Uncountability of the Real Numbers

0. Here's an incomplete proof of the fact that \( \mathbb{R} \) is uncountable. Fill in the details:

**Proof** We proceed by contradiction: assume that \( \mathbb{R} \) is countable

a. Prove that any subset of a countable set is also countable

Thus, \([0, 1] \subseteq \mathbb{R}\) is countable and we can **list** the elements of \([0, 1]\) as \(r_1, r_2, r_3, \ldots\).

b. Verify that every real number has a unique decimal expansion, once we exclude the possibility of a tail end consisting entirely of 9s. For example, 0.5 and 0.49999999999999\(\ldots\) represent the same number; we choose 0.5 as the representative for the decimal expansion.

So, let the decimal expansions of the real numbers \(r_i\) be

\[
\begin{align*}
r_1 &= 0.d_{11}d_{12}d_{13}\ldots \\
r_2 &= 0.d_{21}d_{22}d_{23}\ldots \\
r_3 &= 0.d_{31}d_{32}d_{33}\ldots \\
&\vdots & & \vdots
\end{align*}
\]

with \(d_{ij} \in \{0, \ldots, 9\}\). We will obtain a contradiction by finding a real number \(r \in [0, 1]\) that doesn’t occur on our **list**. To this end, consider \(r = 0.d_1d_2d_3\ldots\) where \(d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}\).

c. Show that \(r \neq r_i\) for all \(i\), by comparing the \(i\)-th term in decimal expansion of \(r\) with the \(i\)-th term in the decimal expansion of \(r_i\).