MATH 113 - HOMEWORK 5

Due in class on Tuesday July 31, 2018

- All rings are assumed to be commutative and contain 1_R.
- Let F be a field. A subring $R \subseteq F$ that's also a field is called a *subfield*.

1. Prove that $\mathbf{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbf{Q}\}$ is a subfield of $\mathbf{C}$. [First show it's a subring, then think about inverses]

2. Which of the following sets are ideals?

(a) $\{p(x,y) : p(x,x) = 0\} \subseteq \mathbf{C}[x,y]$

(b) $\{p(x,y) : p(x,y) = p(y,x)\} \subseteq \mathbf{C}[x,y]$

(c) $\{p(x) : p \text{ has no real roots}\} \subseteq \mathbf{C}[x]$

Note If $p(x,y) = x^2 + y^3$, then $p(x,x) = x^2 + x^3$ and $p(y,x) = y^2 + x^3$. The polynomial $x^2 + 1$ has no real roots.

3. *Ideals*

(a) Consider the set $S = \{x^2 + 1, x^2 - 1\} \subseteq \mathbf{C}[x]$. Determine, precisely, the ideal $(x^2 + 1, x^2 - 1)$.

(b) Let $a, b \in \mathbf{Z}$. Show that $(a, b) = (\gcd(a,b)) \subseteq \mathbf{Z}$. [Hint: Euclidean Algorithm]

4. Let $R = \mathbf{Z}/n\mathbf{Z}$ and $[k] \in R$ a non-zero element. Show that $[k]$ is a zero-divisor iff $\gcd(k, n) > 1$.

5. *Units and ideals*

(a) Let R be a ring. Prove that $u \in R$ is not contained in any proper ideal $\iff$ u is a unit.

(b) Prove that a field has exactly two ideals. Which ideals are they?

(c) What are all the ideals of $\mathbf{Z}/4\mathbf{Z}$?

6. Let R, S be rings and define the set $\text{Hom}(R, S) := \{f : R \longrightarrow S : f \text{ is a ring homomorphism}\}$.

(a) Show that $\text{Hom}(\mathbf{Z}, S)$ contains one element.

(b) Find a ring S for which $\text{Hom}(S, \mathbf{Z}) = \emptyset$.

(c) Show that there's a bijection from $\text{Hom}(\mathbf{Z}[x], S)$ to the set S.

7. *Quotients*

(a) Let $R = \mathbf{Z}[x]$ and $I = (2, x)$. Which familiar ring, up to isomorphism, is R/I.

(b) Let $R = \mathbf{C}[x,y]$ and $I = (y - x)$. Which familiar ring, up to isomorphism, is R/I.

(c) Is the element $\overline{x}$ a unit in $\mathbf{Q}[x]/(x^4 + 1)$?

8. Let R be a ring. We say that $x \in R$ is *nilpotent* if $\exists n \geq 0$ such that $x^n = 0_R$. The element 0_R is called the *trivial nilpotent*.

(a) Prove that the set of nilpotent elements of R, denoted by $\mathfrak{N}(R)$, is an ideal.

(b) Show that the quotient ring $R/\mathfrak{N}(R)$ has no non-trivial nilpotents.

(c) Compute $\mathfrak{N}(R)$ for $R = \mathbf{Z}/540\mathbf{Z}$.