Math 55 - Midterm 2, Summer 2017 - Solutions

1 State whether each of the following statements are True or False. There is no need to provide an explanation, and no credit will be given for explanations!

a (**3 points**) Assume a positive integer from 1 to 100 is chosen at random. The probability that it's divisible by 3 is the same as the probability that it's divisible by 5.

A **False**: The number of integers between 1 and 100 divisible by 3 is not the same as the number of integers divisible by 5.

b (**3 points**) Suppose that E and F are events in a probability space such that $p(E) = 0.8$ and $p(F) = 0.5$. Then we have $p(E \cap F) \geq 0.3$.

A **True**: By inclusion-exclusion, $1 \geq p(E \cup F) = p(E) + p(F) - p(E \cap F)$. Thus, $p(E \cap F) \geq p(E) + p(F) - 1 = 0.8 + 0.5 - 1 = 0.3$.

c (**3 points**) If E and F are independent events in a probability space, then E and $\overline{F}$ are independent.

A **True**: Since $p(E \cap \overline{F}) + p(E \cap F) = p(E)$ we have

$$p(E \cap \overline{F}) = p(E) - p(E \cap F) = p(E) - p(E)p(F) = p(E)(1 - p(F)) = p(E)p(\overline{F}).$$

2a (**5 points**) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.

A Consider the following four pairs, $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$. Then by the pigeonhole principle if we choose five integers, $\lceil \frac{5}{4} \rceil = 2$ of them will be in the same pair. Since two integers in a pair sum to 9, we are done.$\blacksquare$

2b (**5 points**) For any non-negative integer n prove the following identity,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

[Hint: An algebraic solution is to write $(1+x)^{2n}$ in two different ways and compare terms]

A The coefficient of x^n in $(1+x)^{2n}$ is $\binom{2n}{n}$. On the other hand,

$$(1+x)^{2n} = (1+x)^n(1+x)^n = \left(\sum_{k=0}^{n} \binom{n}{k} x^k\right) \left(\sum_{k=0}^{n} \binom{n}{k} x^k\right) = \sum_{m=0}^{2n} \left[\sum_{k=0}^{m} \binom{m}{k} \binom{m}{m-k}\right] x^m.$$

Comparing coefficients we obtain $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$. $\blacksquare$

3

a (**5 points**) For any positive integer n prove that,

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n}.$$

A We prove this by induction; for the base case we have $\frac{1}{\sqrt{1}} \geq \sqrt{1}$. Now assume $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n}$ for some $n \geq 1$. Then,

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}(\sqrt{n+1})+1}{\sqrt{n+1}} \geq \frac{\sqrt{n} \cdot \sqrt{n}+1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}. \blacksquare$$

b (**3 points**) For any positive integer n, prove that

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

A We prove this by induction; for the base case we have $1 = \frac{1(2)}{2} = 1$. Now assume $1+2+\cdots+n = \frac{n(n+1)}{2}$ for some $n \geq 1$. Then,

$$1+2+\cdots+n+(n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+2)(n+1)}{2}. \blacksquare$$

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4 Consider $S = \{1, \ldots, n\}$ with the uniform distribution. Let X be the random variable satisfying $X(i) = i$ for all $i \in S$.

a (**2 points**) Find a closed form for $E(X)$.

A By definition and the previous question,

$$E(X) = \sum_{k=1}^{n} p(X = i) \cdot i = \sum_{k=1}^{n} \frac{1}{n} \cdot i = \frac{1 + 2 + \cdots + n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}. \blacksquare$$

b (**2 points**) Using the definition of expected value, prove that $E((X+1)^3) - E(X^3) = n^2 + 3n + 3$.

A By canceling similar terms we obtain,

$$E((X+1)^3) - E(X^3) = \sum_{k=1}^{n} \frac{1}{n} \cdot (i+1)^3 - \sum_{k=1}^{n} i^3 = \frac{(n+1)^3}{n} - \frac{1}{n} = \frac{n^3 + 3n^2 + 3n}{n} = n^2 + 3n + 3. \blacksquare$$

c (**2 points**) Prove that $E((X+1)^3 - X^3) = 3E(X^2) + 3E(X) + 1$.

