Math 55 - Practice Midterm 1 - Solutions

1 State whether each of the following statements are True or False. There is no need to provide an explanation, and no credit will be given for explanations! (**But I will provide explanations**!)

a (**3 points**) Every subset of an uncountable set is countable.

A **False**: If S was uncountable, then S would be an uncountable subset of S.

b (**3 points**) $(p \vee q) \wedge (\neg p \vee r)$ and $q \vee r$ are logically equivalent

A **True**.

c (**3 points**) Let A and B be finite sets with $|A| = |B|$. If $f : A \longrightarrow B$ is injective (one-to-one), then it's surjective (onto).

A **True**: Let $|A| = |B| = n$. Since f is injective, the n distinct elements of A must map to distinct elements of B. Since B only has n elements, f is onto.

d (**3 points**) $n^3 \equiv n \bmod 4$ for all $n \in \mathbf{Z}$.

A **False**: $0 \equiv 2^3 \not\equiv 2 \bmod 4$.

e (**3 points**) $\forall a, b \in \mathbf{Z}^+ \quad \exists c \in \mathbf{Z} \quad (a^2 + c^2 = b^2)$.

A **False**: There is no $c \in \mathbf{Z}$ such that $1^2 + c^2 = 2^2$

2a (**8 points**) Use the Euclidean algorithm to solve the congruence $7x \equiv 1 \bmod 11$

A Applying the Euclidean algorithm we obtain,

$$\begin{aligned} 11 &= 7 \cdot 1 + 4 \\ 7 &= 4 \cdot 1 + 3 \\ 4 &= 3 \cdot 1 + 1 \\ 3 &= 1 \cdot 3 + 0. \end{aligned}$$

Using this we obtain, $1 = 4 - 3 \cdot 1 = 4 - (7 - 4) = 2 \cdot 4 - 7 = 2 \cdot (11 - 7) - 7 = 2 \cdot 11 - 3 \cdot 7$.
Thus, $1 \equiv 2 \cdot 11 - 3 \cdot 7 \equiv -3 \cdot 7 \bmod 11$ i.e. -3 is a solution modulo 11. ■

2b (**2 points**) Solve the congruence $7x \equiv -2 \mod 11$.

A Notice that $x \equiv 1 \cdot x \equiv (-3) \cdot 7x \equiv 6 \bmod 11$. ■

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3 (**5 points**) Find the largest negative solution to the system of equations: $x \equiv 8 \bmod 12$ and $x \equiv 6^{85} \bmod 13$.

A Applying Fermat's little theorem the second congruence simplifies to $6^{12} \equiv 1 \bmod 13$; $6^{85} \equiv 6^{7\cdot 12+1} \equiv 6 \bmod 13$. Now we just apply the Chinese remainder theorem.

Let $m = 12 \cdot 13 = 156$, $M_1 = 13$ and $M_2 = 12$.

- We want to solve $13y_1 \equiv 1 \bmod 12$. This simplifies to $y_1 \equiv 1 \bmod 12$
- We want to solve $12y_2 \equiv 1 \bmod 13$. This simplifies to $-y_2 \equiv 1 \bmod 13$ or $y_2 \equiv -1 \bmod 13$.

Thus, the solution is $a_1M_1y_1 + a_2M_2y_2 \equiv 8 \cdot 13 \cdot 1 + 6 \cdot 12 \cdot (-1) = 32$ modulo 156. The largest negative solution is just $32 - 156 = -124$. ■

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4 (**5 points**) Prove that the function $f : \mathbf{Z} \times \mathbf{Z} \longrightarrow \mathbf{Z}$ given by $f(m, n) = 2m - 7n$ is surjective (onto).

A I'll provide two solutions; one is intuitive and the other is generalizable.

i Notice that $f(\mathbf{Z} \times 0)$ maps onto the even integers. Given any odd integer n, $n + 7$ is even. Thus, $(\frac{n+7}{2}, 1) \in \mathbf{Z} \times \mathbf{Z}$ and $f(\frac{n+7}{2}, 1) = n + 7 - 7 = n$ i.e. f maps to all the odd integers. ■

ii Since $\gcd(2, 7) = 1$, by Bezout's identity we can find $s, t \in \mathbf{Z}$ such that $2s + 7t = 1$; in particular, $f(s, -t) = 1$.

Now given any $z \in \mathbf{Z}$, notice that $(sz, -tz) \in \mathbf{Z} \times \mathbf{Z}$ and $f(sz, -tz) = 2(sz) - 7(-tz) = z(2s + 7t) = z$. Thus, f is onto. ■

Note The second method shows that $f : \mathbf{Z} \times \mathbf{Z} \longrightarrow \mathbf{Z}$ given by $f(m, n) = am + bn$ is onto if $\gcd(a, b) = 1$.

Question What can you say about the **range** of f when $\gcd(a, b) \neq 1$?

5a (**2 points**) Define what it means for a set S to be countable.

A A set S is countable if it's finite or if there exists a bijection $f : S \longrightarrow \mathbf{Z}^+$.

5b (**7 points**) Let C and D be countable sets. Prove that $C \cup D$ is countable [Hint: Split it into three cases, depending on whether C or D is finite]

A This is Theorem 1 on page 174 in Section 2.5. [You would get some points for each case you prove]

c You may use the result in **a** to justify your answers in the following parts. Find with proof examples of uncountable sets A, B such that

I will provide multiple solutions to each problem.

i (**3 points**) $A - B$ is finite

A

a Take $A = B = \mathbf{R}$. Then $A - B = \emptyset$, which is finite.

b Take $A = (0, 1) \cup \{2\}$ and $B = (0, 1)$. Then $A - B = \{2\}$ is finite.

ii (**3 points**) $A - B$ is countably infinite

A

a Let $A = (0,1) \cup \mathbf{Z}$ and $B = (0,1)$. Since countable subsets of countable sets are countable, we see that A and B must be uncountable. On the other hand, $A - B = \mathbf{Z}$ is countable.

b Let $A = \mathbf{R}$ and $B = \mathbf{R} - \mathbf{Z}$. Clearly, $A - B = \mathbf{R} - (\mathbf{R} - \mathbf{Z}) = \mathbf{Z}$ is countably infinite. To finish we need to show that B is uncountable. If it was countable, then by part a, $B \cup \mathbf{Z} = (\mathbf{R} - \mathbf{Z}) \cup \mathbf{R} = \mathbf{R}$ would be countable. This is a contradiction! Thus, $B = \mathbf{R} - \mathbf{Z}$ is uncountable.

iii (**3 points**) $A - B$ is uncountable

A

a Since $(0,1)$ is uncountable, the bijection $f : (0,1) \longrightarrow (2,3)$ given by $f(x) = x + 2$ shows that $(2,3)$ is also uncountable. Then $A = (0,1) \cup (2,3)$ and $B = (0,1)$ are the required sets.

b Another examples would be to take $A = (0,1)$ and $B = \mathbf{R}$. Since $(2,3) \subseteq \mathbf{R} - (0,1)$ we see that $\mathbf{R} - (0,1) = A - B$ is uncountable.
