How to spot linear factors?

Thm [Fundamental theorem of algebra]
Every non-constant polynomial over $\mathbb{C}$ has a root. (Liouville's theorem to $|p(z)|$ and $\frac{1}{p}$)

Thus $f \in \mathbb{C}[x]$ can be written as
$f(x) = c\,(x-\alpha_1)^{r_1} \cdots (x-\alpha_n)^{r_n}$ with $\alpha_i \in \mathbb{C}$

Defn Let $f(x) = (x-\alpha)^k g(x)$ and $x-\alpha \nmid g(x)$.
- If $k=1$, then α is a simple root
- If $k>1$, then α is a multiple root
- k is called the multiplicity.

Ex Over $\mathbb{R}$, the irreducible polynomials are $x-a$ and ax^2+bx+c with $b^2-4ac<0$.
To show that these are the only ones, let $f(x) \in \mathbb{R}[x]$. View $f(x) \in \mathbb{C}[x]$ and factor it there. Then check that non-real roots come in conjugate pairs $\alpha, \bar{\alpha}$.
Thus $(x-\alpha)(x-\bar{\alpha}) = x^2 - (\alpha+\bar{\alpha})x + |\alpha|^2$.

[Rational roots test]
Thm Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ in $\mathbb{Z}[x]$ with $a_0 \neq 0$. If $f(\alpha)=0$ with $\alpha \in \mathbb{Q}$ then $\alpha \mid a_0$.

Ex Note $f(x) = x^4 - 2$ is irreducible

Show no solutions in $\mathbb{Q}$.

- Rational root test $\Rightarrow$ no linear factors.
- Assume $f(x) = x^4 - 2 = (x^2 + ax + b)(x^2 + cx + d)$

$\Rightarrow a + c = 0,\ ca + d + b = 0,\ ad + bc = 0,\ bd = -2$

Thm [Eisenstein's criterion] Let $f(x) = a_n x^n + \dots + a_0 \in \mathbb{Z}[x]$. Assume $\exists$ a prime p such that $p^2 \nmid a_0$, $p \mid a_i$ for $i = 0, \dots, n-1$ and $p \nmid a_n$. Then f is irreducible over $\mathbb{Q}$.

Pf In your book: Thm 23.15.

Ex. $2x^3 + 6x^2 + 12x + 6$ is irreducible over $\mathbb{Q}[x]$.
($p = 3$ and Eisenstein)

- Eisenstein doesn't apply to $x^2 + 3x + 2$ or $2x^2 + x + 3$. But $x^2 + 3x + 2 = (x+2)(x+1)$ is reducible while $2x^2 + x + 3$ is not
- $x^n - 2$ is irreducible $\forall n \geq 1$.
- $p + px + px^2 + \dots + px^{n-1} + x^n \in \mathbb{Q}[x]$ is irreducible $\forall n \geq 1$ and p a prime.

Defn A field K is said to be algebraically closed if every $f(x) \in K[x]$ has a root in K.

Ex $\mathbb{C}$ is algebraically closed but $\mathbb{Q}, \mathbb{R}$ are not.

Thm [Euclidean Algorithm] Let $f, g \in K[x]$ have no non constant common factors and non-zero.

Then $\exists\, r, s \in K[x]$ s.t. $fr + gs = 1$.

Sketch By division algorithm,

$$f = g a_1 + g_1 \quad \text{with} \quad g_1 = 0 \text{ or } \deg g_1 < \deg g.$$

Note $g_1 \neq 0$ because f, g have no common factors.

So $\deg g_1 < \deg g$. If $\deg g_1 = 0$ we are done

as $f - g a_1 = g_1 \in K$.

Or else, $g = g_1 a_2 + g_2$ with $g_2 = 0$ or $\deg g_2 < \deg g_1$.

If $g_2 = 0$ then, $g = g_1 a_2$ and

$$f = g a_1 + g_1 = g_1 a_2 a_1 + g_1 = g_1(a_2 a_1 + 1)$$

Since f, g have no common factors $\Rightarrow g_1$ is an unit.

Then we are done. So assume $\deg g_2 < \deg g_1$.

If $\deg g_2 = 0$ we have

$$\begin{aligned} f a_2 - g a_1 a_2 &= a_2 (f - g a_1) \\ &= a_2 g_1 \\ &= g - g_2 \end{aligned}$$

$\Rightarrow f a_2 + (-a_1 a_2 - 1) g = -g_2$. YAY!.

Keep going until you win ■

Recall that an integral domain is a ring with no zero divisors. A field is a non-trivial ring in which every non-zero element is a unit.

Ex. • $\mathbb{Z}/n\mathbb{Z}$ domain $\overset{\text{think}}{\Longleftrightarrow}$ n prime

• $\mathbb{R}[x]/(x^2-1)$ not a domain

• $K[x]/(p(x))$ with p irreducible is a domain.

(why: If $\overline{h(x)}\,\overline{g(x)} = \bar{0} \Rightarrow h(x)g(x) \in (p(x))$.

$\Rightarrow h(x)g(x) = p(x)r(x)$.

What can you conclude about $h(x)$ or $g(x)$?)

Defn An integral domain is called a principal ideal domain (PID) if every ideal is principal (generated by one element).

Ex. • $\mathbb{Z}$ is a PID

• $K[x]$ is a PID

$\}$ see next page

• $\mathbb{Z}[x]$ is not a PID

Pf $(2, x)$ is not principally generated.

If $(2,x) = (f(x)) \Rightarrow 2 = f(x)g(x)$.

Think of this over $\mathbb{Q}[x]$ and use "unique factorization" to conclude $f(x), g(x)$ are units in $\mathbb{Q}$. But $f(x) \in \mathbb{Z} \Rightarrow f(x) = 1$ or 2. But then $(2,x) \neq (f(x))$

(note $1 \notin (2,x)$) ∎