Liaison of curves in $\mathbb{P}^3$

0 - Rao modules

Notation: • K is alg. closed
• $H(3)$ denotes subschemes of $\mathbb{P}^3$ that are equidimensional of codim. 2, locally CM and generically a complete intersection

Defn $V_1, V_2 \in H(3)$ are geometrically linked by a c.i. X if
i) V_1, V_2 have no components in common
ii) $V_1 \cup V_2 = X$ as schemes

Thm Algebraic and geometric linkage of elements in $H(3)$ generate the same equivalence relation. The relation is called liaison

The goal of these 2 talks is to study this relation and find a numerical condition in the case of curves in $\mathbb{P}^3$ (elements of $H(3)$).

Defn Given a curve $Y \in H(3)$, the Rao-module is defined to be
$$M(Y) = \bigoplus_{v \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_Y(v)).$$

• $M(Y)$ is a finite S-module: Take a free resolution $0 \to \mathcal{E} \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^3}(-e_i) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Y \to 0$ (Auslander-Buchsbaum on $\mathcal{O}_{\mathbb{P}^3,x}$). Taking the l.e.s we obtain
$$\longrightarrow H^1\Big(\underbrace{\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^3}(v-e_i)}_{0}\Big) \longrightarrow H^1(\mathcal{I}_Y(v)) \to H^2(\mathcal{E}(v)) \longrightarrow H^2\Big(\underbrace{\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^3}(v-e_i)}_{0}\Big)$$
Thus $M(Y) \simeq \bigoplus_{v \in \mathbb{Z}} H^2(\mathcal{E}(v))$.

By vanishing of cohomology, $H^2(\mathcal{E}(v)) = 0$ for $v \gg 0$. By Serre-Duality $H^2(\mathcal{E}(v)) = H^1(\mathcal{E}^\vee(-v) \otimes \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$ for $v \ll 0$. Since each $H^2(\mathcal{E}(v))$ is a f.g. k-module we see that $M(Y)$ is a finite S-module.

② If $Y_1 \sim Y_2$ then $M(Y_1)$ is a shift in grading of $M(Y_2)$ or its dual.

Let $Y_1 \sim Y_2$ via two surfaces of deg. t_1, t_2 with $t = t_1 + t_2$. By last talk we have resolutions

- $0 \to \mathcal{E} \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^3}(-e_i) \to \mathcal{I}_{Y_1} \to 0$
- $0 \to \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^3}(e_i - t) \longrightarrow \mathcal{E}^\vee(-t) \oplus \mathcal{O}_{\mathbb{P}^3}(-t_2) \oplus \mathcal{O}_{\mathbb{P}^3}(-t_1) \to \mathcal{I}_{Y_2} \to 0$

Taking the l.e.s we have $M(Y_2) \simeq \bigoplus_{v \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{E}^\vee(-t+v))$.

By Serre-Duality,
$$\begin{aligned} M(Y_2)_d &\cong H^1(\mathbb{P}^3, \mathcal{E}^\vee(-t+d)) \\ &\simeq H^2(\mathbb{P}^3, \mathcal{E}(t-d-4))^\vee \\ &= (M(Y_1)_{t-d-4})^\vee \end{aligned}$$

Thus we have shown there is the following map of sets:

$$\Gamma : \{\text{liaison classes of curves}\} \longrightarrow \text{Finite length } S\text{-module upto isom.} \,/\, \text{shifts in degree, duals.}$$

Our goal is to show that Γ is an isomorphism. But first lets give some examples:

Ex 1: We have already seen that $M(Y) = 0 \Leftrightarrow Y$ is ACM $\Leftrightarrow$ linked to a complete intersection

Ex 2: · Rational quartic in $\mathbb{P}^3$: $Y_1 = V(x_1x_2 - x_0x_3,\ x_2^3 - x_1x_3^2,\ x_1^3 - x_0^2x_2,\ x_0x_2^2 - x_1^2x_3) \subseteq k[x_0, \dots, x_3]$

$(\psi: \mathbb{P}^1 \to \mathbb{P}^3$ by $[s:t] \mapsto [s^4 : s^3t : st^3 : t^4])$

· Two skew lines in $\mathbb{P}^3$: $Y_2 = V((x_0, x_1) \cap (x_2, x_3))$.

i) $Y_1 \overset{g}{\sim} Y_2$ via $X = V((x_0x_3 - x_1x_2,\ x_0x_2^2 - x_1^2x_3))$.

