

(1)

Surfaces in the positive characteristic: Foliations and inseparable morphisms

O - Intro

Notation: • $K = \bar{K}$ is algebraically closed of char $p > 0$ unless otherwise stated
 • Surface stands for a non-singular dimension 2, connected, projective over K

Thm] IF $c_2(X) < 0$ and $\text{char } K = 0 \Rightarrow K(X) = -\infty$ and X is ruled
 $x_{\text{top}}(X)$ " $c_2(T_X) = c_2(\mathcal{O}_X)$ $c_1(\mathcal{O}_X)$ can be large

[Noether's formula]: $12x_{\text{top}} = c_1(X)^2 + c_2(X)$ [apply Riemann-Roch]

Thm] [BMY (Bogomolov-Miyaoka-Yau)] IF $\text{char } K = 0$, then $c_1^2 \leq 3c_2$
 for any surface X of general type ($K(X) = 2$)
 Kodaira dimension defined above

- This is a ~~too~~ tight bound
- Fails in the positive characteristic (\exists general type surfaces w/ ~~too~~ $c_2(X) < 0$)

However we have the following conjecture (now theorem) of Raynaud:

[conj / Thm] IF $\text{char } K = p > 0$ and X is surface of general type with $c_2 < 0$
 then X is uniruled (admits a dominant birational map from a
 ruled surface) ($Y \times \mathbb{P}^1 \dashrightarrow X$ dominant and doesn't factor through projection onto Y)

One of the goals of this talk is to prove the following weaker statement:

Thm] Let X be a minimal surface of general type. Then either X is
 uniruled or $-\frac{c_2(X)}{c_1(X)^2} \leq \frac{p}{(p-1)^2}$

Note: Shepherd-Barron has given a proof of Raynaud's conjecture
 for $\text{char } K \geq 3$.

- In $\text{char } K = 0$ every ~~too~~ uniruled surface has $K(X) = -\infty$ (converse is a conjecture)

Ex) Consider the sextic $V_+(x_0^6 + x_1^6 + x_2^6 + x_3^6) \subseteq \mathbb{P}_{\mathbb{F}_5}^3$. This is of general type and unirational! (2)

- It's clearly of general type as $\omega_X \simeq \Theta_X(2)$ is very ample
- Here's the classical argument for unirationality:

Reparameterizing to $x_0^6 - x_1^6 = x_2^6 - x_3^6$. Do a change of coordinates to obtain $y_0 = x_0 + x_1$, $y_1 = x_0 - x_1$, $y_2 = x_2 + x_3$, $y_3 = x_2 - x_3$.

Thus our surface is of the form, $y_0 y_1 (y_0^4 + y_1^4) = y_2 y_3 (y_2^4 + y_3^4)$.

Going to the affine open $y_3 = 1$, let $y_2 = uv$, $y_1 = y_0 u$.

Then $K(X) \cong \overline{\mathbb{F}_5}(y_0, u, v) / y_0^6(1+u^4) - v(u^4v^4+1)$

Finally if we let $t = (y_0)^{\frac{1}{5}}$, $K' = \overline{\mathbb{F}_5}(t, u, v) \leftarrow K(X)$ is purely inseparable.

The last thing to do is to consider $s = u(t^6 - v)$ and note

$K' = \overline{\mathbb{F}_5}(t, s) = k(\mathbb{P}^2)$; thus we have a map $\mathbb{P}_{\mathbb{F}_5}^2 \dashrightarrow X$.

Note: This argument is from Shioda's paper "An example of unirational surfaces in char p". Refer to it a more general statement.

I - Foliations

~~Setup 2 A right P-extension of a field K of char = p is a subext~~

Let K be a field of char = p > 0. L is said to be a purely inseparable extension of height 1 if we have $K \subseteq L \subseteq K^{p^n}$.

RH: If K was the function field of a curve over \mathbb{k} , then $K \subseteq K^{p^n}$ is just the extension corresponding to the \mathbb{k} -linear Frobenius

Thm) Let X, Y be non-singular curves over \mathbb{k} and $\varphi: X \rightarrow Y$ a purely inseparable morphism of degree p^n . Then φ is an n-fold composition of \mathbb{k} -linear Frobenius.

