Math 113 - Summer 2018, Midterm

Instructor - Ritvik Ramkumar

July 18, 2018, 4:10PM - 5:30PM, Cory 289

Name: ____

INSTRUCTIONS:

- Write all answers in the provided space. Please write carefully and clearly, in complete sentences.
- You are only allowed to use one sheet of paper with handwritten notes
- **Justify** all your answers. In particular, if you're using a theorem from class, clearly state it's name or contents.
- You're only allowed to use results I've stated in class or on a Homework.

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Grade Breakdown

Question	Points	Maximum
1		10
2		15
3		10
4		15
5		10
Total		**60**

1

a [5 points] Let G be an abelian group of order 200. List, up to isomorphism, all possible groups G. Which of the groups listed are cyclic? Since $200 = 2^3 \cdot 5^2$, we may use the classification theorem.

- $\mathbb{Z}/2^3\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z}$
- $\mathbb{Z}/2^3\mathbb{Z} \times (\mathbb{Z}/5\mathbb{Z})^2$
- $\mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z}$
- $\mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/5\mathbb{Z})^2$
- $(\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}/5^2\mathbb{Z}$
- $(\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/5\mathbb{Z})^2$

There's, up to isomorphism, only one cyclic group of order 200.

It is $\mathbb{Z}/2^3\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z} \cong \mathbb{Z}/200\mathbb{Z}$ as shown in class. ■

b [5 points] Find two non-isomorphic abelian groups of order 200 that contain an element of order 100.

Note that if $(a_1, \dots, a_k) \in G_1 \times \cdots \times G_k$ then

$$\operatorname{ord}(a_1, \dots, a_k) = \operatorname{lcm}(\operatorname{ord}(a_1), \dots, \operatorname{ord}(a_k)).$$

Thus, $([2]_{2^3}, [1]_5) \in \mathbb{Z}/2^3\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z}$ has order $\operatorname{lcm}(4, 25) = 100$

and $([1]_{2^2}, [0]_2, [1]_5) \in \mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5^2\mathbb{Z}$ has order $\operatorname{lcm}(4, 1, 25) = 100$.

By part a), these are non-isomorphic ■

2 An element x of a group G is said to be a *square* if $\exists y \in G$ such that $y^2 = x$.

a [2 points] Find all the squares in $G = \mathbf{Z}/4\mathbf{Z}$.

Note that the squares are just

$$\{2[y]_4 : y \in \mathbb{Z}\} = \{[2y]_4 : y \in \mathbb{Z}\}$$
$$= \{[0], [2]\} \blacksquare$$

b [3 points] Find all the squares in $G = \mathbf{Z}/5\mathbf{Z}$.

This will follow from part d. $\blacksquare$

b [5 points] Let G be a finite, even order cyclic group, find all elements of G that are squares.

Then $G \cong \mathbb{Z}/2k\mathbb{Z}$.

$[y] \in G$ is a square iff $[y] = 2[x]$ for some $x \in \mathbb{Z}$.

$\Leftrightarrow [y-2x] = [0]$

$\Leftrightarrow 2k \mid (y-2x)$

So $2 \mid (y-2x) \Rightarrow 2 \mid y$.

Conversely if $y = 2y'$ then $[y] = [2y'] = 2[y']$.

Thus the squares are $\{[0], [2], [4], \ldots, [2k-2]\}$ ■

c [5 points] Let G be a finite, odd order cyclic group. Prove that every element of G is a square.

Then $G \cong \mathbb{Z}/(2k+1)\mathbb{Z}$.

Note that $[1] = [1] + [2k+1] = [2k+2] = 2[k+2] \Rightarrow [1]$ is a square,

Thus $[m] = m \cdot [1] = m(2[k+2]) = 2[m(k+2)]$ is a square $\forall [m] \in G$.

Thus every element of G is a square. ■

3

a [7 points] Let G be a group of order 62. Prove that every proper subgroup of G is abelian.

By Lagrange's theorem the size of the proper subgroups are 1, 2, 31 (since $62 = 2 \cdot 31$).

The trivial group is abelian. We have shown in class that all groups of prime order are cyclic and thus abelian; thus the subgroups of order 2, 31 are abelian ∎

b [3 points] Is every group of order 62 abelian? If yes prove it, if not give a counterexample.

