where $Ш[p^\infty]$ is the p^∞-torsion part of Ш (see [8]). However, this cannot be applied at the place 3 in our case.

Main objective: We want to find a formula for S_N. By computing the value of S_N, we can determine when we have solutions in (1) and, assuming full BSD, we can find in certain cases the order of Ш:

(i) $S_N \neq 0 \Longrightarrow$ no solutions in (1)
(ii) $S_N \neq 0 \overset{BSD}{\Longrightarrow} S_N = \#Ш$
(iii) $S_N = 0 \overset{BSD}{\Longleftrightarrow}$ we have solutions in (1)

2. Results: Formulas for the L-function $L(E_N,1)$ and the order of $\#Ш$.

Note that each of the elliptic curves E_N is a **cubic twist** of E_1. This means that over $\mathbb{Q}$ the two elliptic curves are not isomorphic; however, they are isomorphic over $\mathbb{Q}[\sqrt[3]{N}]$ as can be easily shown by rewriting $E_N : 1 = \left(\frac{x}{\sqrt[3]{N}}\right)^3 + \left(\frac{y}{\sqrt[3]{N}}\right)^3$.

In the case of quadratic twists of elliptic curves, an important tool in computing the values of the L-functions is the work of Waldspurger [20]. For example, this is used to obtain Tunnell's Theorem for congruent numbers in [19]. However, the cubic twist case proves to be significantly more difficult.

The elliptic curves E_N have *complex multiplication* (CM) by $\mathcal{O}_K$, the ring of integers of the number field $K = \mathbb{Q}[\sqrt{-3}]$. By complex multiplication we mean that $\mathrm{End}\, E_N(\mathbb{C}) \cong \mathcal{O}_K$. From CM-theory this implies that:

$$L(E_N, s) = L(s, \widetilde{\chi}),$$

where $\widetilde{\chi} : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ is a Hecke character, where $\mathbb{A}_K^\times$ is the idele class group of K. Moreover, the character $\widetilde{\chi}$ corresponding to the elliptic curve E_N is the product of a cubic character χ_N and the Hecke character φ corresponding to the elliptic curve E_1 i.e. $\widetilde{\chi} = \chi_N \varphi$.

My first result computes the value of S_N and thus the value of the L-function in the following way:

Theorem 2.1. *Let $K = \mathbb{Q}[\sqrt{-3}]$ and for all positive integers N define $S_N = \frac{1}{\Omega_{N,\infty} R_{E_N} \prod_{p|6N} c_p} L(E_N, 1)$.*

Then S_N is an integer and we have the formula:

$$S_N = \mathrm{Tr}_{H_{3N}/K} \left(N^{1/3} \frac{\Theta_K(N\omega)}{\Theta_K(\omega)} \right), \tag{3}$$

Here:

- *H_{3N} is the ring class field associated to the order $\mathcal{O}_{3N} = \mathbb{Z} + 3N\mathcal{O}_K$,*
- *$\omega = \frac{-1+\sqrt{-3}}{2}$ is a third root of unity, and*
- *$\Theta_K(z) = \frac{1}{2} \sum_{a,b \in \mathbb{Z}} e^{2\pi i z(a^2+b^2+ab)}$ is the theta function associated to the number field K.*

Note that using the formula (3) we can show that a particular N cannot be written as the sum of two cubes by computationally checking whether $L(E_N, 1) \neq 0$. Furthermore, assuming BSD, we have $S_N = \#\text{Ш}$, thus we can compute the expected order of Ш explicitly. The formula (3) above proves that the term S_N is, as expected, an integer.

The idea of proof of Theorem 2.1 is to relate the L-function $L(s, \chi)$ to Tate's Zeta function $Z(s, \chi, \Phi)$ (see [2]). We obtain a formula for the L-function as a linear combination of Eisenstein series. Through a variation of the Siegel-Weil formula (see [11]), we obtain at $s = 0$ linear combinations of theta functions $\Theta_K(z)$. The trace is obtained by using Shimura reciprocity on the modular curve $X_0(3D)$.

In [15], Rodriguez-Villegas and Zagier have proved a similar result for primes $p \equiv 1 \mod 9$: $S_p = \operatorname{Tr} \dfrac{\sqrt[3]{p}\Theta_K(p\delta)}{54\Theta_K(\delta)}$, for $\delta = \frac{-9+\sqrt{-3}}{18}$. Note that my methods are different as they are based on automorphic computations. Moreover, I am generalizing the result of [15] to all integers N.

