Qualifying Exam Syllabus

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Committee: David Eisenbud (Advisor), Martin Olsson, Denis Auroux (Chair), Marjorie Shapiro (External)

I. Algebraic Geometry (Major - Algebra)

1. Scheme Theory

- Schemes, Morphisms between schemes, Separated and Proper morphisms, Valuative Criterion
- (Quasi)-coherent sheaves, Invertible sheaves, Weil divisors, Cartier divisors, Divisor Class group, Picard group
- Projective Morphisms, Blowing up, Kähler Differentials, Non-singular varieties, Bertini's Theorem

2. Cohomology

- Sheaf Cohomology, Grothendieck Vanishing, Cohomology of Noetherian Affine Schemes
- Serre's Affineness criterion, Čech cohomology, Cohomology of Projective Space
- Cohomological criterion of ampleness, Serre Duality (**statement**), Higher Direct Images
- Flat morphisms, Flat families, Hilbert polynomials, Smooth morphisms

3. Curves

- Riemann-Roch Theorem, Hurwitz's Theorem, Linear systems on curves, Embeddings of curves in $\mathbf{P}^n$
- Elliptic Curves (**up to group structure**), Hyperelliptic Curves, Canonical Embeddings
- Clifford's Theorem, Castelnuovo's theorem (**statement**), Classification of low degree curves in $\mathbf{P}^3$

References: [R. Hartshorne, *Algebraic Geometry*: Chapters, I.1 - I.7, II.1 - II.8, III.1 - III.10, IV.1 - IV.6]

II. Commutative Algebra (Major - Algebra)

1. Basic Constructions

- Noetherian Rings, Localization, Associated Primes, Primary Decomposition
- Integral Extensions, Going-up and Going-down theorems, Noether Normalization, Nakayama's Lemma
- Artin-Rees Lemma, Krull's intersection theorem, Flatness, Local Criterion for Flatness

2. Dimension Theory

- Krull Dimension, Principal ideal theorem, Dimension of Base and Fiber
- Normal rings, Discrete Valuation Rings, Dedekind Domains
- Nullstellensatz, Hilbert-Samuel Functions, Main Theorem of Elimination Theory, Differentials

3. Homological Methods

- Ext, Tor, Koszul Complexes, Depth, Cohen-Macaulay Rings
- Serre's Criterion, Projective Dimension, Minimal Resolutions, Hilbert Syzygy Theorem
- Global dimension, Regular Local Rings, Auslander-Buchsbaum Formula

References: [D. Eisenbud, *Commutative Algebra*: Chapters, 1 - 6, 8 - 14, 16.1 - 16.3, 17, 18, 19.1-19.3]

III. Algebraic Topology (Minor - Geometry)

1. Fundamental Group

- van Kampen's theorem, Covering Spaces, Lifting properties

2. Homology

- Simplicial homology, Singular homology, Cellular homology
- Relative homology, Excision, Long exact sequence in homology, Mayer-Vietoris sequence, Axioms for homology

3. Cohomology

- Universal coefficient theorem, Long exact sequence in cohomology
- Cup products, Künneth formula, Poincare duality

References: [A. Hatcher, *Algebraic Topology*: Chapters, 1-3 (**no** Additional Topics)]

Qualifying Exam Transcript

Notes

- My exam was about 2 hours long (**9:15am - 11:11am**).
- David Eisenbud asked me many questions in our meetings before the qualifying exam. He already had a good understanding of what I knew, and I presume he was more easily satisfied.
- Not all of the answers presented here were written on the board; a lot of the times a verbal explanation sufficed.
- Since this transcript was written based on memory a few hours after the exam, the exact words used on the exam might differ from what's on the transcript. It might look very formal, but they were all asked in a friendly manner.
- We took a 5 minute break between each topic.

I. Algebraic Geometry (Major - Algebra)

Olsson Let's warm up by talking about projective morphisms.

