① A_4 has no subgroup of order 6 [Thus the converse of Lagrange's theorem is false]

Pf Let $H \subseteq A_4$ be a subgroup of size 6. By Hwk 4, since $[A_4 : H] = \frac{12}{6} = 2$, H is normal in A_4.

Let $\pi : A_4 \to A_4/H$ be the natural homomorphism.
(observe that $A_4/H \cong \mathbb{Z}/2\mathbb{Z}$)
If $x \in A_4$ is a 3-cycle, $\text{ord}(x) = 3$.
We also have, $\text{ord}\,\pi(x) \mid 2$ ← size of A_4/H.
On the other hand, $\text{ord}\,\pi(x) \mid \text{ord}(x) \Rightarrow \text{ord}\,\pi(x) = 1$
Thus $\pi(x) = xH = H \Rightarrow x \in H$.
Thus $H \supseteq$ all 3-cycles
$\Rightarrow H \supseteq A_4$ (A_4 is generated by 3-cycles).
This is a contradiction.
Thus H doesn't exist ∎

② A_5 is simple

Sketch: Let $N \subseteq A_5$ be normal.

By Lagrange's theorem $|N|$ divides $|A_5| = 60$.

On the other hand N is a union of conjugacy classes.

Lemma The conjugacy classes of A_5 are

- $[(1)]$ size 1
- $[(12345)]$ size 12
- $[(21345)]$ size 12
- $[(12)(34)]$ size 15
- $[(123)]$ size 20

they are conjugate in S_5 but not $\underline{\underline{A_5}}$.

Now N must contain (1) and thus

$$|N| = 1 + a_0 12 + a_1 12 + a_2 15 + a_3 20$$

with $a_i \in \{0,1\}$.

But $|N|$ divides 60 iff $a_i = 0 \ \forall i$ or $a_i = 1 \ \forall i$.

Thus $N = \{e\}$ or $N = A_5$.

Thus A_5 is simple $\blacksquare$

③ Writing groups as product of simpler groups:

If $G \cong H \times K$, then to understand G, it suffices to understand the simpler groups H & K.

$\boxed{\text{Prop}}$ Let G be a group and $H, K \subseteq G$ normal. If $\langle H \cup K \rangle = G$ and $H \cap K = \{e\}$, then $H \times K \cong G$.

subgroup generated by H, K

PF First let's show $hk = kh$ $\forall k \in K, h \in H$.

Well, $hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} \in K$ as K is normal

and $hkh^{-1}k^{-1} = h(kh^{-1}k^{-1}) \in H$ as H is normal.

Since $H \cap K = \{e\} \Rightarrow hkh^{-1}k^{-1} = e$

$\Rightarrow hk = kh$.

Now define $\varphi: H \times K \longrightarrow G$ by $(h,k) \longmapsto hk$.

Then $\varphi((h_1h_2, k_1k_2)) = h_1h_2k_1k_2$

$= h_1k_1h_2k_2$ (above)

$= \varphi(h_1,k_1)\varphi(h_2,k_2)$.

Thus φ is a homomorphism.

- Injective: $\varphi(h,k) = e \Rightarrow hk = e \Rightarrow h = k^{-1} \Rightarrow h, k^{-1} \in H \cap K$

$\Rightarrow h = k^{-1} = e$

- Surjective: The commutativity relation implies

$G = \langle H \cup K \rangle = HK = \{h \cdot k : h \in H, k \in K\} = \varphi(H \times K)$ ∎

One can use this to classify all finite abelian groups:

$\boxed{\text{Ex}}$ Let G be abelian & $|G| = pq$ for distinct primes.
By Sylow 1, there subgroups H of order p and K of order q.
Since G is abelian, H, K are normal.
Since $H \cap K \leq K$ and $H \cap K \leq H \Rightarrow |H \cap K|$ divides $p, q \Rightarrow |H \cap K| = 1 \Rightarrow H \cap K = \{e\}$.

If you think about the proof of the Lemma, we obtain,
$\psi : H \times K \longrightarrow G$, $(h, k) \mapsto (hk)$ an injective homomorphism.
But $|H \times K| = pq$ & $|G| = pq$.
Thus ψ is an isomorphism.
But we know $H \cong \mathbb{Z}/p\mathbb{Z}$ and $K \cong \mathbb{Z}/q\mathbb{Z}$ and this proves the classification theorem in this case. ■

Proceeding in this fashion one obtains the classification of finite abelian groups