A second result of my thesis is a formula similar in nature to the Theorem 2 of [15]. We prove the formula for all integers N that are products of primes that split in $\mathcal{O}_K$:

Theorem 2.2. *Let $N = \prod_{p_i \equiv 1 \mod 3} p_i^{e_i}$ be a positive integer.*

Then S_N is an integer and we have:

$$S_N = \left| \operatorname{Tr}_{H_{\mathcal{O}}/H_0} \left(N^{-1/3} \frac{\theta_1(z_0)}{\theta_0(z_0)} \right) \right|^2 \tag{4}$$

where:

- $\theta_1(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i \left(n + \frac{1}{N} - \frac{1}{6}\right)^2 z}$ *is a 1/2-weight modular form*
- $z_0 = \frac{-b+\sqrt{-3}}{2}$ *a CM-point, with $b^2 \equiv -3 \mod 4N^2$,*
- $H_{\mathcal{O}}$ *is the ray class field of modulus $3N$, and*
- H_0 *is an intermediate field $K \subset H_0 \subset H_{\mathcal{O}}$ that is the fixed field of a certain Galois subgroup $G_0 \cong \mathrm{Cl}(\mathcal{O}_{3D})$.*

Note that the value of S_N is easily computable using mathematical software like SAGE, thus giving us the actual order of the Tate-Shafarevich group when $S_N \neq 0$.

The idea of the proof of Theorem 2.2 is based on factoring each weight one theta function $\Theta_K(z)$ into a product of theta functions of weight 1/2. The method we are using is a factorization lemma of Rodriguez-Villegas and Zagier from [14] applied to the formula in Theorem 2.1 .

We have also proved a similar result in the general case (all integers N). However, the formula is more complicated, as it consists of a sum of several squared traces.

In [15], Rodriguez-Villegas and Zagier have proved a different formula for primes $p \equiv 1 \mod 9$: $S_p = \left(\operatorname{Tr} \dfrac{\sqrt[6]{p}}{\sqrt{\pm 12}} \dfrac{\eta(pz_0)}{\eta(z_0/p)} \right)^2$, where $\eta(z)$ is the Dedekind eta-function and z_0 as above. I am currently working on generalizing their result. Obtaining such a formula would easily show that S_N is an integer square up to multiplying by the Tamagawa numbers, and this would give us the order of Ш precisely when $S_N \neq 0$.

3. Further work: using Eisenstein series over $GL(2)$ and $GSp(4)$

One direction of my research is to generalize the results from Theorem 2.2 to an automorphic setting: we would like to find a general way to write the L-function as the square of a theta function at a CM-point.

In order to do this, we would like to extend the methods of a second proof of Theorem 2.1. The approach is to relate the L-function $L(E_N, s)$ to the Siegel-Eisenstein series for SL_2:

$$E(s, g, \Phi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{Q})} f_s(\gamma g),$$

where we define $P(\mathbb{Q})$ to be the space of upper triangular matrices in $\mathrm{SL}_2(\mathbb{Q})$ and $f_s(g) = r(g)\Phi(0)\delta(g)^s$, for $r(g)$ the Weil representation on $\mathrm{SL}_2(\mathbb{A})$, Φ a Schwartz-Bruhat function in the Schwartz Bruhat space $\mathcal{S}(\mathbb{A}_{\mathbb{Q}})$, and $\delta(g)$ the modulus character for $\mathrm{GL}_2(\mathbb{A})$.

After embedding $\mathbb{A}_K^\times \hookrightarrow \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ through $a + b\sqrt{-3} \hookrightarrow \left(\begin{smallmatrix} a & b \\ -3b & a \end{smallmatrix}\right)$, the proof is based on showing an equality of the form:

$$\int\limits_{\mathbb{A}_{\mathbb{Q}}^\times \backslash \mathbb{A}_K^\times} f_s(g)\widetilde{\chi}(g) dg = \frac{L_K(s, \widetilde{\chi})}{L_{\mathbb{Q}}(2s, \left(\frac{\cdot}{3}\right))} \tag{5}$$

This method can be further generalized to obtain a result for a family of Eisenstein series $E_0(s, g, \Phi)$. For all integers N, I have proved:

$$L(E_N, s) = d_N \sum_{\mathcal{A} \in Cl(\mathcal{O}_{3D^2})} E_0(s, g, \Phi)\widetilde{\chi}(\mathcal{A}) \operatorname{Nm}(\mathcal{A})^{-1/2}(-1, \operatorname{Nm}(\mathcal{A})),$$

where $(-1, \cdot)$ is the Hilbert symbol over $\mathbb{Q}$ and d_N is a constant. The proof is based on on an equality similar to (5).

