MATH 55 - WORKSHEET 7 (FRIDAY)

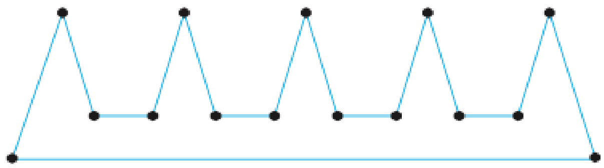
1 The famous Art Gallery Problem asks how many guards are needed to see all parts of an art gallery, where the gallery is the interior and boundary of a polygon with n sides. To state this problem more precisely, we need some terminology. A point x inside or on the boundary of a simple polygon P **covers** or **sees** a point y inside or on P if all points on the line segment xy are in the interior or on the boundary of P. We say that a set of points is a **guarding set** of a simple polygon P if for every point y inside P or on the boundary of P there is a point x in this guarding set that sees y. Denote by $G(P)$ the minimum number of points needed to guard the simple polygon P. The **art gallery problem** asks for the function $g(n)$, which is the maximum value of $G(P)$ over all simple polygons P with n vertices. That is, $g(n)$ is the minimum positive integer for which it is guaranteed that a simple polygon with n vertices can be guarded with $g(n)$ or fewer guards.

a Show that $g(3) = 1$ and $g(4) = 1$ by showing that all triangles and quadrilaterals can be guarded using one point.

b Show that $g(5) = 1$. That is, show that all pentagons can be guarded using one point. [Hint: Show that there are either 0, 1, or 2 vertices with an interior angle greater than 180 degrees and that in each case, one guard suffices.]

c Show that $g(6) = 2$: First using part a and part b as well as Lemma 1 in Section 5.2 to show that $g(6) \leq 2$. Then find a simple hexagon for which two guards are needed.

d Show that $g(n) \geq \lfloor \frac{n}{3} \rfloor$. [Hint: Consider the polygon with $3k$ vertices that resembles a comb with k prongs, such as the polygon with 15 sides shown here.]



e Solve the art gallery problem by proving the art gallery theorem, which states that at most $\lfloor \frac{n}{3} \rfloor$ guards are needed to guard the interior and boundary of a simple polygon with n vertices. [Hint: Use Theorem 1 in Section 5.2 to triangulate the simple polygon into $n - 2$ triangles. Then show that it is possible to color the vertices of the triangulated polygon using three colors so that no two adjacent vertices have the same color; use induction and Exercise 23 in Section 5.2. Finally, put guards at all vertices that are colored red, where red is the color used least in the coloring of the vertices. Show that placing guards at these points is all that is needed.]