A By linearity of expectation we obtain,

$$E((X+1)^3 - X^3) = E(X^3 + 3X^2 + 3X + 1 - X^3) = E(3X^2 + 3X + 1) = 3E(X^2) + 3E(X) + 1. \blacksquare$$

d (**2 points**) By using part b and part c, find a closed form for the sum. $1^2 + 2^2 + \cdots + n^2$.

A Notice that $1^2 + 2^2 + \cdots + n^2 = n \cdot E(X^2)$. We have also shown that $3E(X^2) + 3E(X) + 1 = n^2 + 3n + 3$. Thus,

$$3E(X^2) = n^2 + 3n + 2 - 3E(X) = n^2 + 3n + 2 - \frac{3}{2}(n+1) = n^2 + \frac{3}{2}n + \frac{1}{2}.$$

Finally,

$$1^2 + 2^2 + \cdots + n^2 = nE(X^2) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}. \blacksquare$$

e (**5 points**) Generalize the idea of part b to higher powers to find a recurrence relation for $E(x^k)$ i.e. find integers $c_i \in \mathbf{Z}$ such that $E(X^k) = c_0 + c_1 E(X) + \cdots + c_{k-1} E(X^{k-1})$.

A By definition of expected value as in part b we have

$$E((X+1)^{k+1} - X^{k+1}) = E((X+1)^{k+1}) - E(X^{k+1}) = \frac{(n+1)^k}{n} - \frac{1^k}{n}.$$

Recall that the binomial theorem states, $(X+1)^k = \sum_{m=0}^{k} \binom{k}{m} X^k$. Thus, using linearity of expected value we obtain,

$$E((X+1)^{k+1} - X^{k+1}) = E\left(\sum_{m=0}^{k} \binom{k+1}{m} X^m\right) = \sum_{m=0}^{k} \binom{k+1}{m} E(X^m).$$

Equating the expressions gives us, $\sum_{m=0}^{k} \binom{k+1}{m} E(X^m) = \frac{(n+1)^k - 1}{n}$. Isolating $E(X^k)$ in the left sum we obtain,

$$E(X^k) = \frac{1}{k+1}\left(\frac{(n+1)^k - 1}{n}\right) - \sum_{m=0}^{k-1} \frac{1}{k+1}\binom{k+1}{m} E(X^m). \blacksquare$$

f (**3 points**) Finally using part d, find a closed form expression for the following sum,

$$1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5.$$

A By the previous parts,

$$1^5 + \cdots + n^5 = nE(X^5) = \frac{1}{6}\left(\frac{(n+1)^5 - 1}{n} - \sum_{m=0}^{4} \frac{1}{5}\binom{6}{m} nE(X^m)\right).$$

Thus, we just need to compute $nE(X^m)$ for $0 \le m \le 4$. We have already computed $nE(X^0) = n$, $nE(X) = \frac{n(n+1)}{2}$ and $nE(X^2) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$. Using the formula from the previous part we can compute

$$nE(X^3) = \frac{1}{4}\left[(n+1)^4 - 1 - n - 2n(n+1) - (2n^3 + 3n^2 + n)\right] = \left(\frac{n(n+1)}{2}\right)^2.$$

Similarly, $nE(X^4) = \frac{1}{30} n(n+1)(2n+1)(3n^2+n-1)$. $\blacksquare$

5a (**4 points**) Consider the linear recurrence relation, $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$ and $a_0 = 0$, $a_1 = 4$. Find a closed form expression for a_n.

A The characteristic equation of this recurrence is $x^2 - 2x - 3 = (x-3)(x+1) = 0$. The roots are $-1, 3$ and thus $a_n = \alpha_0(-1)^n + \alpha_1 3^n$ for all $n \geq 0$. We also know $0 = \alpha_0 + \alpha_1$ and $4 = -\alpha_0 + 3\alpha_1$. Solving this system we see that $\alpha_0 = -1$ and $\alpha_1 = 1$ i.e. $a_n = 3^n - (-1)^n$ for all $n \geq 0$. ∎

b (**3 points**) Let S_n denote the number of binary strings of length n that do not contain consecutive 1's. For example, 0110 is a string of length four that is **NOT** allowed. Compute S_0, S_1, S_2, S_3 and S_4.