Taking this approach one step further, we expect this type of result to generalize to the general symplectic group $\mathrm{GSp}(4)$. By twisting by one extra character χ_*, I am expecting to get a result of the form:

Conjecture 3.1. *Let $G \cong \mathbb{A}_{\mathbb{Q}}^\times \setminus \mathbb{A}_K^\times/(K^\times/\mathbb{Q}^\times)$ and $E(s, \overline{g}, \Phi)$ the Siegel-Eisenstein series for GSp_4. We embed $(g_1^2, g_2^2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \hookrightarrow \mathrm{GSp}_4$. Then for some Schwartz-Bruhat functions $\phi_1, \phi_2 \in S(\mathbb{A}_{\mathbb{Q}})$, we conjecture:*

$$\int\limits_G \int\limits_G E\left(-1/2, (g_1^2, g_2^2), \phi_1 \otimes \phi_2\right) \widetilde{\chi}(g_1 g_2^{-1})\chi_*(g_1 g_2^{-1}) dg = L(E_N, 1),$$

Inspired by the Rallis inner product and the results above, we hope that, by properly choosing Schwartz-Bruhat functions $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$, the terms in the double integral above would split. We expect to get a formula of the form $\left|\sum E\left(-1/2, g^2, \phi_1\right)\chi(g)\right|^2 = L(E_N, 1)$, where $E\left(s, g, \phi_1\right)$ are Eisenstein series of weight $1/2$. This would give us a new formula for the L-function and could be used to prove further results about Ш. The main technical difficulty is working with half-integral weight Eisenstein series.

Using a similar setting, we are hoping to recover the formula in Theorem 2.2. Moreover, note that my formula from Theorem 2.2, even if similar, differs substantially from the formula of Rodriguez-Villegas and Zagier from [15] for primes $p \equiv 1 \mod 9$. Obtaining different formulas could be explained by the fact that we made different choices of Schwartz-Bruhat functions in a formula as the one I conjectured.

4. Future work: L-functions as Fourier coefficients of modular functions and other twists of elliptic curves

Finding the L-functions as the Fourier coefficients of a modular form. An integer D is called a *congruent number* if it equals the area of a right triangle with sides of rational value i.e. D congruent if $D = ab/2$, for some $a, b, c \in \mathbb{Q}$ such that $a^2 + b^2 = c^2$. This is equivalent to the existence of rational points on the elliptic curve:

$$E^D : y^2 = x^3 - D^2 x$$

In [19], Tunnell has constructed a cusp form of weight 3/2 and level 128 that has the L-functions $L(E^D,1)$ appearing in the Fourier expansion:

$$R(z)=\sum_{m,n}(-1)^n q^{(4m+1)^2+8n^2}\sum_{m\in\mathbb{Z}}q^{2m^2} \text{ where } q=e^{2\pi iz}.$$

If we write the Fourier expansion $R(z)=\sum_D a(D)q^D$, then we have $a(D)^2=L(E^D,1)$ for D odd. We can construct a similar modular form of weight 3/2 for D even. Such a construction is possible due to the strong result of Waldspurger [20] for quadratic twists of elliptic curves. However, this is not applicable in our case.

Bruinier and Funke have constructed in [1] a meromorphic modular form f of weight 3/2 that has traces of modular functions at CM-points as coefficients in its Fourier expansion :

$$G(\tau,f)=\sum_{D>0}t_f(D)q^D+\text{extra terms involving}(\tau,f),$$

where $t_f(D)=\sum_{\mathcal{A}}f(\alpha_{\mathcal{A}})$ and we are summing over the representatives of certain ring class groups of conductor D at the CM-points corresponding to ideal representatives in the classes of the ring class group.

Note that in our theorems we are taking similar traces $S_D=\sum_{\mathcal{A}}f_D(\alpha_{\mathcal{A}})$. The main difference is that while Bruinier and Funke fix the modular function f, in my setting I choose different modular functions f_D depending on the integers D. By extending the work of Bruinier and Funke, we are hoping to find a modular form F of half-integral weight with Fourier expansion:

$$F(z)=\sum_{n\geq 0}a_nq^n, \text{where } a_n^2=L(E_n,1)$$

While overly optimistic in the general case of any cubic twists, in our case such a construction seems feasible. Moreover such a construction might give us insight into the behavior of other cubic twists of elliptic curves. We would also hope to obtain a simple criterion similar to Tunnell's theorem by looking at the coefficients of the modular form $F(z)$.

Quadratic twists and other cubic twists. For quadratic twists of elliptic curves with CM, Rodriguez-Villegas [12], Rodriguez-Villegas and Yang [13], Pacetti [10] and others constructed several formulas for their L-functions. The general philosophy is: for an elliptic curve with CM, a formula for a quadratic twist $E^{(D)}$ of E is :

$$L(E^{(D)},s)=c_D\left|\sum_{\mathcal{A}\in\mathrm{Cl}(O_K)}\theta(\tau_{\mathcal{A}})\chi^{(D)}(\mathcal{A})\right|^2,$$

where $\chi^{(D)}$ is the Hecke character corresponding to the quadratic twist $E^{(D)}$ of E, θ is some theta function, and c_D is a constant depending on D.

Our methods could be extended to compute similar formulas for the L-functions of quadratic twists of elliptic curves with CM as traces of theta functions.

Another natural continuation of my work is to apply my methods to compute the L-functions of cubic twists of other families of elliptic curves. We expect this to work rather smoothly for elliptic curves with CM.

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