Pf On the quadric  type $(3,3)$ splits as $(1,3)$ and $(2,0)$

Intersecting quadric with cubic is a type $(3,3)$.

Alternatively you can show $(I_X : I_{Y_1}) = I_{Y_2}$. (3)

Let's compute $M(Y_2)$: Use $0 \to I_{Y_2} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{Y_2} \to 0$.

Then we have l.e.s. $H^0(\mathcal{O}_{\mathbb{P}^3}(v)) \xrightarrow{\psi} H^0(\mathcal{O}_{Y_2}(v)) \to H^1(I_{Y_2}(v)) \to 0$

$$\begin{array}{c} \cong \\ H^0(\mathcal{O}_{\mathbb{P}^1}(v))^2 \end{array}$$

union of two disjoint curves

For $v<0$, ψ is clearly surjective.

For $v=0$, ψ isn't surjective as its $k \to k^2$

For $v>0$, ψ is surjective as a function on two disjoint curves is indeed a function from $\mathbb{P}^3$ (more formally every $\omega \in H^0(\mathcal{O}_{Y_2}(v))$ is of the form $\omega_1 + \omega_2$ with $\omega_1 \in k[x_2,x_3]$, $\omega_2 \in k[x_0,x_1]$)

Thus $H^1_*(I_Y) = k = M(Y_2)$. (supported in degree 0)

- Our discussion w/ Serre duality states, $M(Y_2)_d = (M(Y_1)_{t-d-4})^\vee$.

Thus $k = M(Y_2)_0 = (M(Y_1)_{5-0-4})^\vee = M(Y_1)^\vee$ as the CI is a quadric and cubic.

In particular $M(Y_1) = k$ is supported in degree 1.

Of course we can see this by applying Riemann Roch to Y_1.

Indeed $0 \to I_{Y_1}(1) \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_{Y_2}(1) \to 0$ and $h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$.

But by R-R, $h^0(\mathcal{O}_{Y_2}(1)) = \deg \mathcal{O}_{Y_2}(1) + 1 - g(Y_2) = 5$ which is bigger than

Before we show Γ is an isomorphism, let's see if there's an simple class of curves whose Rao-module is "directly" related to the ideal of the curve.

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I - Ideally the intersection of three surfaces

Prop If Y is a curve in $\mathbb{P}^3$ that's ideally the intersection of three surfaces (i.e. $\exists$ a surjection $\bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-a_i) \twoheadrightarrow \mathcal{I}_Y$), then $I(Y)/(f_1,f_2,f_3)$ is upto a dual and shift the same as $M(Y)$. (here f_i cut out the three surfaces).

PF We have an s.e.s of bundles, $0 \to \mathcal{E} \to \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-a_i) \xrightarrow{[f_1,f_2,f_3]} \mathcal{I}_Y \to 0$ with $\mathcal{E}$ of rank 2. Taking the l.e.s we have

$$H^0\Big(\bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-a_i)\Big) \to H^0(\mathcal{I}_Y) \to H^1(\mathcal{E}) \to 0 \quad \text{and}$$

the image of this is (f_1,f_2,f_3)

$$0 \to H^1(\mathcal{I}_Y) \to H^2(\mathcal{E}) \to 0.$$

use this

By Serre-Duality (as $\mathcal{E}$ is rank 2) we obtain,

$$H^1(\mathcal{I}_Y) \simeq H^2(\mathcal{E}) \simeq H^1(\mathcal{E}^\vee \otimes \omega_{\mathbb{P}^3}) \simeq \big[I(Y)/(f_1,f_2,f_3)\big]^\vee(-4) \qquad \blacksquare$$

Thus, if we knew that every liaison class had a curve that is ideally the intersection of 3 surfaces we would be done (above proposition would imply Γ is surjective). Unfortunately this is not the case:

Ex Consider $M = S/(x_0,x_1,x_2,x_3^2)$. We will show that the liaison equivalence class corresponding to this (assuming Rao's theorem) has no curve that's ideally the intersection of three surfaces.