Me A morphism of schemes $f : X \longrightarrow Y$ is projective if it factors as $X \xrightarrow{f'} \mathbf{P}^n_Y \xrightarrow{\pi} Y$ for some n where f' is a closed immersion.

Auroux What does $\mathbf{P}^n_Y$ mean?

Me We have $\mathbf{P}^n_{\mathbf{Z}} := \operatorname{Proj} \mathbf{Z}[x_0, \ldots, x_n]$ and a map $\mathbf{P}^n_{\mathbf{Z}} \longrightarrow \operatorname{Spec} \mathbf{Z}$. Given any other scheme Y there is a unique morphism $Y \longrightarrow \operatorname{Spec} \mathbf{Z}$. We can base change along this morphism to obtain $\mathbf{P}^n_Y := \mathbf{P}^n_{\mathbf{Z}} \times_{\mathbf{Z}} Y$ and π.

Olsson What can you say about maps to projective space?

Me A line bundle $\mathscr{L}$ on X and $n+1$ sections $s_0, \ldots, s_n$ such that all of the s_i are not zero in the residue field $\mathscr{L}_x / \mathfrak{m}_x \mathscr{L}_x$ (by Nakayama's Lemma this implies that the s_i generate the stalk $\mathscr{L}_x$ as an $\mathscr{O}_{X,x}$-module) is in 1-1 correspondence between maps $\varphi : X \longrightarrow \mathbf{P}^n_A$ for which $\varphi^\star \mathscr{O}_{\mathbf{P}^n_A}(1) = \mathscr{L}$ and $\varphi^\star(s_i) = x_i$.

This correspondence is obtained by gluing local maps $X_{s_i} \longrightarrow U_{x_i}$ given by $\frac{s_j}{s_i} \leftarrow\!\shortmid \frac{x_j}{x_i}$.

Olsson Now consider $X = \mathbf{A}^{n+1}_k = \operatorname{Spec} k[x_0, \ldots, x_n]$, $\mathscr{L} = \mathscr{O}_X$ and sections $x_0, \ldots, x_n$. Does this give you a map to $\mathbf{P}^n_k$; what else can you say?

Me The sections $x_0, \ldots, x_n$ all vanish at the point $\mathbf{0} = V(I)$ where $I = (x_0, \ldots, x_n)$. Thus we only have a map $X - \mathbf{0} \longrightarrow \mathbf{P}^n_k$.

Olsson Is there a way to resolve this map?

Me Yes, we can blow up X along $V(I)$ and resolve the map.

Olsson Define blow ups in this case and resolve the map.

Me Since we are blowing up an affine variety we can define the blow-ups locally. Indeed, let $X = \operatorname{Spec} A$ and consider a closed subscheme $Y = V(I)$. We have a graded ring $\widetilde{A} = A \oplus tI \oplus t^2 I^2 \oplus \cdots \subseteq A[t]$ with A in degree 0 and t in degree 1. Thus we can define $\operatorname{Bl}_Y X := \operatorname{Proj} \widetilde{A}$ and this comes with a map $\operatorname{Proj} \widetilde{A} \longrightarrow \operatorname{Spec} A$ induced by the ring map $A \longrightarrow A \oplus tI \oplus \cdots$.

In our case, $I = (x_0, \ldots, x_n)$ is an ideal generated by a regular sequence and this admits a nice description. Namely, the map $A[y_0, \ldots, y_n] \longrightarrow A \oplus tI \oplus t^2 I^2 \oplus \cdots$ given by $y_i \mapsto x_i t$ has kernel precisely $(\{x_j y_i - x_i y_j\}_{i,j})$.
Thus, $\operatorname{Bl}_Y X \simeq V_+(\{x_i y_i - x_j y_i\}_{i,j}) \subseteq \mathbf{A}^{n+1} \times_k \mathbf{P}^n_k$ and finally the map $\varphi : \operatorname{Bl}_Y X \longrightarrow \mathbf{P}^n_k$ is given (in coordinates) by $(x_0, \ldots, x_n) \times [y_0 : \cdots : y_n] \mapsto [y_0 : \cdots : y_n]$.

