

Midterm Review

This is a sketch of Monday's lecture.

It's missing many of the definitions.

This is a REVIEW not a list of what's
on the exam!

and
slightly
messy

- Midterm is 80 minutes long in class.
- 6 - 8 questions

Midterm Review

① Define a group G \leftarrow identity, inverse unique

Ex $(\mathbb{Z}, +)$, (\mathbb{Q}^*, \cdot) , $(M_n(\mathbb{R}), +)$, $(\mathbb{Z}/n\mathbb{Z}, +)$,
 S_n , $GL_n(\mathbb{R})$, D_n .
 \leftarrow non-abelian

② Subgroup $H \subseteq G$: $e \in H$, $x \in H \Rightarrow x^{-1} \in H$
 $x, y \in H \Rightarrow xy \in H$

③ left cosets of H in G as xH for $x \in G$.

- $\{xH : x \in G\}$ forms a partition of G
- $yH = xH \iff y^{-1}x \in H$
- there's a bijection $H \rightarrow xH$

④ Lagrange's theorem : $|G| < \infty \Rightarrow |H| \mid |G|$.

⑤ Given $S \subseteq G$, $\langle S \rangle = \{ \text{finite products of elements in } S \cup \{e\} \cup S^{-1} \}$.

- Finitely generated G
- Cyclic G (\mathbb{Z} or $\mathbb{Z}/n\mathbb{Z}$).
- orders of elements.

⑥ Cyclic groups G and $|G| < \infty$:

- Every subgroup of cyclic group is cyclic
- $\forall m \mid |G|, \exists$ a subgroup of size m .
- $\text{ord}([a]) = \frac{n}{\text{gcd}(a, n)}$ for $[a] \in \mathbb{Z}/n\mathbb{Z}$.

⑦ G, H groups $\rightarrow G \times H$ product

- orders of elements in $G \times H$.
- Finely generated abelian groups
- $G \cong N \times K$ if N, K normal, $K \cap N = \{e\}$, $\langle K \cup N \rangle = G$.

⑧ G, H groups. A homomorphism from $G \rightarrow H$ is a map $\phi: G \rightarrow H$ s.t. $\phi(xy) = \phi(x)\phi(y) \forall x, y \in G$.

- Trivial hom, identity hom, inclusion hom
 - Ker ϕ , im ϕ are subgroups.
 - Isomorphism = bijective + homomorphism
 - $\text{Ker } \phi = \{e\} \Leftrightarrow \phi$ injective
- In this case $G \cong \text{im } \phi$.

⑨ S a set and G a group

- $\text{Sym}(S) = (\{\text{bijections of } S\}, \text{composition})$
- S_n

- Action of G on S is a map $G \times S \rightarrow S$, $(g, s) \mapsto g(s)$
 $e(s) = s$ and $(gh)(s) = g(h(s))$.

This is equivalent to

{ homomorphisms $\rho: G \rightarrow \text{Sym}(S)$ }

- Action is faithful if ρ is injective
- Trivial, conjugation, left regular

- Cayley's theorem: G is isomorphic to a subgroup of $\text{Sym}(G)$. If $|G| = n$, then $G \hookrightarrow S_n$.

- orb(s) = $\{g(s) : g \in G\} \subseteq S$

Orbits form a partition of S .

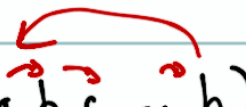
- Action is transitive iff $\text{orb}(s) = S \quad \forall s \in S$.

- stab(s) = $\{g \in G : g(s) = s\} \subseteq G$

- Orbit-Stabilizer : $|G| < \infty$, then

$$|G| = |\text{stab}(s)| \cdot |\text{orb}(s)|$$

and $|\text{orb}(s)|, |\text{stab}(s)|$ divide $|G|$.

⑩  $(a \ b \ c \ \dots \ h)$, $b = \sigma(a)$, $\sigma(b) = c, \dots, \sigma(h) = a$.

- Every $\sigma \in S_n$ is a product of disjoint cycles.
- Cycle structure
- Generated by transpositions
- $A_n = \{\sigma \in S_n : \sigma \text{ even}\}$

- $\sigma \in S_n$ is either a product of even or odd numbers of transpositions (but not both)
- $|S_n| = n!$, $|A_n| = \frac{n!}{2}$
- A_n generated by $\frac{1}{2}$ 3-cycles
- $D_n = \langle \sigma, \tau \rangle \leq S_n$, $\sigma^n = \tau^2 = e$ and $\sigma\tau = \sigma^{-1}\tau$
- $|D_n| = 2n$
- D_n acts on regular n -gon.

(11) $N \leq G$ normal $\Leftrightarrow gNg^{-1} = N$
 $\Leftrightarrow ghg^{-1} \in N \quad \forall g \in G, h \in N$

- Simple groups
- G/N forms a group
- \exists a quotient homomorphism, $\phi: G \rightarrow G/N$
- $\psi: G \rightarrow H$, $\text{Ker } \psi$ is normal

First Isomorphism Theorem $\psi: G \rightarrow H$ a hom.

$\bar{\psi}: G/\text{Ker } \psi \rightarrow \text{im } \psi$, $x \text{Ker } \psi \mapsto \psi(x)$
 is an isomorphism.

- If $|G| < \infty$, then $|G| = |\text{Ker } \psi| \cdot |\text{im } \psi|$.
- A_n is not simple for $n > 5$
- $S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$
- $\mathbb{Z}/n\mathbb{Z}$

(12) Sylow's 1st theorem: Let $|G| < \infty$ and p^n the highest power of a prime p dividing $|G|$, then \exists a subgroup $H \subseteq G$ of size p^n .