MATH 113 - HOMEWORK 2

Due in class on Thursday July 5, 2018

1 *Kernels and Images.* Let $\phi : G \longrightarrow H$ be a homomorphism of groups.

(a) The **kernel** of ϕ is defined to be $\ker\phi := \{g \in G : \phi(g) = e_H\}$. Prove that $\ker\phi$ is a subgroup of G.

(b) Prove that ϕ is injective $\iff$ $\ker\phi = \{e_G\}$.

(c) Let G be a group of prime order. Prove that every **non-trivial** homomorphism, $\psi : G \longrightarrow G$ is an isomorphism.

Note: Similarly, one can show for any subgroup $K \subseteq G$, $\phi(K)$ is a subgroup of H. Also, for any subgroup $J \subseteq H$, $\phi^{-1}(J) = \{g \in G : \phi(g) \in J\}$ is a subgroup of G.

2 *Homomorphisms of cyclic groups*

(a) Let $f : G \longrightarrow H$ be a homomorphism and x an element of G of finite order. Prove that $\operatorname{ord} f(x)$ divides $\operatorname{ord} x$.

(b) Is there a non-trivial homomorphism, $\psi : \mathbf{Z}/m\mathbf{Z} \longrightarrow \mathbf{Z}$ for any integer m?

(c) If m, n are coprime positive integers show that every homomorphisms $\psi : \mathbf{Z}/m\mathbf{Z} \longrightarrow \mathbf{Z}/n\mathbf{Z}$ is trivial.

(d) If m, n are not coprime, construct a non-trivial homomorphism $\psi : \mathbf{Z}/m\mathbf{Z} \longrightarrow \mathbf{Z}/n\mathbf{Z}$. [Hint: Consider defining it on a generator of $\mathbf{Z}/m\mathbf{Z}$. Carefully check that your function is **well defined.**]

3 Let G be an abelian group of order 360. Classify all such G up to isomorphism.

4 *Centers of groups.*

(a) Let G be a group and $g \in G$. Prove that $Z(a) := \{g \in G : ga = ag\}$ is a subgroup of G (this is called the **centralizer of** a).

(b) Prove that $Z(G) := \{x \in G : xa = ax\ \forall a \in G\}$ is a subgroup of G (this is called the **center of** G). [You may use **(a)** or do it directly]

5 *The center of the general linear group (the matrices that commute with ALL the other matrices)*

(a) For each $1 \le i, j \le n$, let $E_{ij} \in \mathrm{Mat}_n(\mathbf{R})$ denote the matrix whose (i, j)-th entry is 1 and the other entires are 0. Prove that $I + E_{ij} \in \mathrm{GL}_n(\mathbf{R})$.

(b) Show that $Z(\mathrm{GL}_n(\mathbf{R})) = \{\lambda I : \lambda \in \mathbf{R}^\star\}$ (and thus, $Z(\mathrm{GL}_n(\mathbf{R})) \cong \mathbf{R}^\star$). [Hint: Use **(a)**]

6 *Conjugacy classes.*

(a) Let G be a group. We say that two elements $a \sim b$ (a and b are said to be **conjugate**) $\iff$ $\exists g \in G$ such that $gag^{-1} = b$. Prove that this is an equivalence relation on G. [If $G = \mathrm{GL}_n(\mathbf{R})$, this equivalence relation is just the relation of matrix similarity i.e. A is similar to B iff $\exists$ an invertible matrix J such that $A = JBJ^{-1}$. J is usually called the change of basis matrix]

(b) Let $a \in G$ and from Problem 4, the centralizer $Z(a)$ is a subgroup of G. Prove for all $g, h \in G$,

$$gag^{-1} = hah^{-1} \iff g, h \text{ belong to the } \textbf{same} \text{ left coset of } Z(a).$$

(c) Use **(b)** to show that there is a bijection of sets $[a] \longrightarrow G/Z(a)$ ($[a]$ is the equivalence of a with respect to the conjugacy relation and $G/Z(a)$ is the set of left cosets of $Z(a)$).

(d) Show that $[g]$ consists of one element $\iff$ $g \in Z(G)$.

Do **NOT** hand in the following problems.

7 *Conjugacy classes of $GL_n(\mathbf{C})$:* (only for those who know the Jordan normal form; this will **never** be tested)

(a) Describe when two invertible matrices in Jordan normal form are conjugate.

(b) Using the fact that every matrix is similar to a matrix in Jordan normal form, describe the unique conjugacy classes

(c) Do part **(b)** explicitly for $\mathrm{GL}_2(\mathbf{C})$ and $\mathrm{GL}_3(\mathbf{C})$

(d) Write down the distinct conjugacy classes of $\mathrm{GL}_2(\mathbf{R})$ [Hint: Be careful, there are matrices that are not upper-triangularizable]

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0 Prove that a homomorphisms $f : G \longrightarrow H$ is an isomorphism $\iff$ $\exists$ a homomorphism $g : H \longrightarrow G$ such that $g \circ f = \mathrm{id}_G$ and $f \circ g = \mathrm{id}_H$.

1 Let G and H be groups, and let S be a set of generators for G. Suppose that $f : G \longrightarrow H$ and $g : G \longrightarrow H$ are both homomorphisms and that $\forall s \in S$, $f(s) = g(s)$. Prove that $f = g$ [This means that two homomorphisms are completely determined by where the generators are mapped. You've seen this with vector spaces; a linear map is uniquely determined by where the basis elements are mapped. On the other hand, unlike vector spaces you cannot define a homomorphism of groups just my defining it on the generators].

2 *Products of groups*: Let H_1 and H_2 be subgroups of G_1 and G_2 respectively. Verify that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$.

3 Let $f : G \longrightarrow H$ be an isomorphism and let **P** be some "reasonable" group theoretic adjective such as cyclic, abelian, finitely generated etc.

(a) Verify that, G is **P** iff H is **P**.

(b) For $x \in G$ show that $\operatorname{ord} x = \operatorname{ord} f(x)$

(c) Assume that G is a finite group. Prove that the number of subgroups of of order d in G is the same as the number of subgroups of order d in H.

(d) Let p be a prime number. Use **(c)** to prove that $(\mathbf{Z}/p^3\mathbf{Z})$, $(\mathbf{Z}/p^2\mathbf{Z}) \times (\mathbf{Z}/p\mathbf{Z})$ and $(\mathbf{Z}/p\mathbf{Z})^3$ are all non-isomorphic [Hint: Consider subgroups of order p]

(e) Show that $\mathbf{Z}^m$ is never cyclic for any $m > 1$. Conclude that $\mathbf{Z}$ is not isomorphic to $\mathbf{Z}^m$ for $m > 1$.

4 Given $n, m \in \mathbf{Z} - \{0\}$, let $p_1, \ldots, p_r$ be the common distinct prime factors of n and m.

(a) Use the uniqueness of prime factorizations to show that $n = p_1^{a_1} \cdots p_r^{a_r} n'$ and $m = p_1^{b_1} \cdots p_r^{b_r} m'$ with $b_i, a_i > 0$ and $\gcd(n', m') = 1$.

(b) Show that $\gcd(n, m) = p_1^{\min(a_1,b_1)} \cdots p_r^{\min(a_r,b_r)}$

(c) Show that $\operatorname{lcm}(n, m) = p_1^{\max(a_1,b_1)} \cdots p_r^{\max(a_r,b_r)} n'm'$.