Uncountability of the real numbers

0 Here's an incomplete proof of the fact that $\mathbf{R}$ is uncountable. Fill in the details:

Proof We proceed by contradiction: assume that $\mathbf{R}$ is countable

a Prove that any subset of a countable set is also countable

Thus, $[0, 1) \subseteq \mathbf{R}$ is countable and we can **list** the elements of $[0, 1)$ as $r_1, r_2, r_3, \ldots$.

b Verify that every real number has a unique decimal expansion, once we exclude the possibility of a tail end consisting entirely of 9s. For example, 0.5 and $0.49999999999999\cdots$ represent the same number; we choose 0.5 as the representative for the decimal expansion.

So, let the decimal expansions of the real numbers r_i be

$$\begin{array}{lcl} r_1 & = & 0.d_{11}d_{12}d_{13}\cdots \\ r_2 & = & 0.d_{21}d_{22}d_{23}\cdots \\ r_3 & = & 0.d_{31}d_{32}d_{33}\cdots \\ \vdots & \vdots & \qquad \vdots \end{array}$$

with $d_{ij} \in \{0, \ldots, 9\}$. We will obtain a contradiction by finding a real number $r \in [0, 1)$ that doesn't occur on our **list**. To this end, consider $r = 0.d_1 d_2 d_3 \cdots$ where $d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$.

c Show that $r \neq r_i$ for all i, by comparing the i-th term in decimal expansion of r with the i-th term in the decimal expansion of r_i.