Polynomial rings

Defn Let R be a ring. A polynomial in R is a formal expression: $f(x) = a_0 + a_1 x + \dots + a_n x^n$, $n \in \mathbb{N}$, $a_i \in R$

- For $i > n$, define $a_i = 0_R$ (we omit terms with coefficient 0_R)
- The a_i are called coefficients of $f(x)$
- $f(x) = g(x) \iff$ all coefficients are equal.
- We write x^k for $1_R \cdot x^k$

The set of all polynomials is denoted by $R[x]$.

(One can think of $R[x]$ as a collection of sequences $a_0, a_1, a_2, \dots$ that are eventually 0)

$+$ is defined: $(a_0 + \dots + a_n x^n) + (b_0 + \dots + b_m x^m) := (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n$ (if $n \geq m$)

$\times$ is defined: $(a_0 + \dots + a_n x^n)(b_0 + \dots + b_m x^m) := a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \dots + \left(\sum_{i+j=k} a_i b_j\right)x^k + \dots + (a_n b_m)x^{m+n}$.

Important Exercise $(R[x], +, \times)$ is a ring, called the polynomial ring.

Note
- R commutative $\Rightarrow$ $R[x]$ commutative
- There's a natural embedding $R \to R[x]$, $a \mapsto a_0 = a$
- There's a natural projection $R[x] \to R$, $x \mapsto 0$.
- Given $f(x) \in R[x]$, there's a map of sets $R \to R$, $a \mapsto f(a)$.

Ex. $2(x^2+2) = 2x^2+4$

- $4+x^2 = 4+x^2+0\cdot x^3$
- $1+x^2+x^3+x^4+\dots = \sum_{k\geq 1} x^k$ is not a polynomial
- $\bar{4}+x^2 = x^2$ in $(\mathbb{Z}/4\mathbb{Z})[x]$

Note: Homomorphisms from $R[x] \longrightarrow R$ are completely determined by the image of x (upto where the scalars go)

Indeed, if $\varphi: R[x] \to R$ is any homomorphism we must have,

$\varphi(1_R) = 1_R$ and let $\varphi(x) = g$

$$\begin{aligned}
\varphi(P(x)) &= \varphi(a_0 + a_1 x + \dots + a_n x^n) \\
&= \varphi(a_0) + \varphi(a_1 x) + \dots + \varphi(a_n x^n) \\
&= \varphi(a_0) + \varphi(a_1)\varphi(x) + \dots + \varphi(a_n)\varphi(x)^n \\
&= \varphi(a_0) + \varphi(a_1) g + \dots + \varphi(a_n) g^n.
\end{aligned}$$

If we assume $\varphi(r) = r \quad \forall r \in R.$

We obtain, $\varphi(P(x)) = a_0 + a_1 g + \dots + a_n g^n.$

Defn. Let $f \in R[x]$ such that $f = a_0 + a_1 x + \dots + a_n x^n$ with $a_n \neq 0$.
Then n is called the degree of f, denoted $\deg f$.

- a_n is called the leading coefficient
- f is said to be monic if $a_n = 1$.
- If $\deg f = 0$, then f is said to be constant.