Koszul res: $0 \to S(-5) \to S(-3) \oplus S(-4)^3 \to S(-2)^3 \oplus S(-3)^3 \to S(-1)^3 \oplus S(-2) \to S \to M \to 0$

Sheafy version: $0 \to \mathcal{O}_{\mathbb{P}^3}(-5) \to \mathcal{O}_{\mathbb{P}^3}(-3) \oplus \mathcal{O}_{\mathbb{P}^3}(-4)^3 \to \mathcal{F} \to 0$

If Y is ideally an intersection of f_1,f_2,f_3 and $M(Y) \simeq M$ (upto twist), then $I(Y)/(f_1,f_2,f_3) \simeq M$ (upto grading).

We may assume $I(Y)$ has four minimal generators with f_1, f_2, f_3 three of them.

If $I(Y)$ has three or fewer gens, the previous prop. would imply $M(Y)=0$ ✗.

We now have s.e.s',

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(d) \arrow[r] & \bigoplus_{i=1}^{4} \mathcal{O}_{\mathbb{P}^3}(-a_i) \arrow[r] & I_Y \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[u, dashed] & \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^3}(-a_i) \arrow[r] \arrow[u, hook] & I_Y \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

(resolution of Rao module gives one for $\mathcal{O}_Y$)

(by assumption / previous prop)

Thus we have, $0 \to \mathcal{E} \to \mathcal{F}(d) \to \mathcal{O}_{\mathbb{P}^3}(-a_4) \to 0$.

Thus
$$c(\mathcal{F}(d+a_4)) = c(\mathcal{E}(a_4))\cdot c(\mathcal{O}_{\mathbb{P}^3})$$
$$= [1 + c_1(\text{—}) + c_2(\text{—})]\cdot 1$$
$$\Rightarrow c_3(\mathcal{F}(d+a_4)) = 0.$$

Using the first s.e.s for $\mathcal{F}$ in this example we obtain,

$$\underbrace{[1+c_1(\mathcal{O}_{\mathbb{P}^3}(r-5))]}_{= c(\mathcal{O}_{\mathbb{P}^3}(r-5))}\cdot\underbrace{[1+c_1(\mathcal{F}(r))+\dots+c_3(\mathcal{F}(r))]}_{= c(\mathcal{F}(r)).} = c\left(\mathcal{O}_{\mathbb{P}^3}(r-3)\oplus\mathcal{O}_{\mathbb{P}^3}(r-4)^3\right),$$

Solving for c_3 we obtain, $c_3(\mathcal{F}(r)) = r^3 - 10r^2 + 34r - 38$.
This has exactly one real root and its strictly between 2 and 3.
Thus $c_3(\mathcal{F}(d+a_4)) = 0$ is a contradiction i.e. Y cannot exist ∎

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II – Surjectivity of Γ

Thm] Let M be any non-zero finite length S-module. Then $\exists$ a non-singular curve $Y \subseteq \mathbb{P}^3$ such that $M \simeq M(Y)$ up to a shift in grading.

PF ① Let $0 \to L_4 \to \cdots \to L_0 \to M \to 0$ be a free resolution. Since M has finite length, the map of sheaves $\tilde{L}_1 \to \tilde{L}_0$ is surjective and we obtain an s.e.s $0 \longrightarrow \mathcal{E} \longrightarrow \tilde{L}_1 \longrightarrow \tilde{L}_0 \longrightarrow 0$ w/ $\mathcal{E}$ locally free of rank r.

Taking the l.e.s we obtain $\to H^0_*(\tilde{L}_1) \to H^0_*(\tilde{L}_0) \longrightarrow H^1_*(\mathcal{E}) \longrightarrow 0$

$\left(H^i_*(-) = \bigoplus_{i \in \mathbb{Z}} H^i(-)\right)$; thus $H^1_*(\mathcal{E}) = \text{cokernel}(L_1 \to L_0) = M$.

② Find a rank 3 bundle G with $H^1_*(G) = M$:

Note that $\mathcal{E}(p)$ is gen. by global sections for $p >> 0$. If $r > 3$, since $\dim \mathbb{P}^3 = 3$, a general section ω of $\mathcal{E}(p)$ is nowhere vanishing [equivalently $c_r(\mathcal{E}(p)) = 0$ on $\mathbb{P}^3$ for $r > 3$] (also Lemma 5.2 in [3264]).