Olsson What are the fibers of φ?

Me The fibers are isomorphic to $\mathbf{A}^1$.

Auroux So what can you say about the map φ? In particular, what's the line bundle associated to φ?

Me Well, it corresponds to the exceptional divisor which is $\mathscr{O}(-1)$.

Olsson Ok, but can you see that from the explicit map φ you've written down?

Me By the description in coordinates the map is either $\mathscr{O}(1)$ or $\mathscr{O}(-1)$. I explained that y_i are acting as functionals and thus the line bundle cannot be $\mathscr{O}(1)$.

Olsson Here's another way to see this: What can you tell me about maps from $\mathbf{P}^n_k$ to $\mathbf{A}^{n+1}_k$?

Me Call the morphism $\psi : \mathbf{P}^n_k \longrightarrow \mathbf{A}^{n+1}_k$. Since $\mathbf{P}^n_k$ is proper and irreducible, the image $\psi(\mathbf{P}^n_k)$ is proper and irreducible over k. Since $\psi(\mathbf{P}^n_k)$ is closed it's also affine. The only variety over k that's both proper, affine and irreducible is a point. Thus, ψ is just constant.

Olsson Now what does tell you about the sections of the bundle.

Me Oh, if the line bundle, call it $\mathscr{L}'$, associated to $\operatorname{Bl}_Y X \longrightarrow \mathbf{P}^n_k$ had a section, then we would a map from $\mathbf{P}^n_k \longrightarrow \mathbf{A}^{n+1}_k$. But the only such maps are constant. Thus the global section of $\mathscr{L}'$ must be the zero section (it lies on the same point in every $\mathbf{A}^1$-fiber). This forces $\mathscr{L}' = \mathscr{O}(-1)$.

Eisenbud Can you tell us about Hurwitz's theorem

Me By a curve I mean an integral, one-dimensional, projective scheme defined over $\bar{k}$. Consider a finite morphism of curves $f : X \longrightarrow Y$ of degree $n = [K(X) : K(Y)]$. If f is separable, we have $2g(X) - 2 = n(2g(Y) - 2) + \deg R$ where $R = \sum_P (\text{length}\, \Omega_{X/Y})_P [P]$ is the ramification divisor. If f is purely inseparable, then $g(X) = g(Y)$.

Eisenbud What's an example of an unramified map?

Me Well, take any isomorphism of $\mathbf{P}^1_k \longrightarrow \mathbf{P}^1_k$.

(Everyone starts laughing. Martin comments that's the Bourbaki example!)

Eisenbud How about something that's not an isomorphism.

Me We clearly can't take maps to $\mathbf{P}^1$.

Olsson Why can't you do that?

Me Well, $g(\mathbf{P}^1) = 0$ so we have $\deg R = 2g(X) - 2 + 2n > 0$ unless $n = 1$ and $g(X) = 0$; this was my first example.

Auroux So try a higher genus.

Me Oh, if $g(X) = g(Y) = 1$, then by Hurwitz's theorem, $\deg R = 0$. So we want to find an isogeny of an elliptic curve E that's not an isomorphism. One can take the squaring map $2_E : E \longrightarrow E$; this has non trivial kernel.

Auroux What's the kernel?

Me There are many ways to find this; here's a nice way to see this pictorially assume $k = \mathbf{C}$. Represent E as a torus $\mathbf{C}/\Lambda$ with $\Lambda = \text{span}_{\mathbf{Z}}\{\omega_1, \omega_2\}$ the lattice. Then we want to find the number of points P in the fundamental parallelogram such that $P + P = \mathbf{0}$. There are clearly four points; $\mathbf{0}, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$. This also shows that the kernel is the group $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

More generally multiplication by $n_E : E \longrightarrow E$ has degree n^2 and this is the size of the kernel in "most" cases. So as long as $\text{char}\, k \neq 2$ we still have $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ as the kernel.

