Normal subgroups & Quotients

Let G, H be groups and $f: G \to H$ a homomorphism.
From Hwk 2, $\ker f = \{ g \in G \mid f(g) = e_H \}$ and $\operatorname{im} f$ are subgroups of G, H respectively.
We also saw f is injective $\iff \ker f = \{e_G\}$.

~~Consider~~

We constructed $\mathbb{Z}/m\mathbb{Z}$ by showing the set of left cosets forms a group. We will now describe a general procedure of doing this.

Defn Let G be a group and $N \subseteq G$ a subgroup. N is called <u>normal</u> if $h \in N, g \in G \Rightarrow ghg^{-1} \in N$. We denote this by $N \trianglelefteq G$.

Ex • $\ker f$ is normal: $h \in \ker f, g \in G \Rightarrow \phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1}$

$$= \phi(g) \cdot e_H \phi(g)^{-1}$$
$$= e_H$$
$$\Rightarrow ghg^{-1} \in \ker f.$$

• N is normal $\iff$ N is a union of (some) conjugacy classes of G.

• $\{e\}, G$ are always normal in G

• Every subgroup of an <u>abelian</u> group is normal
($ghg^{-1} = gg^{-1}h = h \ \forall h \in N, g \in G$ abelian)

Non example • $\operatorname{im}(f)$ need <u>not</u> be normal

• $H = \langle (12) \rangle \subseteq S_3$ is a 2-element group that's not normal:
$(13)(12)(13)^{-1} = (13)(12)(13) = (1)(23) \notin H$

Defn A group G is said to be *simple* if it's only normal subgroups are $\{e\}$ and G.

- $(\mathbb{Z}/p\mathbb{Z}, +)$ for any prime p is simple.

$\boxed{\text{Big}}$ Theorem : Let $N \subseteq G$ be a normal subgroup. Then, the binary operation $\cdot : G/N \times G/N \longrightarrow G/N$,

$$(xN, yN) \longmapsto (xy)N$$

is well defined and gives G/N the structure of a group.

Pf Let $x_1, x_2, y_1, y_2 \in G$ such that $x_1N = x_2N$ and $y_1N = y_2N$.
We need to show $x_1y_1N = x_2y_2N$.

Well, we know $x_1^{-1}x_2$, $y_1^{-1}y_2 \in N$, thus

$$(x_1y_1)^{-1}(x_2y_2) = y_1^{-1}x_1^{-1}x_2y_2 = \underbrace{y_1^{-1}\underbrace{(x_1^{-1}x_2)}_{\in N}y_1}_{\in N \text{ by normality}} \cdot \underbrace{y_1^{-1}y_2}_{\in N}$$

Thus $(x_1y_1)^{-1}(x_2y_2) \in N \Rightarrow x_1y_1N = x_2y_2N$.
Thus the binary operation $\cdot$ is well defined. To check that $\cdot$ satisfies the group axioms, notice:

① Associativity: $\forall x, y, z \in G$,
$$(xN \cdot yN)\cdot(zN) = xyN \cdot zN = (xy)zN = x(yz)N = xN \cdot (yN \cdot zN).$$

② $\exists$ Identity: Notice that $xN \cdot eN = xeN = xN = exN = eN \cdot xN$.
Thus $eN = N$ is the identity.

③ $\exists$ Inverses: $xN \cdot x^{-1}N = x \cdot x^{-1}N = N = x^{-1}xN = x^{-1}N\,xN$.
$x^{-1}N = (xN)^{-1}$

Defn If $N \subseteq G$ is normal, G/N is called the quotient group.

Ex • $\underset{\substack{\| \\ N}}{m\mathbb{Z}} \subseteq \underset{\substack{\| \\ G}}{\mathbb{Z}}$ is normal, thus $G/N = \mathbb{Z}/m\mathbb{Z}$ is a group!

• $A_n \subseteq S_n$ is normal (If $\sigma \in A_n$, $\tau \in S_n \Rightarrow \tau\sigma\tau^{-1} \in A_n$ because $\operatorname{sgn}(\tau\sigma\tau^{-1}) = \operatorname{sgn}(\tau)\operatorname{sgn}(\tau^{-1})\operatorname{sgn}\sigma = 1$)

Thus $S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$.

Prop The natural map $f: G \to G/N$, $x \mapsto xN$ is a homomorphism. Moreover $\ker f = N$.

Pf f is a homomorphism because, $f(xy) = xyN = xN \cdot yN = f(x)f(y)$.