Thus we have an s.e.s $0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-p) \xrightarrow{\omega} \mathcal{E} \longrightarrow \mathcal{E}_1 \longrightarrow 0$, with $\mathcal{E}_1$ locally free of rank $r-1$ and $H^1_*(\mathcal{E}_1) = H^1_*(\mathcal{E})$! Inductively we obtain G.

③ Finding a curve as the degeneracy locus of two sections of $G(-)$:

Thm: (Bertini) For a generic morphism $f: \mathcal{O}^2_{\mathbb{P}^3} \to G(a)$, ← twist till global generation, the degeneracy locus

$D_k(f) = \{x \in \mathbb{P}^3 : \text{rank}(f_x) \leq k\}$ is empty or has the expected codimension $(m-k)(n-k)$ and the singular locus of $D_k(f)$ is contained in $D_{k-1}(f)$.

(m: rk $\mathcal{O}^2_{\mathbb{P}^3}$; n: rk $G(a)$)

In our case, rk $G(a) = 3$ and thus $D_1(f)$ has expected codimension $(2-1)(3-1) = 2$ and $D_0(f) = \emptyset$!. Thus $\exists$ two sections s_1, s_2 of $G(a)$ such that

$s_1 \wedge s_2 \in H^0(\mathbb{P}^3, (\Lambda^2 G)(2a))$ has zero scheme Y.

Note: It's much easier to show that D_k(f) is Cohen-Macaulay (c.f. ACGH, Vol 1, II.4). One might then hope for an alternative way of finding a non-singular curve linked to a CM curve Y. For example, along the lines of (Proposition 4.1 in Peskine & Szpiro).

(4) Show $M(Y) \simeq H^1_*(\mathbb{P}^3, G(-c_1-2a)) \simeq M$: established in (1)

By defn of chern class, $(\Lambda^2 G)^\vee = G(-c_1)$: [There's a natural perfect pairing
$$\Lambda^2 G \otimes \Lambda^1 G \to \underset{\underset{\mathcal{O}_{\mathbb{P}^3}(c_1)}{\shortparallel}}{\Lambda^3 G} \implies \Lambda^2 G \simeq (\Lambda^1 G)^\vee \otimes \Lambda^3 G = G^\vee(-c_1).]$$
(on $\mathbb{P}^3$ all line bundles are given by their chern class)

So we have an exact sequence $G(-c_1-2a) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Y \to 0$ where $\alpha^\vee$ is the one mapping to $s_1 \wedge s_2$.

If we had an s.e.s of the form $0 \to \mathcal{L}_1 \oplus \mathcal{L}_2 \to G(-c_1-2a) \to I_Y \to 0$, taking the l.e.s we would get
$$H^1_*(\cancel{\mathcal{L}_1 \oplus \mathcal{L}_2})^{0} \to H^1_*(G(-c_1-2a)) \xrightarrow{\sim} H^1_*(I_Y) \to H^2_*(\cancel{\mathcal{L}_1 \oplus \mathcal{L}_2})^{0}.$$

This is exactly what we need!

Let's finish by showing the following sequence is an exact complex:
$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c_1-3a) \oplus \mathcal{O}_{\mathbb{P}^3}(-c_1-3a) \xrightarrow{[s_1, s_2]} G(-c_1-2a) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}.$$

Locally this is $0 \to R^2 \xrightarrow{\beta} R^3 \xrightarrow{\alpha} R$ with $\alpha = (\Lambda^2 \beta)^\vee$.

(1) This is a complex

(2) This is exact: We apply the Hilbert Burch theorem [Thm 3.2 in [E]] as we are in the setting $0 \longrightarrow R^2 \xrightarrow{M} R^3 \longrightarrow I_2(M) \longrightarrow 0$.

So we only need to show grade $I_2(M) \geq 2$. But this is true as Y is a non-singular curve $\Rightarrow$ Y is Cohen-Macaulay $\Rightarrow$ depth $(\underset{\underset{I(Y)}{\shortparallel}}{I_2(M)}, R) = 0$

This completes the proof! ■