MATH 113 - HOMEWORK 3

Due in class on Thursday July 12, 2018

1. Section 9: **9, 11, 23**

2. *The group S_6*

(a) Determine the order of all elements in S_6.

(b) How many elements of S_6 have order 2?

(c) How many conjugacy classes are there in S_6?

(d) Which of the conjugacy classes in part (c) are contained in A_6?

3. *Subgroups of S_4.*

(a) Using a theorem I've stated, verify that there exists subgroups of order 8 and 3 in S_4

(b) Find a subgroup of order 3.

(c) Find a subgroup of order 8. [Hint: It's not going to be cyclic; so try 2 generated subgroups]

4. *Group actions Part I - Basic Examples*

Q Are the following actions faithful? Are they transitive?

(a) Let $G = \mathrm{GL}_2(\mathbf{R})$ act on $\mathbf{R}^2$ by matrix multiplication i.e. $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} ax+by \\ cx+dy \end{smallmatrix}\right)$.

(b) Let $G = \mathbf{Z}$ act on $\mathbf{Z}/9\mathbf{Z}$ by $n(m) = [m] + [n]$.

(c) Let $G = S_3$ act on $\mathbf{R}^3$ by permuting coordinates i.e. $\sigma((x_1, x_2, x_3)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. For example, (12)(3) sends $(x, y, z) \mapsto (y, x, z)$ and (132) sends $(x, y, z) \mapsto (z, x, y)$.

5. *Group actions Part II - More Examples*

(a) Let G be a group and let $\mathrm{Sub}(G)$ denote the set of subgroups of G. Let G act on $\mathrm{Sub}(G)$ by $g(H) = gHg^{-1} = \{ghg^{-1} : h \in H\}$. Prove that this is an action.

(b) Let $G = S_3$. Describe all the orbits of the action. [In particular, start by listing all subgroups of S_3]

(c) Compute stab $\langle(12)\rangle$ and stab $\langle(123)\rangle)$.

6. *Group actions Part III - Theory*

(a) Let G be a group of order 12 and X a size of set 10. Does there exist a transitive action of G on X?

(b) Does there exist a faithful action of $\mathbf{Z}/7\mathbf{Z}$ on $\{1, 2, 3, 4, 5, 6\}$? [Hint: Consider the associated homomorphism and use Problem 2].

(c) Let $X = \{1, 2, 3\}$ and $G = \mathbf{Z}/2\mathbf{Z}$. Classify all the group actions of G on X.

7. *Dihedral Groups*

(a) List all the conjugacy classes of D_4 (with proof).

(b) List all conjugacy classes of D_5 (with proof). [One can directly extend this to D_n]

Other problems, **NOT** to be handed in:

9. *Small number of conjugacy classes*

(a) Prove that the only group with exactly 1 conjugacy class is the trivial group

(b) Prove that the only finite group with exactly 2 conjugacy classes is $\mathbf{Z}/2\mathbf{Z}$. [Hint: Think about the larger conjugacy class]

10. *Centers of D_n and S_n*

(a) Prove that center of S_n for $n \geq 3$ is trivial.

(b) Compute $Z(D_n)$ for $n \geq 3$. [Hint: It depends on whether n is even or odd]

12. *Inversion Action*

Definition Given an action of a group G on a set X, we say $x \in X$ is **fixed** by G if $Gx := \{g(x) : g \in G\} = \{x\}$.

(a) Let G be a group and let $\mathbf{Z}/2\mathbf{Z}$ act on G by $[0](g) = g$ and $[1](g) = g^{-1}$. Prove that this is an action.

(b) Let $G = S_n$. Find all elements of S_n that are fixed by the action of **part (a)**.