A A direct computation shows $S_0 = 1$, $S_1 = 2$, $S_2 = 3$, $S_3 = 5$ and $S_4 = 8$.

c (**4 points**) Find a recurrence relations for S_n. Justify your answer (you don't need to solve the recurrence!)

A Looking at the last two digits of a *valid* string we see that it's either 00, 10 or 01. Thus, any *valid* string of length n is of the form **x**0 with **x** any *valid* string of length $n-1$ or **y**01 with **y** a *valid* string of length $n-2$. This shows that $S_n = S_{n-1} + S_{n-2}$ for $n \geq 2$. ∎

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6 Here are a series of questions that describe the daily life of a certain Calculus 2 instructor.

a (**3 points**) She gave a midterm to 33 students and she needs to assign a grade of A,B, C, D or F to each student. How many ways are there to assign grades to students, assuming that the students are all distinct.

A Each student gets one of the five grades, thus there are 5^{33} assignments.

b (**3 points**) The instructor decides to assign only an A, B or C as she doesn't like low grades. To be fair she gives eleven A's, eleven B's and eleven C's. How many ways are there to do this, still assuming the students are distinct.

A We need to choose eleven students to get As, eleven more for the B' and the remaining get C's. By the product rule this is just

$$\binom{33}{11}\binom{22}{11}\binom{11}{11} = \frac{33!}{11!11!11!}.$$

c (**5 points**) She just finished grading the final exam. She has ten students who she needs to assign a pass or fail to. Unfortunately for the students she's very angry and has decided to fail **seven** of the students. She also decides to flip a coin to assign grades. In particular, she will assign a pass to the student if the coin comes up heads and a fail if the coin comes up tails. However the other instructors feel bad and give her a weighted coin that comes up heads $\frac{2}{3}$ of the time and tails the other times. She will only stop tossing the coin once seven students have been failed. Assuming the students are indistinguishable (she's really angry!), what is the expected number of coin tosses? (You don't need to simplify the sum)

A1 The statement that there are ten students is irrelevant; you only need to know that there are at least seven students. She also needs to toss the coin at least seven times. If she tosses the coin m times, she must have tossed a tails on the last toss. Thus, the number of such tosses of length m is $\binom{m-1}{6}$; we have six spots for the tails to occur in the first $m-1$ tosses. If X denotes the number of coin tosses, taking the sum over all m we obtain,

$$\begin{aligned} E(X) &= \sum_{m=7}^{\infty} m \cdot p(X=m) \\ &= 7 \cdot \left(\frac{1}{3}\right)^7 + 8 \cdot \binom{7}{6}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^7 + 9 \cdot \binom{8}{6}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^7 + \cdots \\ &= \sum_{m=7}^{\infty} m\binom{m-1}{6}\left(\frac{2}{3}\right)^{m-7}\left(\frac{1}{3}\right)^7 . \blacksquare \end{aligned}$$

A2 We will now describe a slicker method to solve such problems. Let X_1 be the number of tosses until the first tails, X_2 the number of tosses between the first and second tails and in general X_i the number of tosses between the $i-$th and $(i+1)-$th tails. Notice that $X = X_1 + \cdots + X_7$ and thus $E(X) = E(X_1) + \cdots + E(X_7)$. But each X_i is just a random variable with a geometric distribution with parameter $p = \frac{1}{3}$ (keep tossing until you obtain the first tail). Thus, as noted in class, $E(X_i) = \frac{1}{p} = 3$ and we have $E(X) = 7E(X_1) = 7 \cdot 3 = 21$.

