Defn $\sigma \in S_n$ is called *even*, if it can be expressed as a product of an even number of transpositions. Otherwise it's called *odd*.

$\operatorname{sgn}(\sigma) := \begin{cases} 1 & \text{if } \sigma \text{ even} \\ -1 & \text{if } \sigma \text{ odd} \end{cases}$ is called the *sign* of a permutation.

Ex
- Any transposition σ is odd
- Identity is even (no transpositions)
- $(12)(53)$ is even
- (13564) is even

Prop Let $\sigma, \tau \in S_n$. Then
- σ even, τ even $\Rightarrow$ $\sigma\tau$, $\tau\sigma$ are even
- σ odd, τ even $\Rightarrow$ $\sigma\tau$, $\tau\sigma$ are odd
- σ odd, τ odd $\Rightarrow$ $\sigma\tau$, $\tau\sigma$ are even

Thus, $\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$

In particular, if σ is even the $(a\,b)\sigma$ is odd
σ is odd then $(ab)\sigma$ is even.

Defn The alternating *group* of rank n is ~~$\{\sigma \in S_n \mid \sigma \text{ even}\}$~~

$$\mathrm{Alt}_n = A_n := \{\sigma \in S_n \mid \sigma \text{ is even}\}$$

$\mathrm{Alt}_n \leq S_n$ is a subgroup:
- ~~e is even~~ e is even
- $\sigma \in \mathrm{Alt}_n \Rightarrow \sigma \cdot \sigma^{-1} = e$ ~~...~~
  By proposition, since σ is even and e even $\Rightarrow$ σ^{-1} even.
- $\sigma, \tau \in \mathrm{Alt}_n \Rightarrow \sigma, \tau$ even $\Rightarrow$ $\sigma\tau$ even (by proposition).

Just as S_n is generated by transpositions we have a similar result for A_n:

Prop A_n is generated by 3-cycles

PF Every $\sigma \in A_n$ is a product of an even # of transpositions.
It suffices to show that a product of two transpositions is a generated by 3-cycles. $(ij)(kl) = (kil)(ijk)$ and $(ij)(ik) = (ikj)$ ∎

Prop $|A_n| = \frac{n!}{2}$

Pf Since $|S_n| = n!$, it suffices to show $[S_n : A_n] = 2$.

Let $\sigma, \tau \in S_n$. Then $\sigma A_n = \tau A_n \iff \sigma^{-1}\tau \in A_n$

$$\iff \operatorname{sgn}(\sigma^{-1}\tau) = 1$$
$$\iff \operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\tau)$$

Thus A_n has two left cosets, A_n and σA_n = set of odd permutations ∎

(σ: odd permutation)

Dihedral groups

We may view S_n as $\operatorname{Sym}(\{1, 2, \ldots, n\}) = \operatorname{Sym}\Big($  $\Big)$ ← regular n-gon.

Let $X \subseteq \mathbb{R}^2$ be a regular n-gon centred at $(0,0)$.

Although $r \in S_n$ is a bijection from $\{1, \ldots, n\}$ ↺, it doesn't necessarily preserve the geometry of the polygon.

A "geometric" symmetry of a regular n-gon would be an invertible linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(X) = X$

i.e. a matrix $A \in GL_2(\mathbb{R})$ s.t. $A(X) = X$.

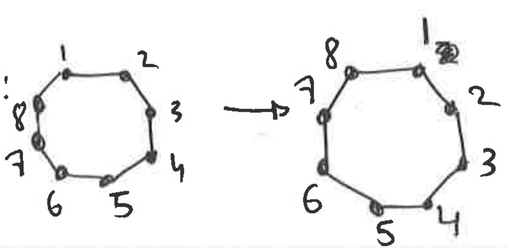
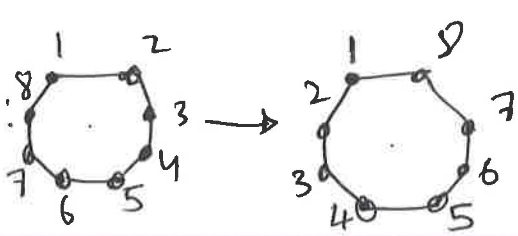
It turns out that if we restrict S_n to the linear symmetries, we get the Dihedral group.

Let $\sigma \in S_n$ correspond to rotating X by $\frac{2\pi}{n}$ clockwise around the origin.

Let $\tau \in S_n$ correspond to reflecting X along the line through the vertex 1 and the origin.

Explicitly σ is $1 \mapsto 2, 2 \mapsto 3, \ldots, n \mapsto 1$

τ is $1 \mapsto 1, 2 \mapsto n, 3 \mapsto n-1, \ldots$

Ex σ:  and τ: 

It follows that
$$\operatorname{ord}(\sigma) = n$$
$$\operatorname{ord}(\tau) = 2$$
$$\tau\sigma = \sigma^{-1}\tau \quad *$$

We write $D_n = \langle \sigma, \tau \rangle \subseteq S_n$.

Note that $D_n = \{e, \sigma, \dots, \sigma^{n-1}, \tau, \sigma\tau, \dots, \sigma^{n-1}\tau\}$.

$*$ To prove this, it's helpful to think of $\tau\sigma$ and $\sigma^{-1}\tau$ as acting on .

Then $(\tau\sigma)(1) = n$ $\qquad$ $(\sigma^{-1}\tau)(1) = n$

$(\tau\sigma)(2) = \tau(3) = n-1$ $\qquad$ $(\sigma^{-1}\tau)(2) = \sigma^{-1}(n) = n-1$

$\vdots$ $\qquad\qquad\qquad$ $\vdots$

As a consequence, D_n is non-abelian for $n \geq 3$.

If $\tau\sigma = \sigma\tau$, then by $*$,
$$\sigma\tau = \sigma^{-1}\tau \Rightarrow \sigma = \sigma^{-1} \Rightarrow \sigma^2 = e \Rightarrow n = 2.$$

Observe that $D_n = \langle \sigma\tau, \tau \rangle$ where $\operatorname{ord}(\sigma\tau) = \operatorname{ord}(\tau) = 2$. and $|D_n| = 2n$. Thus the order of the group, in general, has little to do with the order of the generators.

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