

The symmetric groups  $S_n$

Given  $\sigma \in S_n$  and  $a \in \{1, \dots, n\}$  we can consider ~~the~~  
 $a, \sigma(a), \sigma^2(a), \dots$ . Since  $\sigma$  has finite order  $\exists m > 0$  such that  
 $\sigma^m(a) = a$ . This gives us a cycle  $(a \ \sigma(a) \ \sigma^2(a) \ \dots \ \sigma^{m-1}(a))$

In this way every  $\sigma \in S_n$  can be written as a "product"  
of disjoint cycles:  $\sigma = (a \ \sigma(a) \ \dots \ \sigma^{m_1-1}(a)) (b \ \sigma(b) \ \dots \ \sigma^{m_2-1}(b)) \dots (e \ \sigma(e) \ \dots \ \sigma^{m_r-1}(e))$

$\uparrow$  choose  $b$  next

This representation is unique to re ordering of cycles and shifting internally in a cycle.

Ex.  $\sigma = (123)(456) \in S_6$  is the map  $\sigma:$   
 $1 \rightarrow 2$   
 $2 \rightarrow 3$   
 $3 \rightarrow 1$   
 $4 \rightarrow 5$   
 $5 \rightarrow 6$   
 $6 \rightarrow 4$

•  $\sigma = (12)(53)(4)(6) \in S_6$  is the map  $\sigma:$   
 $1 \rightarrow 2$   
 $2 \rightarrow 1$   
 $3 \rightarrow 5$   
 $4 \rightarrow 4$   
 $5 \rightarrow 3$   
 $6 \rightarrow 6$

• Try some by yourself!

Note ① We usually suppress fixed points of  $\sigma$  i.e.  
 $(12)(53)(4)(6)$  is usually written  $(12)(53)$  and it's understood that the rest is fixed.

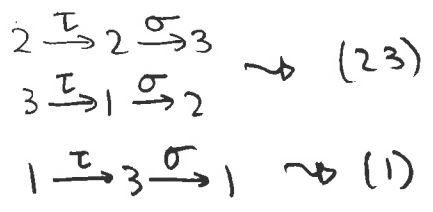
② Now composition of cycles is much clearer than composition of functions:

Ex .  $\sigma = (123)$  ,  $\tau = (13)(2) \in S_3$

$\sigma\tau = (123)(13)(2)$

$\curvearrowright = (23)(1)$

Work right to left



$\tau\sigma = (13)(123)$

$\curvearrowright = (12)(3)$

If even works when you ignore fixed points

Since  $\sigma\tau \neq \tau\sigma \Rightarrow S_3$  is not abelian!

Cor  $S_n$  is not abelian for  $n > 3$ .

PF  $\sigma = (123)(4) \dots (n) \in S_n$

$\tau = (13)(2)(4) \dots (n) \in S_n$

$\sigma\tau = (23)(1)$  and  $\tau\sigma = (12)(3)$  are distinct  $\square$

(This is the power of cycle notation).

•  $S_1$  is trivial and  $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ .

③ A cycle  $(a_1 \dots a_m)$  is said to have length  $m$

④ Given  $\sigma \in S_n$ , the lengths of each disjoint cycle gives a partition of  $n$ . This is called the cycle structure

So  $(12)(53)(4)(6)$  has cycle structure  $1, 1, 2, 2$ .

Prop Let  $\sigma \in S_n$  decompose as a disjoint product of cycles of lengths  $n_1, \dots, n_m$  (so  $\sum n_i = n$ ). Then  $\text{ord}(\sigma) = \text{lcm}(n_1, \dots, n_m)$ .

PF ~~Homework~~ / (check that  $\sigma^n = (\sigma_1 \dots \sigma_m)^n = \sigma_1^n \dots \sigma_m^n$ ).  
Exercise

Theorem Two permutations are conjugate in  $S_n \iff$  they have the same cycle structure

PF If  $\sigma = (a_1 \dots a_r) \dots (a_k \dots a_n)$  and  $\tau = (b_1 \dots b_r) \dots (b_k \dots b_n)$   
Define  $\alpha \in S_n$  by  $\alpha(a_i) = b_i \forall i$ . Notice that  $\alpha^{-1} \tau \alpha = \sigma$

Defn A transposition is a cycle of length 2.

- Every cycle is a product of transpositions,  $(a_1 \dots a_r) = (a_1 a_r) \dots (a_1 a_4)(a_1 a_3)(a_1 a_2)$
- $\uparrow$  representation is not unique: Take  $n = 4$  and  
 $(12)(34)(42)(23) = (124) = (14)(12)$

Thm Let  $\sigma \in S_n$  be expressed as a product of transpositions in two potentially different ways. If the first has  $m$  transpositions and the second has  $n$  transpositions, then  $2 \mid (m-n)$  i.e.  $m, n$  are both even or both odd.

Proof omitted.

Since  $(a_1 \dots a_m) = (a_1 a_m) \dots (a_1 a_3)(a_1 a_2)$ ,

- even length cycles are always a product of an odd number of transpositions
- odd length cycles are always a product of an even number of transpositions.