Midterm Review

This is a sketch of Monday's lecture. It's missing many of the definitions and slightly messy.
This is a REVIEW not a list of what's on the exam!

- Midterm is 80 minutes long in class.
- 6 – 8 questions

Midterm Review

① Define a group G ← identity, inverse unique

Ex $(\mathbb{Z}, +)$, $(\mathbb{Q}^*, \times)$, $(M_n(\mathbb{R}), +)$, $(\mathbb{Z}/n\mathbb{Z}, +)$,
S_n, $GL_n(\mathbb{R})$, D_n.
non-abelian

② Subgroup $H \leq G$: $e \in H$, $x \in H \Rightarrow x^{-1} \in H$
, $x, y \in H \Rightarrow xy \in H$

③ Left cosets of H in G as xH for $x \in G$.
- $\{xH : x \in G\}$ forms a partition of G
- $yH = xH \iff y^{-1}x \in H$
- there's a bijection $H \longrightarrow xH$

④ Lagrange's theorem : $|G| < \infty \Rightarrow |H| \mid |G|$.

⑤ Given $S \subseteq G$, $\langle S \rangle = \{$finite products of elements in $S \cup \{e\} \cup S^{-1}\}$.
- Finitely generated G
- cyclic G ($\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$).
- orders of elements.

(6) Cyclic groups G and $|G| < \infty$:

- Every subgroup of cyclic group is cyclic
- $\forall m \mid |G|$, $\exists$ a subgroup of size m.
- $\text{ord}([a]) = \dfrac{n}{\gcd(a,n)}$ for $[a] \in \mathbb{Z}/n\mathbb{Z}$.

(7) G, H groups $\to$ $G \times H$ product

- orders of elements in $G \times H$.
- Finitely generated abelian groups
- $G \cong N \times K$ if N, K normal, $K \cap N = \{e\}$, $\langle K \cup N \rangle = G$.

(8) G, H groups. A homomorphism from $G \to H$ is a map $\phi : G \to H$ s.t. $\phi(xy) = \phi(x)\phi(y)$ $\forall x, y \in G$.

- Trivial hom, identity hom, inclusion hom
- $\ker \phi$, $\text{im}\,\phi$ are subgroups.
- Isomorphism = bijective + homomorphism
- $\ker \phi = \{e\} \iff \phi$ injective

  In this case $G \cong \text{im}\,\phi$.

(9) S a set and G a group

- $\text{Sym}(S) = (\{\text{bijections of } S\}, \text{composition})$
- S_n

- Action of G on S is a map $G \times S \to S$, $(g,s) \mapsto g(s)$
  $e(s) = s$ and $(gh)(s) = g(h(s))$.
  This is equivalent to
  $\{$ homomorphisms $\phi : G \to \mathrm{Sym}(S) \}$
- Action is faithful if ϕ is injective
- Trivial, conjugation, left regular

- Cayley's theorem: G is isomorphic to a subgroup of $\mathrm{Sym}(G)$. If $|G| = n$, then $G \hookrightarrow S_n$.
- $\mathrm{orb}(s) = \{ g(s) : g \in G \} \subseteq S$
  Orbits form a partition of S.
- Action is transitive iff $\mathrm{orb}(s) = S$ $\forall s \in S$.
- $\mathrm{stab}(s) = \{ g \in G : g(s) = s \} \leq G$

- Orbit-Stabilizer: $|G| < \infty$, then
  $|G| = |\mathrm{stab}(s)| \cdot |\mathrm{orb}(s)|$
  and $|\mathrm{orb}(s)|$, $|\mathrm{stab}(s)|$ divide $|G|$.

(10) $(a\ b\ c \dots h)$, $b = \sigma(a)$, $\sigma(b) = c, \dots, \sigma(h) = a$.
- Every $\sigma \in S_n$ is a product of disjoint cycles.
- Cycle structure
- Generated by transpositions
- $A_n = \{ \sigma \in S_n : \sigma \text{ even} \}$

- $\sigma \in S_n$ is either a product of even or odd number of transpositions (but not both)
- $|S_n| = n!$, $|A_n| = \frac{n!}{2}$
- A_n generated by 3-cycles
- $D_n = \langle \sigma, \tau \rangle \leq S_n$, $\sigma^n = \tau^2 = e$ and $\sigma\tau = \sigma^{-1}\tau$
- $|D_n| = 2n$
- D_n acts on regular n-gon.

(11) $N \leq G$ normal $\Leftrightarrow gNg^{-1} = G$

$\Leftrightarrow ghg^{-1} \in G \ \forall g \in G, h \in N$

- Simple groups
- G/N forms a group
- $\exists$ a quotient homomorphism, $\phi : G \to G/N$
- $\psi : G \to H$, $\ker \psi$ is normal

First Isomorphism Theorem $\psi : G \to H$ a hom.

$\overline{\psi} : G/\ker\psi \to \operatorname{im}\psi$, $x \ker\psi \longmapsto \psi(x)$

is an isomorphism.

- If $|G| < \infty$, then $|G| = |\ker\psi| \cdot |\operatorname{im}\psi|$.
- A_n is not simple for $n \geq 5$
- $S_n/A_n \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$
- $\mathbb{Z}/n\mathbb{Z}$

(12) Sylow's 1^{st} theorem: Let $|G| < \infty$ and p^n the highest power of a prime p dividing G, then $\exists$ a subgroup $H \subseteq G$ of size p^n.