Auroux What can you say about degree 3, genus 1 curves X in $\mathbf{P}^3$.

Me They are all embedded as plane cubic curves (in particular, it's more than an abstract isomorphism). The overkill way to see this is to use Castelnuovo's bound. It states that if $X \hookrightarrow \mathbf{P}^3$ is a degree $d \geq 3$ curve of genus g that does **not** lie in a plane, then $g \leq \frac{1}{4}(d^2 - 1) - d + 1$ (when d is odd). Plugging in $d = 3$ we obtain, $g \leq \frac{1}{4}(3^2 - 1) - 3 + 1 = 0$. Thus the only non-planar degree 3 space curve is projectively isomorphic to the twisted cubic.

Auroux What's a more elementary way to see this?

Me Let D be the degree 3 divisor on X corresponding to the embedding $X \hookrightarrow \mathbf{P}^3$. Note that K_X has degree 0 and Riemann-Roch implies that,

$$h^0(D) = h^1(K_X - D) + \deg D + 1 - g(X) = 0 + \deg D + 1 - 1 = \deg D.$$

In other words, D has only $\deg D = 3$ linearly independent sections. Thus the corresponding map to $\mathbf{P}^3$ factors as $X \hookrightarrow \mathbf{P}^2 \longrightarrow \mathbf{P}^3$.

II. Commutative Algebra (Major - Algebra)

Eisenbud Define Krull dimension of a ring R.

Me It's the length of the "longest" chain of prime ideal i.e. $\dim R = \sup\{n : P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n : P_i \text{ prime}\}$.

Eisenbud Is there another characterization of this when R is an affine domain (over k)?

Me Using noether normalization one can show that $\dim R = \mathrm{trdeg}_k K(R)$.

Eisenbud Now consider $I = (x,y) \cdot (u,v,w) \subseteq k[x,y,u,v,w]$. What's the dimension of $k[x,y,u,v,w]/I$?

Me I is the product of the ideal cutting out a 3 dimensional plane and a 2 dimensional plane. Their product, set theoretically, cuts out the union of a 3-plane and a 2-plane. Thus the dimension is just 3.

Eisenbud That's the set theoretic locus, is it also the scheme theoretic locus?

Me In this case, $(x,y) \cdot (u,v,w) = (x,y) \cap (u,v,w)$, so it's indeed the scheme theoretic locus.

Eisenbud Now let's consider $I = (x,y) \cdot (x,v,w) \subseteq k[x,y,v,w]$. What is $k[x,y,v,w]/I$ as a scheme?

Me Let's write out the generators of I and find a primary decomposition. Since we are dealing with monomial ideals, recall that a monomial ideal $J \subseteq k[x_1,\ldots,x_p]$ is $(x_{a_1},\ldots,x_{a_k})-$primary if it's of the form $(x_{a_1}^{b_1},\ldots,x_{a_k}^{b_k},m_1,\ldots,m_\ell)$ with $b_i > 0$ and m_i just monomials in $x_{a_1},\ldots,x_{a_k}$. Moreover, for monomial ideals, if $uv \in I$ with u,v coprime, then $I = (I+u) \cap (I+v)$. Using these we can find a primary decomposition for I as follows,

$$\begin{aligned}
I &= (x^2, xv, xw, yx, vy, wy) \\
&= (I+x) \cap (I+y) \\
&= (x, vy, wy) \cap (x^2, xv, xw, y) \\
&\vdots \quad \vdots \\
&= (x,y) \cap (x,v,w) \cap (x^2, v, w, y).
\end{aligned}$$

Eisenbud Consider the subalgebra $R = k[s^4, s^3t, st^3, t^4]$ (homogenous coordinate ring) associated to the rational quartic in $\mathbf{P}^3$. Find a noether normalization.

