MATH 55 - WORKSHEET 2 (MONDAY)

1 [Understanding definitions] Let $f : A \to B$ be a bijection and $f^{-1} : B \to A$ its inverse. Verify that $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

2 Determine whether each of these functions is a bijection from **R** to **R**.

a $f(x) = -3x + 4$

b $f(x) = -3x^2 + 7$

c $f(x) = \frac{x+1}{x+2}$

d $f(x) = x^5 + 1$

3 If f and $f \circ g$ are onto, does it follow that g is onto? [Hint: Test a few examples]

4 Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

a $a_n = 0$

b $a_n = 1$

c $a_n = (-4)^n$

d $a_n = 2(-4)^n + 3$.

4 Find the following values:

$\lceil \frac{3}{4} \rceil$

b $\lfloor \frac{7}{8} \rfloor$

c $\lceil -\frac{3}{4} \rceil$

d $\lfloor -\frac{7}{8} \rfloor$

e $\left\lfloor \frac{1}{2} + \left\lceil \frac{1}{2} \right\rceil \right\rfloor$

f $\left\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \right\rfloor$

5 For any real number x, show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \lfloor x + \frac{2}{3} \rfloor$ [Hint: Prove this by considering different cases]

a Consider any real number $x \in [0, 1)$. Prove the above equality in each of the following cases:

i $0 \leq x < \frac{1}{3}$:

ii $\frac{1}{3} \leq x < \frac{2}{3}$:

iii $\frac{2}{3} \leq x < 1$:

b Part **a** has proven the equality for all real numbers $x \in [0, 1)$. Now choose an arbitrary real number $y \in \mathbf{R}$ and write $y = n + \epsilon$ where $n = \lfloor y \rfloor$ and $0 \leq \epsilon < 1$ (convince yourself that such an expression exists and is unique!). Now use **a** to prove the equality for y.

6 **[Challenge]** Show that if a_n denotes the n-th positive integer that is not a perfect square, then $a_n = n + \left\{\sqrt{n}\right\}$ where $\{x\}$ denotes the integer closest to the real number x. We use the convention that half integers are rounded up i.e. $\{1.5\} = 2$ and $\{-4.5\} = -4$; although, this issue will not occur as $\sqrt{n}$ is never a half integer for any positive integer n.