MATH 113 - HOMEWORK 4

Due in class on Thursday July 19, 2018

NOTE You only need to submit solutions to **Problems 1 to 5**. However, you will need to know the contents of Problem 6, 7 and 8a for the midterm.

1. *Example I*

Let $G = (\mathbf{Q}^+, \cdot)$ be the group of positive rational numbers under multiplication.

(a) List the elements of $N = \langle 9 \rangle$ and explain why N is normal.

(b) Find an element of order 2 in G/N.

(c) Find an element of infinite order in G/N.

2. *Example II*

(a) Prove that $\mathrm{GL}_n(\mathbf{R})/\mathrm{SL}_n(\mathbf{R}) \cong \mathbf{R}^\star$. [Hint: Find a homomorphism from $\mathrm{GL}_n(\mathbf{R})$ to.....]

Let $\mathcal{U}$ be the subgroup of upper triangular matrices in $\mathrm{GL}_2(\mathbf{R})$. Let $\mathcal{D}$ denote the subgroup of diagonal matrices in $\mathrm{GL}_2(\mathbf{R})$.

(b) Prove that $\mathcal{U}$ is not a normal subgroup of $\mathrm{GL}_2(\mathbf{R})$.

(c) Show that $N = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbf{R} \right\}$ is a normal subgroup of $\mathcal{U}$.

(d) Prove that $\mathcal{U}/N \cong \mathcal{D}$. [Hint: Find a homomorphism from.....]

3. Find all normal subgroups of D_4 and D_5. [Hint: Use something from one of the Homeworks]

4. *Subgroups of index 2 are normal [**Very useful!**]*

Let G be a group and H a subgroup. You may assume that the right cosets of H in G are of the form Hg for $g \in G$. You may also assume that the **set** of left cosets are in bijection with the **set** of right cosets.

(a) Prove that H is normal if and only if $Hg = gH$ for all $g \in G$ i.e. the right coset is the same as the left coset.

(b) Show that if $H \subseteq G$ has index 2, then H is normal in G.

5. *All groups of order p^2 are abelian*:

(a) Verify that $Z(G)$ is a normal subgroup of G.

(b) Prove that if $G/Z(G)$ is cyclic, then G is abelian [Hint: The cosets are of the form $a^n Z(G)$]

(c) Now let G be a group of order p^2 for p a prime. Using part (b) and results from class show that G is abelian.

Conclusion We know, up to isomorphism, all groups of order p^2 (p a prime) are just $\mathbf{Z}/p^2\mathbf{Z}$ or $(\mathbf{Z}/p\mathbf{Z})^2$.

6. *Groups of order 6*

(a) Let G be a group and H a subgroup. Let G act on $X = G/H$ (set of left cosets) by left multiplication i.e. $x : gH \mapsto xgH$. Let ϕ be the induced homomorphism $\phi : G \longrightarrow \mathrm{Sym}(X)$. Prove that $\ker \phi \subseteq H$.

Let G be a group of order 6. The goal is to show that $G \cong S_3$ or $G \cong \mathbf{Z}/6\mathbf{Z}$ following these steps:

(b) Prove that there are subgroups H, K of order $2, 3$ respectively.

(c) If H is normal, show that $G \cong (\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/3\mathbf{Z})$. [Hint: What can you say about $H \times K$?]

(d) If H is not normal in G, use part (a) to show that $\ker \phi = \{e_G\}$. Conclude that $G \cong S_3$.

Conclusion Using results from class, Problem 5 and Problem 6, we have found all the groups of orders $1, 2, 3, 4, 5, 6, 7, 9$.

7. *A_n is simple for $n \geq 5$:*

In class I showed that A_5 is simple. One can extend this argument and show that A_6 is simple. Assuming that A_6 is simple, prove that A_n is simple for $n \geq 7$ following these steps:

(a) Prove that any two 3-cycles in A_n are conjugate in A_n. [Hint: We know $\exists g \in S_n$ such that $g(123)g^{-1} = (ijk)$. If $g \notin A_n$, modify g slightly]

(b) Let $N \trianglelefteq A_n$ be a non-trivial normal subgroup and $\gamma \in N$ non-identity. Find a 3-cycle σ that doesn't commute with γ i.e. $\gamma\sigma \neq \sigma\gamma$.

(c) Explain why $\tau = \gamma^{-1}(\sigma^{-1}\gamma\sigma)$ moves (at most) 6 elements in $\{1, \dots, n\}$ and leaves the rest fixed. Conclude that there is a subgroup $H \subseteq G$, isomorphic to A_6 that contains τ.

(d) Prove that $N \cap A_6 = A_6$.

(e) Conclude that $N = A_n$.

Conclusion A_n is simple, $0 \trianglelefteq A_n \trianglelefteq S_n$ is a composition series and S_n is NOT solvable for $n \geq 5$!

8. *Composition series of S_3 and S_4*

(a) Prove that $N = \langle (12)(34), (13)(24) \rangle$ is a normal subgroup of S_4. Deduce that $N \cong (\mathbf{Z}/2\mathbf{Z})^2$.

(b) Find a composition series of S_4 [Note: Make sure it's maximal]

(c) Conclude that S_4 is solvable.

(d) Show that S_3 is solvable.