

**Defn / theorem** The functor  $\text{hilb}_{\mathbb{P}^n} : (\text{locally noetherian schemes})^{\text{op}} \rightarrow \text{Sets}$  given by

$$\text{hilb}_{\mathbb{P}^n}(S) = \{ Y \subseteq \mathbb{P}^n \times S \mid Y \text{ is flat over } S \}$$

is representable by a locally noetherian scheme  $\mathcal{H}$ . ( $= \text{Hilb}_{\mathbb{P}^n}$ )

① We have the following universal property

$$\{ \text{families over } S \} \leftrightarrow \text{Hom}(S, \mathcal{H}) \quad \text{and there is}$$

a universal family

$$\begin{array}{ccc} C & \longrightarrow & \mathcal{H} \times \mathbb{P}^n \\ & \searrow & \downarrow q_H \\ & & S \end{array}$$

(induced by  $\text{id} : \mathcal{H} \rightarrow \mathcal{H}$ )

such that every other family  $X \longrightarrow S \times \mathbb{P}^n$  is obtained from the universal family by base change:

In particular if the family corresponds to  $e : S \rightarrow \mathcal{H}$ , then  $X = C \times_{\mathcal{H}} S$

② This is some huge disconnected thing, the following theorem lets us zoom into a component:

**Thm:** Let  $X \subseteq \mathbb{P}^n \times T$  be a projective variety over an integral scheme  $T$ . Then  $X$  is flat over  $T$  iff the Hilbert polynomial of all the closed fibers of  $X \rightarrow T$  are the same.

The restricted functor  $\text{hilb}_{\mathbb{P}^n}^{P(t)} : S \mapsto \{ Y \subseteq \mathbb{P}^n \times S \mid Y \text{ is flat over } S \text{ and has Hilbert poly. } P(t) \}$  is representable. (This is a connected component of  $\text{hilb}_{\mathbb{P}^n}$ )

③ The natural generalization  $\text{hilb}_X(S) = \{ Y \subseteq X \times S \mid Y \text{ is flat over } S \}$  is also representable, where  $X$  is any projective scheme.

④ We can even ask for  $X$  to be quasi-projective (Hartshorne's definition)

Ex 1 Recall the classic correspondence: {hypersurfaces of degree d in  $\mathbb{P}^n$ } (2)

$$\uparrow \\ \mathbb{P}^{(\binom{n+d}{d}-1)} = \mathbb{P} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)).$$

- Such a hypersurface  $X$  is cut out by a degree d homog. poly.  $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ . For example in  $n=2, d=3$ :  $X$  corresponds to a poly. of the form,
- $$a_{300}x^3 + a_{201}y^3 + a_{003}z^3 + a_{210}x^2y + a_{120}x^2z + a_{102}xy^2 + a_{012}xz^2 + a_{001}xyz \\ + a_{012}yz^2 + a_{021}y^2z \quad (*)$$

$$\binom{3+2}{2} = 10 \text{ varying coefficients}$$

- The hilbert polynomial of ANY hypersurface of degree d in  $\mathbb{P}^n$  is  $\binom{n+t}{n} - \binom{n-d+t}{n}$ .

Conversely any closed subscheme of  $\mathbb{P}^n$  with this Hilbert polynomial is a hypersurface of degree d

Thus  $\text{hilb}_{\mathbb{P}^n}^{Q(t)}$  is represented by  $\mathbb{P}^{(\binom{n+d}{d}-1)}$

- Intuitively giving a family of hypersurfaces in  $\mathbb{P}^n$  should correspond to varying the coefficients of the ~~"general"~~ "general" polynomial (\*)
- This Hilbert scheme is ~~singular~~ non-singular

Fact A zero dimensional subscheme  $X \subseteq \mathbb{P}_k^n$  has Hilbert polynomial  $P_X(t) = d$  where  $d = \text{length } \mathcal{O}_X$ .

