Linear equation : $ax+b=0 \longrightarrow x=-\frac{b}{a}$

Quadratic equation: $ax^2+bx+c=0 \longrightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Cubic / Quartic equations: I'll post it on my website.

Quintic equations: $a_5x^5+a_4x^4+\dots+a_0=0 \longrightarrow ?$

Of course there is SOME function that takes in the coefficients and outputs all the roots. But we want a "natural" function. So we formulate this:

Q) Does there exist a function in the coefficients of a quintic polynomial (coefficients in $\mathbb{C}$) that outputs the roots only using "algebraic" operations: addition, subtraction, multiplication, division, taking n^{th} roots, for any n.

A) No. An example is x^5-x-1. (Buzzwords: The galois group, S_5 is not solvable)

6/19 ~~A crash course~~ Sets and Functions

Defn A **set** is a collection of objects called **elements**.

Ex $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

Notation: Let X be a set ~~and y an element of X~~.

- $y \in X$ means y is an element of X.
- $y \notin X$ means y is not an element of X.
- $\forall z \in X$ means for every element of X.
- If X has finite size, $|X|$ means the # of elements of X.

Now let Y be another set

- $X \subseteq Y$ means X is a subset of Y : $\{1,2\} \subseteq \mathbb{Z}$.
  (every element of X is an element of Y)

- Set builder notation: $\{$ elements of S : property the elements need to satisfy $\}$

Ex: Even integers $= \{x \in \mathbb{Z} : 2 \text{ divides } x\}$
Odd integers $= \{y \in \mathbb{Z} : 2 \text{ doesn't divide } y\}$

- Intersection of sets: $X \cap Y = \{z \in X : z \in X \text{ and } z \in Y\}$
$= \{z \in Y : z \in X\}$ etc..
$= \{z : z \in X \text{ and } z \in Y\}$
"place holder"

- Union of sets: $X \cup Y = \{z : z \in X \text{ or } z \in Y\}$
- Direct / Cartesian product: $X \times Y = \{(x, y) : x \in X, y \in Y\}$
- $\emptyset$ is empty set
- X, Y disjoint $\longrightarrow$ $X \cap Y = \emptyset$.

Defn A function, f, from a set X to a set Y is a rule that assigns to each $x \in X$ a single element $f(x) \in Y$.
This is denoted by $f : X \longrightarrow Y$, $x \longmapsto f(x)$.

Ex
- $\cos : \mathbb{R} \longrightarrow \mathbb{R}$ by $x \longmapsto \cos x$
- $f : \mathbb{R} \longrightarrow \mathbb{C}$, $f(x) = x^3$
- $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \longmapsto Ax$ where A is an $n \times n$ matrix.

Properties of functions:
- $f : X \longrightarrow Y$, X is called the domain, Y the codomain
- $\mathrm{Im}(f) = \{y \in Y : \exists x \in X \text{ with } f(x) = y\}$
there exists.
- f is said to injective / one to one iff $f(a) = f(b) \Rightarrow a = b \ \forall a, b \in X$.
"implies"

- $P \Rightarrow Q$ means whenever P is true, so is Q.
- $P \Leftrightarrow Q$ means $P \Rightarrow Q$ and $Q \Rightarrow P$.

Ex • $f: \mathbb{Z} \longrightarrow \mathbb{Z}$, $x \longmapsto 2x$ is injective

Pf $f(a) = f(b)$ implies $2a = 2b \Rightarrow a = b$.

• $f: \mathbb{Z} \longrightarrow \mathbb{Z}$, $x \longmapsto x^2$ is not injective

Pf Notice that $f(-2) = 4 = f(2)$ but $2 \neq -2$.

• f is said to surjective / onto $\iff$ $\mathrm{Im}(f) = Y$.

Ex. Both the examples above are not surjective

• $f: \mathbb{Z} \longrightarrow \mathbb{Z}$, $x \longmapsto -x + 5$ is surjective.

Pf Given $y \in \underset{\text{codomain}}{\mathbb{Z}}$, we may take $x = -y + 5$. Then,

$f(x) = -(-y+5) + 5 = y - 5 + 5 = y$.

• f is bijective iff f is both injective & and surjective

Thm If X, Y are finite sets, then $\exists$ a bijection $f: X \to Y$ $\iff$ $|X| = |Y|$.

Ex We always have a ~~map~~ function $\mathrm{id}_X : X \longrightarrow X$ by $x \longmapsto x$. This is a bijection

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Equivalence relation

Idea: Consider a set X of humans. Although each person is different we might say two people are equivalent if they have the same gender. This gives us a way to categorize the elements of a set. If $x, y \in X$ are two people with the same gender, we denote equivalence by $x \sim y$.

Notice that • $x \sim x$

• $x \sim y \iff y \sim x$

• If $x \sim y$ and $y \sim z \Rightarrow x \sim z$

Moreover, the set X can be split up into a disjoint union of sets determined by gender.

Defn An equivalence relation on a set S is a subset $\mathcal{U} \subseteq S \times S$ such that

Reflexive
① $(x,x) \in \mathcal{U}$ for all $x \in S$

Symmetric
② $(x,y) \in \mathcal{U} \Rightarrow (y,x) \in \mathcal{U}$

Transitive
③ $(x,y), (y,z) \in \mathcal{U} \Rightarrow (x,z) \in \mathcal{U}$

We notate $(x,y) \in \mathcal{U}$ by $x \sim y$.

Defn Let $\sim$ be an equivalence relation on a set S. Given $x \in S$, we define $[x] = \{ y \in S : x \sim y \}$. This is called the equivalence class containing x.

- $x \in [x]$
- $y \in [x] \Rightarrow [y] = [x]$
- Thus $[x] = [y]$ or $[x] \cap [y] = \emptyset$

} Verify these.

~~So the equivalence classes form a partition parti~~

Defn A partition of a set S is a collection $\{S_i\}$ (set) of subsets of S satisfying

① $S_i \neq \emptyset$

② $\bigcup_i S_i = S$

③ $S_i \cap S_j = \emptyset$ for $i \neq j$ (i.e. they are disjoint)

Ex • ~~$\mathbb{Z}$ $\{S_1, S_2\}$~~ Consider $S = \mathbb{Z}$ and S_1 = even integers
S_2 = odd integers

$\{S_1, S_2\}$ is a partition of $\mathbb{Z}$

• $S_1 = \{\ldots, -4, -3, -2\}$, $S_2 = \{-1, 2, 3\}$, $S_3 = \{0, 1\}$, $S_4 = \{4\}$, $S_5 = \{5\}$, … $S_n = \{n\}$ for all $n \geq 4$.

Then $\{S_i\}_{i \geq 1}$ is a partition of $\mathbb{Z}$.