$x \in \ker f \iff xN = eN = N$

$\iff e^{-1} \cdot x = x \in N$ ∎

Given a homomorphism $f: G \to H$ we know that f maps $\ker f$ to e_H.
Moreover, f is actually constant on the cosets of $\ker f$:

Lemma $x \ker f = y \ker f \iff f(x) = f(y)$

Pf $x\ker f = y \ker f \iff x^{-1}y \in \ker f$

$\iff f(x^{-1}y) = e_H \iff f(x)^{-1}f(y) = e_H \iff f(y) = f(x)$ ∎

Thus we get a map of sets,

$$G/\ker f \longrightarrow H,$$
$$x \ker f \longmapsto f(x)$$

This is actually a homomorphism [Exercise]

[First Isomorphism Theorem] Let G, H be groups and $f: G \to H$ a homomorphism. Then the induced map,

$$\bar{f} : G/\ker f \longrightarrow \mathrm{Im}(f)$$
$$x \ker f \longmapsto f(x)$$

is an isomorphism of groups.

Pf $\bar{f}$ is well defined injective and a homomorphism from above. It's surjective by definition of $\mathrm{Im}(f)$

- Generally, given a homomorphism $f: G \to H$, we obtain an injective map $\bar{f}: G/_{\ker f} \to H$. It's image is the subgroup $\mathrm{im}(f) \subseteq H$.

<u>Cor</u> Let $f: G \to H$ be a homomorphism and $|G| < \infty$. Then $|G| = ~~|\ker f|\,|\mathrm{Im} f|~~ \; |\ker f| \cdot |\mathrm{Im} f|$

Pf By first isomorphism, $G/_{\ker f} \cong \mathrm{Im} f$.
But $|\mathrm{Im} f| = |G/_{\ker f}| = \dfrac{|G|}{|\ker f|} \Rightarrow |G| = |\mathrm{Im} f| \cdot |\ker f|$ ∎

Ex Let $n \mid m$. Then consider $f: \mathbb{Z} \longrightarrow \mathbb{Z}/_{m\mathbb{Z}}$, $1 \longmapsto [k]$ and $m = nk$.
Notice that $\ker f = n\mathbb{Z}$.
Thus we get a homomorphism, $\mathbb{Z}/_{n\mathbb{Z}} \longrightarrow \mathbb{Z}/_{m\mathbb{Z}}$ and it's not trivial.

- If $N \subseteq \ker f$ is a normal subgroup of G we get a homomorphism $f: G/_N \longrightarrow H$.
Using this you can find nontrivial homomorphisms $\mathbb{Z}/_{n\mathbb{Z}} \longrightarrow \mathbb{Z}/_{m\mathbb{Z}}$ $\forall\, n, m$ not coprime ∎

Second Isomorphism Theorem

Thm Let G be a group and $H \subseteq G$ a subgroup.
If $N \trianglelefteq G$ is normal, then $H \cap N$ is normal in N.
Moreover $S/_{S \cap N} \cong SN/_N \subseteq G/_N$.

Normality is stable under subgroups.

Third Isomorphism Theorem

Thm Let $N \subseteq G$ be a normal subgroup. There's an inclusion preserving bijection,

$$\Psi : \{\text{subgroups of } G \text{ containing } N\} \longleftrightarrow \{\text{subgroups of } G/N\}$$

$$H \longmapsto H/N$$

Moreover, if $H \subseteq G$ is a subgroup then
$H/_N \subseteq G/_N$ is normal $\Longleftrightarrow$ H is normal in G

We have an isomorphism, $G/N\,/\,_{H/N} \cong G/_H$.

Let's do this step by step.

① If $N \subseteq G$ is normal and $N \subseteq H$, then N is normal in H
Thus $H/_N$ is a group and moreover a subgroup of $G/_N$.

- $e \in H \Rightarrow eN \in H/_N$
- If $xN \in H/_N \Rightarrow x \in H \Rightarrow x^{-1} \in H \Rightarrow x^{-1}N \in H/_N$
- If $xN, yH \in H/_N \Rightarrow x, y \in H \Rightarrow xy \in H \Rightarrow xN \cdot yN = (xy)N \in H/_N$.

So we see that Ψ is defined

② Ψ has an inverse: Given $K \leq G/N$ a subgroup,

let $\Psi^{-1}(K)$ = union of elements in K
= union of left cosets in K
$$= \bigcup_{xN \in K} xN .$$

This is a subset of G. Since $eN \in K \Rightarrow$ it contains $eN = N$.
It's a subgroup. Let $H_K = \Psi^{-1}(K)$.

- $x, y \in H_K \Rightarrow xN, yN \in K \Rightarrow xyN \in K$ (as K is a subgroup)
$\Rightarrow xy \in H_K$
- Similarly $x \in H_K \Rightarrow x^{-1} \in H_K$ $\blacksquare$

③ Ψ is a bijection: Let ϕ be the inverse map above given by $K \leq G/N \longmapsto H_K$.

Notice that $(\phi \circ \Psi)(H) = \phi(H/N) = H = \mathrm{id}(H)$ and
$(\Psi \circ \phi)(K) = \Psi(H_K) = H_K/N = K = \mathrm{id}(K)$.

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