Math 55 - Summer 2017, Midterm 2

Instructor - Ritvik Ramkumar

July 28, 2017, 10:20AM - 12:00AM, Etcheverry Hall 3111

Name: ______________________________

SID: ______________________________

INSTRUCTIONS:

- Write all answers in the provided space. Please write carefully and clearly, in complete English sentences.
- You are not allowed to use any notes, books, electronic devices, or your own scratch paper.
- Questions $2-7$ require you to **justify** your answers

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Grade Breakdown

Question	Points	Maximum
1		9
2		10
3		8
4		16
5		11
6		20
7		6
Total		**80**

1 State whether each of the following statements are True or False. There is no need to provide an explanation, and no credit will be given for explanations!

a (**3 points**) Assume a positive integer from 1 to 100 is chosen at random. The probability that it's divisible by 3 is the same as the probability that it's divisible by 5.

b (**3 points**) Suppose that E and F are events in a probability space such that $p(E) = 0.8$ and $p(F) = 0.5$. Then we have $p(E \cap F) \geq 0.3$.

c (**3 points**) If E and F are independent events in a probability space, then E and $\overline{F}$ are independent.

2a (**5 points**) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.

2b (**5 points**) For any non-negative integer n prove the following identity,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

[Hint: An algebraic solution is to write $(1+x)^{2n}$ in two different ways and compare terms]

3

a (**5 points**) For any positive integer n prove that,

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n}.$$

b (**3 points**) For any positive integer n, prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

4 Consider $S = \{1, \dots, n\}$ with the uniform distribution. Let X be the random variable satisfying $X(i) = i$ for all $i \in S$.

a (**2 points**) Find a closed form for $E(X)$.

b (**2 points**) Using the definition of expected value, prove that $E((X+1)^3) - E(X^3) = n^2 + 3n + 3$.

c (**2 points**) Prove that $E((X+1)^3 - X^3) = 3E(X^2) + 3E(X) + 1$.

d (**2 points**) By using part b and part c, find a closed form for the sum. $1^2 + 2^2 + \cdots + n^2$.

e (**5 points**) Generalize the idea of part b to higher powers to find a recurrence relation for $E(x^k)$ i.e. find integers $c_i \in \mathbf{Z}$ such that $E(X^k) = c_0 + c_1 E(X) + \cdots + c_{k-1} E(X^{k-1})$.

f (**3 points**) Finally using part d, find a closed form expression for the following sum,

$$1^5 + 2^5 + 3^5 + 4^5 + \cdots + n^5.$$

5a (**4 points**) Consider the linear recurrence relation, $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$ and $a_0 = 0$, $a_1 = 4$. Find a closed form expression for a_n.

b (**3 points**) Let S_n denote the number of binary strings of length n that do not contain consecutive 1's. For example, 0110 is a string of length four that is **NOT** allowed. Compute S_0, S_1, S_2, S_3 and S_4.

c (**4 points**) Find a recurrence relations for S_n. Justify your answer (you don't need to solve the recurrence!)

6 Here are a series of questions that describe the daily life of a certain Calculus 2 instructor.

a (**3 points**) She gave a midterm to 33 students and she needs to assign a grade of A,B, C, D or F to each student. How many ways are there to assign grades to students, assuming that the students are all distinct.

b (**3 points**) The instructor decides to assign only an A, B or C as she doesn't like low grades. To be fair she gives eleven A's, eleven B's and eleven C's. How many ways are there to do this, still assuming the students are distinct.

c (**5 points**) She just finished grading the final exam. She has ten students who she needs to assign a pass or fail to. Unfortunately for the students she's very angry and has decided to fail **seven** of the students. She also decides to flip a coin to assign grades. In particular, she will assign a pass to the student if the coin comes up heads and a fail if the coin comes up tails. However the other instructors feel bad and give her a weighted coin that comes up heads $\frac{2}{3}$ of the time and tails the other times. She will only stop tossing the coin once seven students have been failed. Assuming the students are indistinguishable (she's really angry!), what is the expected number of coin tosses? (You don't need to simplify the sum)

d (**5 points**) Sometime before the instructor graded her second midterm, she tried to convince me to write my midterm; you can blame her for this midterm. The probability that she tells me to write is $\frac{1}{4}$. The probability that I write my midterm if she told me to write it is $\frac{3}{4}$. The probability that I don't write my midterm is $\frac{1}{10}$. What's the probability that she didn't tell me to write given that I have written the midterm.

e (**4 points**) For any positive integers $n, r \geq 0$, give a combinatorial proof of the identity

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}.$$

[Hint: The right hand side is the number of ways to put $n+1$ indistinguishable balls into $r+1$ distinguishable boxes. Count this in another way]

7a (**4 points**) Show that if n is any positive integer, then

$$\binom{-1/2}{n} = \frac{\binom{2n}{n}}{(-4)^n}.$$

b (**2 points**) Using part b deduce that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$