$\text{Hilb}_{\mathbb{P}^2}^d$  is ~~singular~~ non-singular i.e. the hilbert scheme of "d points in  $\mathbb{P}^2$ " is non-singular

- $\text{Hilb}_{\mathbb{P}^2}^d$  is irreducible and in the closure of the open subscheme of d general points in  $\mathbb{P}^2$  [H, Theorem 8.11]

- Ex 1 showed that plane curves of degree d is non-singular.

So let's study the Hilbert scheme of curves in  $\mathbb{P}^3$

## II - compactification of the space of curves of degree d in $\mathbb{P}^3$

Defn A twisted cubic is a non-singular curve of degree 3 in  $\mathbb{P}^3$  not contained in the plane.

① It has Hilbert polynomial  $P(t) = 3t + 1$

② It's obtained by embedding  $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$  via  $\mathcal{O}_{\mathbb{P}^1}(3)$ .

One specific curve is  $[s:t] \mapsto [s^3: s^2t: st^2: t^3]$ .

③ The space of twisted cubics can also be viewed as a homogeneous space  $\mathbb{P}GL(4)/\mathbb{PGL}(2)$ . Its dimension is  $(4^2 - 1) - (2^2 - 1) = 15 - 3 = 12$ .

$\uparrow$  automorphisms of  $\mathbb{P}^3$        $\nwarrow$  automorphisms of  $\mathbb{P}^1$

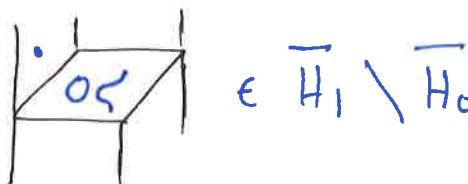
Thus twisted cubics form an open 12 dimensional component of  $\overline{\text{Hilb}}_{\mathbb{P}^3}^{3t+1}$ .  
 Call this component  $H_0$ .

Question) Is  $\overline{H}_0 = \overline{\text{Hilb}}_{\mathbb{P}^3}^{3t+1}$ ?

A) No we have another irreducible component:

$H_1 = \{ C \in \mathbb{P}^3 \mid C \text{ non-singular plane cubic and } p \text{ a point not on } C \}$

So



$$\in \overline{H}_1 \setminus \overline{H}_0$$

Fact  $\overline{H}_0 \cup \overline{H}_1 = \overline{\text{Hilb}}_{\mathbb{P}^3}^{3t+1}$

- Thus  $\overline{\text{Hilb}}_{\mathbb{P}^3}^{3t+1}$  is ~~singular~~ along  $\overline{H}_0 \cap \overline{H}_1$  birational to  $\mathbb{P}^n$
- But  $\overline{H}_0$  and  $\overline{H}_1$  are both non-singular; they are in fact rational (~~a non-trivial fact~~)

~~What are the other components?~~

~~What are the components of the third section?~~

Since the Hilbert polynomial of a curve in  $\mathbb{P}^3$  is determined by its degree ( $d$ ) and arithmetic genus ( $g$ ), we will write  $\text{Hilb}_{\mathbb{P}^3}^{g,d}$  for the corresponding component. As in the twisted cubic case we can look at the locus of all non-singular irreducible curves in  $\text{Hilb}_{\mathbb{P}^3}^{g,d}$ ; call this  $U$ .

We now state the main question: Is the component  $U \subseteq \text{Hilb}_{\mathbb{P}^3}^{g,d}$  non-singular? (assuming  $U$  is open and irreducible)

The ~~expected~~ answer is no, and we will show this by considering  $\text{Hilb}_{\mathbb{P}^3}^{24,14}$

### III - Expected Dimension of $U$

From the previous talk.

