To understand $K[x]$ a bit more, we will need to talk about prime/maximal ideals.

Defn
- Let $R$ be a ring. $(0)$ is called the trivial ideal and $R = (1)$ is called the improper ideal. An ideal $I$ is said to be proper if $I \neq R$.
- A principal ideal $I \subseteq R$ is an ideal generated by one element i.e. $I = (x)$ for $x \in R$.
- A maximal ideal is an ideal $\mathfrak{m} \neq R$ such that if $I$ is an ideal that is strictly bigger than $\mathfrak{m}$ i.e. $I \supsetneq \mathfrak{m}$ then $I = R$.
- A prime ideal is an ideal $P \neq R$ such that if $a, b \in R$ and $ab \in P \Rightarrow a \in P$ or $b \in P$.

Ex.
- $(p) \subseteq \mathbb{Z}$ is maximal (and prime)
- In a PID every ideal generated by an irreducible element is prime
- $\mathbb{C}[x,y]$: $(x)$ is prime but not maximal
  $(x,y)$ is maximal.
- $(4)$ in $\mathbb{Z}[x]$ (or $\mathbb{Z}$) is not prime
- $(x)$ is prime, but not maximal in $\mathbb{Z}[x]$
- $(2,x)$ is maximal in $\mathbb{Z}[x]$

We will prove part of the third isomorphism theorem for rings:

Thm Let $R$ be a ring and $I$ an ideal. Then ideals of $R/I$ are in bijection with ideals of $R$ containing $I$.

Pf We have a canonical ring hom. $\pi: R \longrightarrow R/I$.
Given an ideal $\bar{J}$ of $R/I$, $\pi^{-1}(\bar{J})$ is an ideal of $R$ (check). It contains $I$ as $\pi(I) = 0$.
Conversely if $J \subseteq R$ is an ideal containing $I$ lets show $\pi(J)$ is an ideal of $R/I$.
If $x \in R/I$ and $y \in \pi(J)$, then $x = \pi(z)$ and $y = \pi(w)$ $\Rightarrow$ $xy = \pi(z)\pi(w) = \pi(zw) \in \pi(J)$.
$\pi(J)$ is also closed under addition.
Check that this a bijection ■

Thm Let $I \subseteq R$ be an ideal.
① $I$ is prime $\Leftrightarrow$ $R/I$ is an integral domain
② $I$ is maximal $\Leftrightarrow$ $R/I$ is a field.

Pf ① $\bar{a}\bar{b} = 0$ in $R/I$ $\Leftrightarrow$ $ab \in I$ and $\bar{a}, \bar{b} \neq 0$ in $R/I$ $\Leftrightarrow$ $a, b \notin I$.
② If $R/I$ is a field, the only ideals are $0$ and $R/I$. Thus by the previous theorem, the only ideals containing $I$ are $I$ and $R \Rightarrow I$ maximal.

Conversely, if $I$ is maximal, the only ideals of $R/I$ are $0$ and $R/I$ $\Rightarrow R/I$ is a field (Why? Think about the ideal $(u)$ for any $u \neq 0$ in $R/I$) ∎

Cor Maximal ideals are prime

Cor In a PID, every non-zero ideal generated by an irreducible element is maximal.

Defn If $E, F$ are fields and there's an injective homomorphism $E \to F$, we say $F$ is a field extension of $E$ (and $E$ is a subfield of $F$, where we identify $E$ with its image).

Exercise Homomorphisms of fields $E \to F$ are either injective or zero [Hint: what's the kernel?]

Q) Given a polynomial $f \in K[x]$ that's irreducible can we find a bigger field $K'$ containing $f$ where $f$ will factor completely?

Defn $f \in K[x]$ is said to factor completely in $K$ if $f(x) = (x-\alpha_1) \cdots (x-\alpha_n)$ for $\alpha_j \in K$