This is not true. Consider D_{31}. ∎

a b c

4 Let $X = \{(12)(34), (13)(24), (14)(23)\} \subseteq S_4$ be a set. Let S_4 act on X by conjugation i.e. $g(x) = gxg^{-1}$ (you may assume that this is an action).

a [5 points] Let ϕ_g be the permutation of X associated to g. Compute $\phi_{(12)}$, $\phi_{(13)}$ and $\phi_{(23)}$.

- $\phi_{(12)}((12)(34)) = (12)(12)(34)(12) = (12)(34)$

$\phi_{(12)}((13)(24)) = (12)(13)(24)(12) = (14)(23)$

$\phi_{(12)}((14)(23)) = (12)(14)(23)(12) = (13)(24)$.

Similarly, $\phi_{(13)}(a) = c$, $\phi_{(13)}(b) = b$, $\phi_{(13)}(c) = a$

$\phi_{(23)}(a) = b$, $\phi_{(23)}(b) = a$, $\phi_{(23)}(c) = c$

b [5 points] Show that there's a surjective group homomorphism $\Phi : S_4 \longrightarrow S_3$.

Let $\Phi : S_4 \longrightarrow \mathrm{Sym}(X) \cong S_3$ be the induced homomorphism. Then $\mathrm{im}\,\Phi \leq S_3$ is a subgroup. Moreover, by part (a), $\mathrm{im}\,\Phi$ contains the transpositions of S_3. Since S_3 is generated by transpositions, $\mathrm{im}\,\Phi = S_3$

c [5 points] Consider the subgroup $N = \{e, (12)(34), (13)(24), (14)(23)\}$ of S_4. Prove that N is normal and show $S_4/N \cong S_3$.

Since Φ is surjective, the first isomorphism theorem implies,

$$S_4 / \operatorname{Ker} \Phi \cong \operatorname{im} \Phi = S_3 .$$

It suffices to show $N = \operatorname{Ker} \Phi$ (Kernels are always normal).

Since $|\operatorname{Ker} \Phi| = \dfrac{|S_4|}{|S_3|} = 4$, it suffices to show $N \subseteq \operatorname{Ker} \Phi$.

This can be checked directly: $\phi_{(12)(34)}((12)(34)) = (12)(34)$

$$\phi_{(12)(34)}((13)(24)) = (12)(34)(13)(24)(12)(34) = (13)(34)$$

$$\Rightarrow \phi_{(12)(34)} = id_X .$$

Similarly, $\phi_{(13)(24)}$, $\phi_{(14)(23)}$ are all id_X.

Thus $N \subseteq \operatorname{Ker} \Phi = \{ g \in S_4 : \phi_g = id \}$ $\blacksquare$

5 *Isomorphic subgroups can have non-isomorphic quotients*

a [5 points] Let $G = (\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/4\mathbf{Z})$. Find all subgroups of G of order 2.

All order 2 subgroups will be cyclic.

Thus $\langle ([a]_2, [b]_4) \rangle \subseteq G$ is a subgroup of order 2

$\Rightarrow$ $2[a]_2 = [0]_2$ and $2[b]_4 = [0]_4$

$\Leftrightarrow$ $[a]_2 = [0]_2, [1]_2$ and $[b]_4 = [0]_4, [2]_4$

Thus we have 3 subgroups,

$$\langle [0], [2] \rangle, \quad \langle [1], [2] \rangle, \quad \langle [1], [0] \rangle$$

b [5 points] Find two subgroups $H, K \subseteq G$ such that $H \cong K$ but $G/H \ncong G/K$.

Let $H = \langle [1], [0] \rangle$ and $K = \langle [0], [2] \rangle$.

One can check $G/H \cong \mathbb{Z}/4\mathbb{Z}$ and $G/K \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,

but it's simpler to consider order of elements:

- $G/H = \{ ([0], [1]) + H,\ (0,2) + H,\ (0,3) + H,\ H \}$
  has an element of order 4 $\left((0,1) + H \right)$

- $G/K = \{ (0,0) + K,\ (1,0) + K,\ (0,1) + K,\ (1,1) + K \}$
  has no elements of order 4

  Indeed, $2[(a,b) + K] = (2a, 2b) + K$
  $= (0,0) + K$ as $(2a, 2b) \in K$