If  $U$  was non-singular and irreducible we would have

$$\dim U = \dim_K T_x|_U = \dim_K \left\{ \begin{array}{l} \text{maps } K[\varepsilon]/\varepsilon^2 \rightarrow U \\ \downarrow \kappa \quad \leftarrow \\ \text{s.t. } (\varepsilon) \mapsto x \end{array} \right\} = \dim_K \left\{ \begin{array}{l} \{x\}_+ \\ \downarrow \\ K[\varepsilon]/\varepsilon^2 \end{array} \right. \text{ with } x_0 \simeq x \left. \right\} = h^0(X, N_{X/\mathbb{P}^3})$$

• For ~~new~~ new  $X$  (high enough degree, fixed genus),  $h^1(X, N_{X/\mathbb{P}^3}) = 0$ .  
(Assuming  $\Theta_X(1)$  to be non-special in sufficient)

Thus  $h^0(X, N_{X/\mathbb{P}^3}) = \chi(N_{X/\mathbb{P}^3})$ . ~~so~~ Let's compute this:

We have two exact sequences,

$$0 \rightarrow T_x \rightarrow T_{\mathbb{P}^3}|_x \rightarrow N_{X/\mathbb{P}^3} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \Theta_X \rightarrow \Theta_X(1)^4 \rightarrow T_{\mathbb{P}^3}|_x \rightarrow 0$$

(tensor the Euler sequence by  $\Theta_X$ )

$$\text{Thus we have } h^0(X, N_{X/\mathbb{P}^3}) = \chi(T_{\mathbb{P}^3}|_x) - \chi(T_x)$$

$$\begin{aligned} \textcircled{1} \text{ Using the second sequence, } \chi(T_{\mathbb{P}^3}|_x) &= \chi(\Theta_X(1)^4) - \chi(\Theta_X) \\ &= 4\chi(\Theta_X(1)) - \chi(\Theta_X) \quad \leftarrow \text{Riemann-Roch} \\ &= 4(d+1-g) - (1-g) \\ &= 4d + 3 - 3g \end{aligned}$$

$$\textcircled{2} \text{ By Riemann-Roch, } \chi(T_x) = \chi(\omega_X) = (2-2g) + 1 - g = 3 - 3g.$$

Thus the \*expected dimension ~~is~~ of  $U$  is

$$\boxed{\dim U = (4d + 3 - 3g) - (3 - 3g) = 4d}$$

This is independent of the genus!

Example Consider  $g=0$

- ①  $\text{Hilb}_{\mathbb{P}^3}^{0,1} = \{ \text{lines in } \mathbb{P}^3 \} \cong \mathbb{G}(1,3) \rightsquigarrow 4 \text{ dimensional}$
- ②  $\text{Hilb}_{\mathbb{P}^3}^{0,2} = \{ \text{conics in } \mathbb{P}^3 \}$ . Every conic lies in a plane; For each plane there is a  $\binom{2+2}{2}-1 = 5$  dimensional family of conics. The space of planes in  $\mathbb{P}^3$  is  $\mathbb{G}(2,3)$ , which is 3 dimensional. Thus  $\dim \text{Hilb}_{\mathbb{P}^3}^{0,2} = 3 + 5 = 8$ .
- ③  $\text{Hilb}_{\mathbb{P}^3}^{0,3} = \{ \text{twisted cubics in } \mathbb{P}^3 \} \rightsquigarrow$  We have seen that this is 12 dimensional  
 $\vdots$

#### IV - Mumford's Example (Part 2)

Theorem. There is an irreducible component of the Hilbert scheme of non-singular, irreducible curves in  $\mathbb{P}^3$  of degree 14 and genus 24 that is generically non-reduced!

① Let's first find an explicit family of degree 14, genus 24 curves. One place to search for this is on a non-singular cubic surface.

Let  $S$  be a non-singular cubic surface,  $H$  a hyperplane section and  $L$  one of its lines (there are 27 to choose from).

i) Consider the linear system  $|4H+2L|$  on  $S$

ii) The formulas in [H, Chapter V.4] tell us that this divisor class is very ample. Thus, the generic element is irreducible and non-singular. We also see that the degree is 14 and genus is 24.