Q) What if K was the function field of a surface? Can we characterize all extensions $K \subseteq L \subseteq K^{F^1}$? ③

A) Yes we can and to do this we will define foliations

Dfn) A purely inseparable morphism $F: X \rightarrow Y$ of char. p schemes is of height 1 if there exists a morphism $g: Y \rightarrow X$ s.t. $g \circ F = \text{Fr}$

Recall: $\begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow & F & \downarrow \pi \\ \text{spec } K & \xrightarrow{\text{Fr}} & \text{spec } K \end{array} = \text{Fr}$

$\text{Fr} = \text{K-Linear Frob.}$

Dfn) A Foliation on a variety X is a saturated subsheaf $E \subseteq T_{X/k}$ that is closed under taking p -th powers and the lie bracket.

- A subsheaf $E \subseteq T_X$ is said to be saturated if T_X/E is torsion free
- Note that $T_X = \text{Hom}(\Omega_X^1, \Theta_X)$ is a space of differential operators of order ≤ 1 . The lie-bracket $[\cdot, \cdot]$ is just $[D_1, D_2] = D_1 D_2 - D_2 D_1$, and it's a differential operator of order $\leq i+j-1$ if $D_1 \in \text{Diff}^i$ and $D_2 \in \text{Diff}^j$.

(Define D to be a differential operator of degree $\leq i$ if $[D, a]$ is a differential operator of degree $\leq i-1$ for all $a \in \Theta_X$ (acting by multiplication)).

- Note $[T_X, T_X] \subseteq T_X$.

Thus E being closed under $[\cdot, \cdot]$ is just $[E, E] \subseteq E$.

- Given $r \in T_X$, notice that $r^p \in T_X$ $(r^p(ab)) = \sum_{\alpha=0}^p \binom{p}{\alpha} r^\alpha(a)r^{p-\alpha}(b) = ar^p(b) + br^p(a)$

Thus being closed under p -th powers means $r^p \in E \Rightarrow r \in E$.

Here's the main theorem connecting the two concepts.

Thm] There's a 1-1 correspondence between the following two sets: (4)

{ finite morphisms $X \rightarrow Y$ of height one with Y normal } + (Flat)

{ foliations \mathcal{E} of X/Y + (\mathcal{E} is a subbundle of T_X) }

The correspondence is

$$\begin{array}{ccc} Y & & \text{Spec}_X \text{Ann}(\mathcal{F}) = X/\mathcal{F} \\ \downarrow & & \uparrow \\ \text{Ann}(\mathcal{O}_Y) = T_{X/Y} & & \mathcal{F} \end{array}$$

- $\text{Ann}(\mathcal{O}_Y) := \{ \mathcal{G} \in T_X : \mathcal{G}(\mathcal{O}_Y) = 0 \}$ is just " $T_{X/Y}$ ".
- $\text{Ann}(\mathcal{F}) = \{ a \in \mathcal{O}_X : \mathcal{G}(a) = 0 \forall \mathcal{G} \in \mathcal{F} \}$

\mathcal{F} is a subbundle of T_X

We clearly have $\mathcal{O}_X^P \subseteq \text{Ann}(\mathcal{F}) \subseteq \mathcal{O}_X$

Thus $X/\mathcal{F} := \text{Spec}_X \text{Ann}(\mathcal{F})$ sits in between X and $f(X)$ i.e.

$X \rightarrow \text{Spec}_X \text{Ann}(\mathcal{F})$ is a height one morphism.

PF [Prop 2.4 in Ekedahl's paper] \square

Ex] Let $\text{Spec } K[x_0, x_1] = A^2$ and $Y = \text{Spec } K[x_0^P, x_1]$.

Then $A^2 \rightarrow Y$ is a finite height one morphism corresponding to $(\frac{\partial}{\partial x}) \subseteq T_{A^2}$ (Lie-bracket is zero and it's P -closed!)

More generally, height 1 morphisms $A^2 \rightarrow Y = \text{Spec } S$ with Y normal correspond to ring extensions $K[x_0, x_1] \supseteq S \supseteq K[x_0^P, x_1]$. As long as S is strictly between, the corresponding foliation is a line bundle. Thus,

$S \hookrightarrow S = f(x_0, x_1) \frac{\partial}{\partial x_0} + g(x_0, x_1) \frac{\partial}{\partial x_1}$, satisfying $S^P = h(x_0, x_1) S$ for some $h \in K[x_0, x_1]$.

may differ from coproduct

Note that S is singular along $V(f, g)$ which is of codimension 2 (as it should be)

Ex] Let $f: Y \rightarrow \mathbb{P}^n$ be a finite height 1 morphism of degree p with Y non-singular. Then $p=2$, n is odd and f is isomorphic to the projection $V + (y^2 - x_1x_2 - x_3x_4 - \dots - x_nx_{n+1}) \xrightarrow{\pi^{n+1}} \mathbb{P}^n$

• For $n=1$ we indeed obtain the Frobenius.