Me The subring $S = k[s^4,t^4] \subseteq R$ gives us the noether normalization. Indeed, s^3t is annihilated by $p(T) = T^4 - (s^4)^3(t^4) \in R[T]$ and ts^3 is annihilated by $q(T) = T^4 - (s^4)(t^4)^3 \in R[T]$.

Olsson Is there something special about the choices s^4, t^4? Could we have chosen s^3t instead of s^4?

Me We have to be slightly careful if we want to choose other monomials. For example $k[s^3t, t^4] \subseteq R$ is not a noether normalization as s^4 is NOT integral over this subring (consider the degree of t's appearing in the coefficients of any polynomial over this subring). By choosing the monomial with the lowest degree in t and lowest degree in s we are guaranteed to not run into this issue.

Eisenbud What's the integral closure of R?

Me Notice that $s^2t^2 \in K(R)$; in particular, $s^2t^2 = (s^3t)(st^3)^{-1}t^4 = \frac{s^2}{t^2}t^4 = s^2t^2$. It's also annihilated by $T^2 - (s^4)(t^4) \in R[T]$. Thus, $k[s^4, s^3t, s^2t^2, st^3, t^4]$ is the integral closure of R.

Eisenbud What does Serre's criterion say?

Me It states that a ring R is a normal ring iff R satisfies the following two conditions:

(R1) This means that R_P is regular for all codimension 1 primes P.

(S2) This means that $\mathrm{depth}(P,R) \geq 2$ for all primes P of codimension (at least) 2.

The parenthetical remark in **(S2)** follows from the fact that $\mathrm{depth}(-,R)$ can only increase when you localize.

Eisenbud Why does it not apply to our algebra R?

Me Since the projection of the rational quartic from $\mathbf{P}^4 \dashrightarrow \mathbf{P}^3$ is non-singular, (R1) is satisfied for R. Let $P = (s^4, s^3t, st^3, t^4)$ be the codimension 2 maximal ideal of R. To show that (S2) fails, it suffices to show that $\operatorname{depth}(P, R) = 1$. This will be true if we find an element $a \in R$ such that $\operatorname{depth}(P + (a), R/(a)) = 0$.

Before we do that it might be helpful to have some equations for R as a subvariety of $\mathbf{P}^3$ i.e. write I as an ideal of $k[x, y, z, w]$. (Now everyone joined in on the fun and started to find minimal equations for the quartic!)

After a few minutes we found that $I = (x^2z - y^3, xw - yz, y^2w - xz^2, yw^2 - z^3)$. So let's quotient by x and obtain,

$$R/(x) = k[x, y, z, w]/(I, x) = k[y, z, w]/(y^3, yz, y^2w, yw^2 - z^3).$$

Showing that $\operatorname{depth}((y, z, w), R/(x)) = 0$ is equivalent to showing that (y, z, w) is associated. In other words, we want $\ell \in R/(x)$ such that $\operatorname{ann}(\ell) = (y, z, w)$. Looking at our relations we see that we can take $\ell = (yw^2)$!

Auroux Geometrically what went wrong? In other words, what happened when we quotiented out by $x = s^4$?

Me The main issue is that we embedded $\mathbf{P}^1 \hookrightarrow \mathbf{P}^3$ by an incomplete linear series. More generally, any such incomplete linear series is not going to be projectively normal. In our case, since $\dim R = 2$, normality is equivalent to being Cohen-Macaulay. So if R were Cohen-Macaulay, we could find a regular sequence of length 2 which would act like our "parameters" up to some artinian quotient. We took x to be our first one and then found no others.

Eisenbud Is $R \to S$ (normalization denoted above) flat?

Me More generally, any non-trivial normalization $\varphi : A \longrightarrow \widetilde{A}$ is never flat. My favorite way to see this to prove the more general fact that the properties (R1) and (S2) descend via faithfully flat extensions. In particular, let $(A, P) \longrightarrow (B, Q)$ be a local homomorphism that's flat.