Note The infinite sum in the first answer can be shown to equal 21 as follows: Notice that $m\binom{m-1}{6} = \frac{m(m-1)\cdots(m-5)(m-6)}{6!}$ and with some generating function manipulation we obtain,

$$\frac{7!}{(1-x)^8} = \frac{d^7}{dx^7}\left(\frac{1}{1-x}\right) = \frac{d^7}{dx^7}\sum_{m=0}^{\infty} x^m = \sum_{m=0}^{\infty}\frac{d^7}{dx^7}x^m = \sum_{m=7}^{\infty} m(m-1)\cdots(m-6)x^{m-7}.$$

Thus,

$$\begin{aligned} E(X) &= \sum_{m=7}^{\infty} m\binom{m-1}{6}\left(\frac{2}{3}\right)^{m-7}\left(\frac{1}{3}\right)^7 \\ &= \left(\frac{1}{3}\right)^7 \frac{1}{6!}\sum_{m=7}^{\infty} m(m-1)\cdots(m-6)\left(\frac{2}{3}\right)^{m-7} \\ &= \left(\frac{1}{3}\right)^7 \frac{1}{6!}\frac{7!}{(1-\frac{2}{3})^8} \\ &= \frac{7!}{6!}\left(\frac{1}{3}\right)^7\left(\frac{3}{1}\right)^8 \\ &= 21. \end{aligned}$$

d (**5 points**) Sometime before the instructor graded her second midterm, she tried to convince me to write my midterm; you can blame her for this midterm. The probability that she tells me to write is $\frac{1}{4}$. The probability that I write my midterm if she told me to write it is $\frac{3}{4}$. The probability that I don't write my midterm is $\frac{1}{10}$. What's the probability that she didn't tell me to write given that I have written the midterm.

A We will give a solution that doesn't use Baye's theorem: Let E be the event that she tells me to write my midterm and F the event that I write my midterm. We want to compute $p(\overline{E}|F)$; by definition this is just $\frac{p(\overline{E}\cap F)}{p(F)} = \frac{p(F|\overline{E})p(\overline{E})}{p(F)}$. We will now compute the relevant quantities:

i $p(\overline{E}) = \frac{3}{4}$ as we are given $p(E) = \frac{1}{4}$.

ii $p(F) = \frac{9}{10}$ as we are given $p(\overline{F}) = \frac{1}{10}$.

iii To compute $p(F|\overline{E})$ recall that $p(F|\overline{E})p(\overline{E}) + p(F|E)p(E) = p(F)$; you can prove this using the definition of conditional probability. Thus, $p(F|\overline{E}) = (\frac{3}{4})^{-1}\left(\frac{9}{10} - \frac{3}{4}\cdot\frac{1}{4}\right) = \frac{4}{3}\cdot\left(\frac{9}{10} - \frac{3}{16}\right) = \frac{12}{10} - \frac{1}{4} = \frac{19}{20}$.

Finally, $p(\overline{E}|F) = \frac{\frac{19}{20}\cdot\frac{3}{4}}{\frac{9}{10}} = \frac{57}{9\cdot 8} = \frac{57}{72} = \frac{19}{24}$. ■

7a (**4 points**) Show that if n is a positive integer, then

$$\binom{-1/2}{n} = \frac{\binom{2n}{n}}{(-4)^n}.$$

A While one can directly manipulate the left hand side to get the right hand side, the most straightforward way is to prove this by induction on n: The base case of $n = 0$ is clearly true. Now assume $\binom{-1/2}{n} = \frac{1}{(-4)^n}\binom{2n}{n}$ for some $n \geq 1$ and notice that

$$\begin{aligned}
\binom{-1/2}{n+1} &= \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-n+1)(-\frac{1}{2}-n)}{(n+1)!} \\
&= \binom{-1/2}{n}\frac{(-\frac{1}{2}-n)}{n+1} \\
&= \frac{1}{(-4)^n}\binom{2n}{n}\frac{2n+1}{2(n+1)} \\
&= \frac{1}{(-4)^n}\frac{(2n)!}{n!n!}\left(\frac{2n+1}{2n+2}\right)\left(\frac{2n+2}{2n+2}\right) \\
&= \frac{1}{(-4)^{n+1}}\frac{(2n)!(2n+1)(2n+2)}{(n+1)!(n+1)!} \\
&= \frac{1}{(-4)^{n+1}}\binom{2(n+1)}{n+1}. \blacksquare
\end{aligned}$$

b (**2 points**) Using part b deduce that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty}\binom{2n}{n}x^n.$$

A By part a and the binomial theorem,

$$\frac{1}{\sqrt{1-4x}} = (1-4x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-4)^n x^n = \sum_{n=0}^{\infty}\binom{2n}{n}x^n. \blacksquare$$