(Choosing  $L$  to be  $E_6$  we have that any element in  $|4H+2L|$  is of the form

$12l - 4\sum_{i=1}^5 e_i - 2e_6 \in \text{Pic } X \cong \mathbb{Z}^7$ . Here  $l$  is the class of the pullback of a line in  $\mathbb{P}^2$ )

② It can be shown that the family of curves in  $|4H+2L|$  is irreducible.

③ Thus we define  $U$  to be space of non-singular curves in  $|4H+2L|$   
~~over~~ for all choices of non-singular cubics  $X$  and all lines  $L$  on  $X$ .

So  $U$  is a projective bundle over the space of non-singular cubics

$$\hookrightarrow \mathbb{P}^{\binom{3+3}{2}-1} = \mathbb{P}^{19}$$

(Since degree  $> 9$ , each curve in the class lies on a unique cubic)

④ It can be shown that  $U$  is irreducible

The last thing we need to check is that  $U$  is open. Another way of phrasing this is, is there another family  $U'$  of genus 24, degree 14 curves with  $\dim U' > \dim U$ . ~~exists~~

It will turn out that there is no such  $U'$ . But first let's compute  $\dim U$  and deduce Mumford's theorem assuming  $U$  is open.

$$\begin{aligned} \textcircled{1} \quad \dim U &= 19 + \dim_{\mathbb{K}} |4H + 2L| = 19 + (h^0(S, \mathcal{O}_S(C) - 1)) \\ &= 56 \end{aligned}$$

where  $C$  is a non-singular curve in the linear system and  $S$  is a fixed cubic surface

Use the exact sequence  $0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(L) \rightarrow \mathcal{O}_C(C) \rightarrow 0$

The problem will occur with  $T_{[C]}U$

$$\textcircled{2} \quad \dim T_{[C]}U = h^0(C, N_{C/P^3}) = 57!$$

This computed using the exact sequence  $0 \rightarrow N_{C/S} \rightarrow N_{C/P^3} \rightarrow N_{S/C} \rightarrow 0$

i) Show  $h^1(N_{C/S}) = 0$  i.e.  $N_{C/S}$  is non-special

ii) Then  $h^0(N_{C/P^3}) = h^0(N_{C/S}) + h^0(N_{S/C})$

The computations done in ① will imply  $h^0(N_{C/S}) = 37$  and  $h^0(N_{S/C}) = 20$ .

Why the discrepancy? There is a 56 dimensional space of infinitesimal deformations that are embedded in a cubic. Unfortunately there's this extra dimension of deformations that are NOT embedded in the cubic.

Since  $\dim T_{[C]}U > \dim U + [C]$ , we see that  $U$  is singular.

We know that  $U$  is irreducible. If  $U$  was reduced, then it would be integral and by generic smoothness,  $U$  has an open dense non-singular subset. This is a contradiction. Thus  $U$  is generically non-reduced.

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## V - Mumford's example (Part 2)

Let's show that there is no family  $U'$  with  $\dim U' > \dim U$ .

(7)

Assume there's a non-singular, general curve  $C'$  of degree 14, genus 24 that doesn't lie on a cubic surface. Let  $U'$  be the family that contains  $C'$

(1) Show  $C'$  lies on two distinct quartics  $F, F'$

Take the I.e.s. associated to  $0 \rightarrow I_{C'}(4) \rightarrow \mathcal{O}_{\mathbb{P}^3}(4) \rightarrow \mathcal{O}_{C'}(4) \rightarrow 0$  and show  $h^0(I_{C'}(4)) \geq 2$ . (Riemann-Roch)

(2)  $C'$  is linked to a plane conic.

Since  $C'$  is not on a cubic (or lower degree surface),  $F$  and  $F'$  are irreducible. Thus we have a residual curve  $D = F \cap F' - C'$ .