- **(R1)** Assume that (B, Q) satisfies (R1). Then for any codimension 1 prime $P_1 \in \operatorname{Spec} A$ we can find a prime Q_1 lying over it in B of codimension 1 (faithfully flat). By localizing we may assume $P_1 = P$ and $Q_1 = Q$. To show that A is regular it suffices to show that it has finite global dimension. This is equivalent to showing that $\operatorname{Tor}_i^A(A/P, A/P) = 0$ for $i \gg 0$. On the other hand since B is regular it has finite global dimension and thus we have, $\operatorname{Tor}_i^A(A/P, A/P) \otimes_A B = \operatorname{Tor}_i^B(B/PB, B/PB) = 0$ for $i \gg 0$. Since $A \longrightarrow B$ is faithfully flat we obtain the desired result.
- **(S2)** Now assume (B, Q) satisfied (S2). Without loss of generality assume P is a codimension 2 prime and Q is a codimension 2 prime minimal with respect to lying over P. Then we have $\operatorname{depth}(Q + PB, B/PB) \le \operatorname{codim}_{B/PB}(Q + PB) = 0$. Thus we obtain,
$$2 \le \operatorname{depth}(Q, B) = \operatorname{depth}(Q, A) + \operatorname{depth}(Q + PB, B/PB) = \operatorname{depth}(Q, A).$$
This shows that (A, P) also satisfies (S2).

In our case, if $\varphi : A \longrightarrow \widetilde{A}$ were flat, every localization $A_P \longrightarrow \widetilde{A}_Q$ with $\varphi^{-1}(Q) = P$ would be flat. Since $\widetilde{A}_Q$ satisfies Serre's criterion so would A_P. Thus $A_P \xrightarrow{\sim} \widetilde{A}_Q$ and we have that all the local rings of A are integrally closed. This implies that $A \simeq \widetilde{A}$.

Olsson Isn't there a simpler answer in our case as $A \longrightarrow \widetilde{A}$ is finite?

Me Sure, since $\varphi : A \longrightarrow \widetilde{A}$ is finite, it's locally finite and flat. Since flatness and freeness are equivalent for finite modules over local rings, $\widetilde{A}$ is a locally free $A-$module. Thus $(\widetilde{A})_P$ is a free A_P module for all $P \in \operatorname{Spec} A$. Since the normalization map is birational, $A_P \longrightarrow (\widetilde{A})_P$ is an isomorphism on a dense open subset of $\operatorname{Spec} A$. Thus, $\widetilde{A}$ is a locally free $A-$module of rank 1; this forces $\widetilde{A} \simeq A$.

Eisenbud Can you talk about the Artin-Rees Lemma.

Me A filtration $M_\bullet : M = M_0 \supseteq M_1 \supseteq \cdots$ is said to be an I-filtration if $IM_n \subseteq M_{n+1}$ for all n. The filtration $M_\bullet$ is said to be I-stable if $IM_n = M_{n+1}$ for all $n \gg 0$. Given a noetherian ring, finitely generated modules $M' \subseteq M$, the Artin-Rees Lemma states that the $I-$stable filtration $M_\bullet$ of M restricts to an $I-$stable filtration $M' \cap M_\bullet$ of M'.

A nice applications of this is in the proof of Krull's intersection theorem.

Eisenbud This is not on your syllabus, but can you say something about an application of Artin-Rees Lemma to completions?

Me It can be used to show that taking completions with respect to the $\mathfrak{m}-$adic filtration is exact.

III. Algebraic Topology (Minor - Geometry)

Shapiro What is a fundamental group? (Outside member)

A A loop based at $x_0 \in X$ is a morphism $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = \gamma(1) = x_0$. The fundamental group $\pi_1(X, x_0)$ is a group on the set of all loops based at x_0 up to homotopy. The multiplication operation is concatenation of loops (I drew a few pictures).

