Solvable groups & classification of finite groups

Strategy : Take a group G and decompose it into "simple / easy to understand" pieces

Finite abelian groups : $G \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p_n^{a_n}\mathbb{Z}$

Defn Let G be a finite group. A composition series for G is a nested collection of subgroups,

$$\{e\} = G_0 \trianglelefteq G_1 \triangleleft \dots \trianglelefteq G_r = G \quad \text{where,}$$

- $G_i \neq G_{i+1}$
- G_i is normal in G_{i+1}
- G_{i+1}/G_i is simple $\forall i$

- Third isomorphism theorem implies that a composition series is maximal

Thm Any finite group G has a composition series

- If G is simple $\{e\} \trianglelefteq G$ is a composition series
- $\langle 0 \rangle \trianglelefteq \langle p^{n-1} \rangle \trianglelefteq \langle p^{n-2} \rangle \trianglelefteq \dots \triangleleft \langle p \rangle \trianglelefteq \mathbb{Z}/p^n\mathbb{Z}$ is a composition series

  $\langle p^{i+1} \rangle / \langle p^i \rangle \cong \mathbb{Z}/p\mathbb{Z}$ is simple

- By hwk, $\{e\} \trianglelefteq A_5 \triangleleft S_5$ is a composition series

  [More generally $\{e\} \triangleleft A_n \triangleleft S_n$ is a composition series for $n \geq 5$]

- $\{e\} \trianglelefteq A_3 \triangleleft S_3$ ~~and~~ ~~$\{e\} \triangleleft \dots$~~
- Refer to Hwk for S_4.

Ex $(\mathbb{Z}, +)$ has no composition series.

Assume ~~$\mathbb{Z}$~~ $0 = G_0 \trianglelefteq \cdots \trianglelefteq G_n = \mathbb{Z}$.

Then $G_1 = n\mathbb{Z}$, but $G_1/G_0 \cong n\mathbb{Z} \cong \mathbb{Z}$ is not simple.

Jordan-Hölder Theorem Let G be a finite group. Suppose

$$\{e\} \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G \quad \text{and}$$

$$\{e\} \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{m-1} \trianglelefteq H_m = G$$

are composition series.

Then $m = n$ and the quotient groups

$\{G_1/G_0, \ldots, G_n/G_{n-1}\}$, $\{H_1/H_0, \ldots, H_m/H_{m-1}\}$ are pairwise isomorphic up to reordering.

Defn The elements of the set $\{G_1/G_0, \ldots, G_n/G_{n-1}\}$ are called simple components.

Thus a first step to classify groups would be to understand the simple components of G. (Unfortunately S_3 and $\mathbb{Z}/6\mathbb{Z}$ have the same simple components, but $S_3 \not\cong \mathbb{Z}/6\mathbb{Z}$).

Defn A finite group is solvable if the simple components are abelian. (It suffices to just know G_i normal in G_{i+1} & G_{i+1}/G_i abelian)

- S_3 is solvable, S_4 is solvable
- S_n for $n \geq 5$ is not solvable [This will imply later on that deg ≥ 5 polynomial eq. have no formula for roots in general]

Thm A finite group of odd order is solvable
[Feit - Thompson]

Thm All finite simple groups have been classified.