Linkage relates degree and genus of  $D$  to degree and genus of  $C'$  and ~~of~~ degree of  $F, F'$ :

$$\deg D = 4 \cdot 4 - \deg C' = 16 - 14 = 2$$

$$\text{genus}(D) = \frac{1}{2} (\deg D - \deg C') (4+4-4) + \text{genus}(C') = \frac{1}{2} (-12)(4) + 24 = 0.$$

Degree 2, genus 0 curves are plane conics!

(3)  $\dim U' = 56$

We will count the dimension of the residual conic:

- i) Hilbert scheme of ~~the~~ plane conics in  $\mathbb{P}^3$  is 8 dimensional
- ii) The space of quartics containing  $D$  is  $H^0(I_D(4))$ .

Consider  $0 \rightarrow I_D(4) \rightarrow \mathcal{O}_{\mathbb{P}^3}(4) \rightarrow \mathcal{O}_D(4) \rightarrow 0$

The property of being arithmetically Cohen-Macaulay <sup>(ACM)</sup> is preserved by linkage [H, Exercise 8.14]. Thus  $D$  is ACM.

The main property we need is  $H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow H^0(\mathcal{O}_D(4))$ .

$$\begin{aligned} \text{Thus } h^1(I_D(4)) &= 0 \Rightarrow h^0(I_D(4)) = h^0(\mathcal{O}_{\mathbb{P}^3}(4)) - h^0(\mathcal{O}_D(4)) \\ &= \binom{4+3}{3} - (4 \cdot 2 + 1 - 0) \\ &= 35 - 9 \\ &= 26. \end{aligned}$$

Riemann-Roch as  
 $D \hookrightarrow \mathbb{P}^3$  has degree 2  
and  $h^1(\mathcal{O}_D(4)) = 0$

- iii) The space of 2 quartics containing  $D$  is ~~is~~  $\text{Gr}(2, H^0(I_D(4)))$ .  
This has dimension  $(26-2) \cdot 2 = 48$

Thus  $\dim U' = 8 + 48 = 56!$

This coincides with the dimension of  $U$ .  
completing our proof of Mumford's theorem

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## VI - Singularities in general

- Define an equivalence relation on pointed schemes:  $(X, p) \sim (Y, q)$  if there is a smooth morphism  $(X, p) \rightarrow (Y, q)$ . The equivalence classes are called singularity types.
- A moduli space is said to satisfy Murphy's law if every singularity of finite type over  $\mathbb{Z}$  appears on that moduli space.

Theorem [V, Theorem 1.1] The Hilbert scheme of nonsingular curves in projective space satisfies Murphy's law.

In fact it gets a lot worse. Most moduli spaces satisfy Murphy's law

The key result to prove this theorem is Mnev's universality theorem.

Theorem (Mnev) Every singularity type of finite type over  $\mathbb{Z}$  appears on some incidence scheme

Definition An incidence scheme of points and lines in  $\mathbb{P}^2$  is a locally closed subscheme of  $(\mathbb{P}^2)^m \times (\mathbb{P}^2)^n = \{(P_1, \dots, P_m, l_1, \dots, l_n)\}$  parameterizing  $m \geq 4$  marked points and  $n$  marked lines as follows

- $P_1 = [1:0:0]$ ,  $P_2 = [0:1:0]$ ,  $P_3 = [0:0:1]$ ,  $P_4 = [1:1:1]$
  - Pairs  $(P_i, l_j)$ , either  $P_i$  lies on  $l_j$  or  $P_i$  does not lie on  $l_j$ .
  - Given two marked lines, there is a marked point on both of them
  - Every marked line contains three marked points.
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## References

- [H] Robin Hartshorne - Deformation Theory - Springer 2010
- [H'] Robin Hartshorne - Algebraic Geometry - Springer 1977
- [V] Ravi Vakil - Murphy's law in Algebraic geometry... - Inventiones 2006