Auroux Okay tell me about the fundamental group of the three spaces on the board?

Me While I was explaining the fundamental group, I "drew" $\mathbf{R}^2$, S^1 with a generating loop x, and $\mathbf{T}^2$ with generators x, y. Well, $\pi_1(\mathbf{R}^2) = \{0\}$, $\pi_1(S^1) = \mathbf{Z}x$ and $\pi_1(\mathbf{T}^2) = \langle x, y : xyx^{-1}y^{-1}\rangle$.

(Everyone found my presentation of $\pi_1(\mathbf{T}^2)$ very entertaining (because it's just $\mathbf{Z} \oplus \mathbf{Z}$). So Auroux asked the following question)

Auroux Present the universal cover of $\mathbf{T}^2$ in a nice geometric way.

Me The universal cover of $\mathbf{T}^2$ is $\mathbf{R}^2$. I drew the plane, subdivided it into tori and identified edges. In this way, the covering map is given by $\mathbf{R}^2 \longrightarrow \mathbf{R}^2/\mathbf{Z}e_1 \oplus \mathbf{Z}e_2$.

Auroux Now can you see what the fundamental group of $\mathbf{T}^2$ is from the picture?

Me Sure, it's $\mathbf{Z} \oplus \mathbf{Z}$ and the loop is $e_1e_2e_1^{-1}e_2^{-1}$.

Olsson The fundamental group has a natural action. What is it?

Me If $f : (\widetilde{X}, \widetilde{x}) \longrightarrow (X, x)$ is the universal cover, $\pi_1(X, x)$ has an action on the fibers of $f^{-1}(x)$.

Eisenbud Is the universal cover unique?

Me Yes, this follows from the homotopy lifting property.

Auroux State that more precisely. In general if you have a covering map $S \longrightarrow X$ what can you say about maps from S to $\widetilde{X}$ or $\widetilde{X}$ to S?

Me Say we had a covering $p : (S, s_0) \longrightarrow (X, x_0)$, Y a nice enough space and a map $\varphi : (Y, y_0) \longrightarrow (X, x_0)$. Then we obtain a factorization of φ as $(Y, y_0) \xrightarrow{\varphi'} (S, s_0) \longrightarrow (X, x_0)$ iff $\varphi_\star \pi_1(Y, s_0) \subseteq p_\star \pi_1(S, s_0)$.

So in the case when $\widetilde{X}$ is the universal cover, we have $\pi_1(\widetilde{X}, \widetilde{x}_0) = 0$ and thus we always obtain a map $\widetilde{X} \longrightarrow S$. If S was another universal cover we will obtain maps in both directions and from this we have an isomorphism (mapping base points to base points).

Auroux Let's consider $\mathbf{RP}^3$ topologically. Compute it's homology, fundamental group and cohomology (with $\mathbf{Z}$ coefficients).

Me

1 *Homology*: First of all $\mathbf{RP}^3$ admits a cell decomposition $e^0 \cup e^1 \cup e^2 \cup e^3$. We obtain a cellular chain complex, $0 \longrightarrow \mathbf{Z} \xrightarrow{d_3} \mathbf{Z} \xrightarrow{d_2} \mathbf{Z} \xrightarrow{d_1} \mathbf{Z} \longrightarrow 0$. To figure out the differentials, recall that to obtain $\mathbf{RP}^n$ from $\mathbf{RP}^{n-1}$ we attach an $n-$cell via the $2:1$ cover, $S^{n-1} \longrightarrow \mathbf{RP}^{n-1}$. The degree of composition $\eta : S^{n-1} \longrightarrow \mathbf{RP}^{n-1} \longrightarrow \mathbf{RP}^{n-1}/\mathbf{RP}^{n-2} \simeq S^{n-1}$ is the map d_n.

