MATH 113 - Practice Questions

1. Assorted things

(a) [5 points] Find prime ideals I, J such that $I \cap J$ is not a prime ideal

(b) [5 points] Find a ring homomorphism $f : R \longrightarrow S$ and $\mathfrak{m}$ a maximal ideal, such that $f^{-1}(\mathfrak{m})$ is not a maximal ideal.

(c) [5 points] Prove that $\mathbf{Z}[x]/(3, 3x-1)$ is the trivial ring.

(d) [5 points] Given $a, b \in R$, is it true that $(a) = (b)$ in R iff $a = b$? Prove it or give a counterexample

2.0

(a) [5 points] Let I be an ideal of R. Prove that $\sqrt{I} = \{a \in R : a^n \in I\}$ is an ideal of R

(b) [5 points] Let $\mathfrak{p}$ be a prime ideal of R. Prove that $\sqrt{\mathfrak{p}} = \mathfrak{p}$.

(c) [5 points] Find an example of an ideal I and a ring R such that $I \neq \sqrt{I}$.

2.1 An element $x \in R$ is called an idempotent if $x^2 = x$.

(a) [5 points] Prove if x is an idempotent then $1 - x$ is also an idempotent

For the rest of the question assume that $x \in R$ is an idempotent.

(b) [5 points] Let S be the ideal $(x) = \{rx : r \in R\}$. Show that S is a ring with $1_S := x$ (careful, I don't claim S is a subring of R). Conclude that $T = (1-x)$ is also a ring with identity $1_T := 1 - x$.

(c) [5 points] Prove that the map $\psi : R \longrightarrow S \times T$ given by $r \mapsto (rx, r(1-x))$ is a ring homomorphism.

2.2

(a) [3 points] Prove that the the "evaluation" map $\varphi : \mathbf{R}[x] \longrightarrow \mathbf{C}$ that sends $f(x)$ to $f(1+3i)$ is a ring homomorphism.

(b) [7 points] Find the kernel of φ.

3.1

(a) [5 points] Prove that $f(x) = x^3 - 2x^2 + 3x + 5$ in $\mathbf{Q}[x]$ is irreducible.

(b) [5 points] Find a prime p such that $x^3 - 2x^2 + 3x + 5$ is not irreducible in $\mathbf{Z}/p\mathbf{Z}$.

3.2

(a) [5 points] Determine all the irreducible factors of $g(x) = x^3 - 2x^2 - x + 2$ in $\mathbf{Q}[x]$.

(b) [5 points] Find three *distinct* zero divisors in $\mathbf{Q}[x]/(g)$.

4.1

(a) [5 points] Prove that if R is a PID and I a prime ideal of R, then R/I is a PID.

(b) [5 points] Find an example of a ring R that's not a PID and I prime such that R/I is a PID.

4.2

(a) [3 points] Find an example of a domain R and an ideal I such that R/I is not an integral domain.

(b) [7 points] Let $\mathbf{Z}_{12}$ denote the ring $\mathbf{Z}/12\mathbf{Z}$. Show that $\mathbf{Z}_{12}/(3)$ is an integral domain.

4.3

(a) [5 points] Prove that a ring homomorphism between two fields (i.e. $\psi : K \longrightarrow L$ with K, L fields) is always injective.

(b) [5 points] Show that there is no homomorphism from $\mathbf{Q} \longrightarrow \mathbf{Z}/p\mathbf{Z}$.

(c) [5 points] Show that there is no ring homomorphism from $\mathbf{C} \longrightarrow \mathbf{Q}$ [Hint: Homomorphisms are multiplicative].

5.1 Let $\alpha = \sqrt{2+\sqrt{2}} \in \mathbf{C}$.

(a) [5 points] Compute the minimal polynomial of α over $\mathbf{Q}$ and determine $[\mathbf{Q}(\alpha) : \mathbf{Q}]$.

(b) [5 points] Show that $\mathbf{Q}(\sqrt[3]{2})$ is not a subfield of $\mathbf{Q}(\alpha)$.

(c) [5 points] Write a basis for $\mathbf{Q}(\alpha)$ over $\mathbf{Q}$ and express α^6 in terms of that basis

5.2 Consider the extensions $\mathbf{Q} \subseteq K \subseteq L$ where $K = \mathbf{Q}(\sqrt{2})$ and $L = \mathbf{Q}(i\sqrt[4]{2})$

(a) [5 points] Prove that $K \subseteq L$ but $K \neq L$.

(b) [5 points] Find the minimal polynomial of $i\sqrt[4]{2}$ over $\mathbf{Q}$.

(c) [5 points] Compute $[K : \mathbf{Q}]$ and find all embeddings of K into $\mathbf{C}$