The degree of the map η is the sum of the relative degrees of a generic fiber. The fiber of a point x in the codomain is $\{x, -x\}$. Thus, the degree is just $\deg(\mathrm{id} : S^{n-1} \longrightarrow S^{n-1}) + \deg(\text{antipodal} : S^{n-1} \longrightarrow S^{n-1})$. Since the antipodal map $S^{n-1} \longrightarrow S^{n-1}$ has degree $(-1)^n$ we obtain $d_n = 1 + (-1)^n$. The complex is just

$$0 \longrightarrow \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \longrightarrow 0.$$

Thus the groups $H_i(X)$ for $i \geq 0$ are $\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}, 0, \mathbf{Z}, 0, 0, \ldots$

2 *Fundamental group*: When one attaches a 2−cell to a space X, it corresponds to adding new relation to $\pi_1(X)$ coming from the attaching map. Attaching a higher cell does nothing because the boundary is not a 1-cell. All of this is essentially a consequence of Van-Kampen's theorem. Thus, by our cell decomposition we see that

$$\pi_1(\mathbf{RP}^3) = \pi_1(\mathbf{RP}^1)/(\text{relation from } d_2) = \mathbf{Z}x/(x^2) \simeq \mathbf{Z}/2\mathbf{Z}.$$

Auroux There's an easier way to see the fundamental group, in this case, from the picture you drew.

Me Of course, we have the universal cover $S^n \longrightarrow \mathbf{RP}^n$. This is a $2:1$ cover and thus $\pi_1(\mathbf{RP}^n) = \mathbf{Z}/2\mathbf{Z}$.

3 *Cohomology*: We can take the dual of the original complex or use the universal coefficient theorem. If we dualize we obtain,

$$0 \longleftarrow \mathbf{Z} \xleftarrow{0} \mathbf{Z} \xleftarrow{2} \mathbf{Z} \xleftarrow{0} \mathbf{Z} \longleftarrow 0.$$

Thus the groups $H^i(X)$ for $i \geq 0$ are $\mathbf{Z}, 0, \mathbf{Z}/2\mathbf{Z}, \mathbf{Z}, 0, 0 \ldots$.

Auroux What happens if you use the universal coefficient theorem?

Me It states that the exact sequence $0 \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}}(H_{k-1}(\mathbf{RP}^3), \mathbf{Z}) \longrightarrow H^k(\mathbf{RP}^3, \mathbf{Z}) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(H_k(\mathbf{RP}^3), \mathbf{Z}) \longrightarrow 0$ splits. Thus,

$$\begin{aligned} H^k(\mathbf{RP}^3, \mathbf{Z}) &\simeq \mathrm{Hom}_{\mathbf{Z}}(H_k(\mathbf{RP}^3), \mathbf{Z}) \oplus \mathrm{Ext}^1_{\mathbf{Z}}(H_{k-1}(\mathbf{RP}^3), \mathbf{Z}) \\ &= \begin{cases} \mathbf{Z} & \text{if } k = 0, 3 \\ 0 & \text{else} \end{cases} \oplus \begin{cases} \mathbf{Z}/2\mathbf{Z} & \text{if } k = 2 \\ 0 & \text{else} \end{cases}. \end{aligned}$$

We get the same result as before.

Auroux Is the conclusion of Poincaré duality satisfied in this case?

Me Indeed, abstractly, $H^i(\mathbf{RP}^3, \mathbf{Z}) \simeq H_{3-i}(\mathbf{RP}^3, \mathbf{Z})$ for all i.

Auroux State Poincaré duality.

Me Let M be a compact, orientable, manifold of dimension n. Then we have an isomorphism $H^i(M, \mathbf{Z}) \xrightarrow{\sim} H_{n-i}(M, \mathbf{Z})$ given by $\alpha \mapsto [M] \cap \alpha$.

END After the statement of Poincare duality everyone was satisfied, and they told me to leave the room. Approximately 2 minutes later, the door opened and they said "Congratulations, you passed"!