Qn] If $f: X \rightarrow Y$ is a finite height 1 morphism (X non-singular and Y normal), then $\omega_X = f^*\omega_Y \otimes (\det T_{X/Y})^{p-1}$

$$\text{use } 0 \rightarrow T_{X/Y} \rightarrow T_X \rightarrow f^*T_Y \rightarrow F^*T_{X/Y} \rightarrow 0$$

Take determinants to obtain the result.

- This is true more generally (with the usual changes) to height n - foliations

II - Albanese morphisms

Rabin] Let X be a non-singular projective variety. \exists an abelian variety A and a map $\alpha: X \rightarrow A$ satisfying $X \xrightarrow{f} B \leftarrow$ for any other abelian variety (Fix a pt. $x \in X$ and assume $f(x)$ is the identity in B)
It's called the Albanese

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ & \downarrow \alpha & \uparrow \exists! \\ & & A \end{array}$$

- Over \mathbb{C} , choosing a base point $p \in X$, we have $\alpha: X \rightarrow A = H^0(X, \Omega_X)/H_1(X, \mathbb{Z})$
- Generally $X \rightarrow \text{Pic}^0(X)_{\text{red}}$.

Prop] For every abelian surface A over k \exists infinitely many surfaces of general type, whose Albanese morphisms are purely inseparable onto A .

Sketch: Choose a generic section $s \in H^0(A, \mathcal{L}^{\otimes p})$ where \mathcal{L} is very ample (+ other stuff). Consider the "p-th root cover" $X = \text{Spec}_A \bigoplus_{i=0}^{p-1} \mathcal{L}^{\otimes p-i} \xrightarrow{\pi} A$ given by

$$\begin{cases} \mathcal{L}^{\otimes p-i} \rightarrow \mathcal{L}^{-i} & \text{if } i \leq p \\ \mathcal{L}^{-i} \otimes s \rightarrow \mathcal{L}^{-i-p} & \text{if } i > p \end{cases}$$

↑ Note X is general type as $\omega_X = \pi^*(\omega_A \otimes \mathcal{L}^{p-1})$.

It's a pullback of ample along a finite map.

Then $\tilde{X} \rightarrow X \rightarrow A$ is going to be the Albanese morphism ■

We end by discussing two applications given by Ekedahl

Prop) Let Y be connected, non-singular, proper of $K(Y) \leq 0$. If the image of Y^* in $\text{Alb} Y$ is 2-dimensional, then $Y \rightarrow \text{im}(Y)$ is separable.

PF Proposition 4.3 in Ekedahl ■

Prop) Let X be a minimal surface of general type. Then either X is uniruled or $-\frac{c_2(X)}{c_1^2(X)} \leq \frac{P}{(P-1)^2}$.

PF) If $c_2(X) > 0$ we are done. So assume $c_2 < 0$ and $\text{Alb} X \neq 0$.

Fact: The image of $f^* \mathcal{J}^1_{\text{Alb} X/k} \rightarrow \mathcal{J}^1_{X/k}$ has a non-zero global section.

Let \mathcal{L} be the saturation of the section i.e. make $\mathcal{J}^1_{X/k}/\mathcal{L}$ torsion free.

Applying the Chern character to $0 \rightarrow \mathcal{L} \rightarrow \mathcal{J}^1 \rightarrow \mathcal{J}^1/k \rightarrow 0$ and

$$0 \rightarrow \mathcal{J}^1/k \rightarrow (\mathcal{J}^1/k)^{\vee} \rightarrow T \rightarrow 0,$$

we obtain that $c_2 = c_2(\mathcal{J}^1) \geq (\mathcal{L}, \mathcal{L}) - (\omega_X, \mathcal{L})$

The main point here is that \mathcal{J}^1/k is not locally free.

Since X is minimal of general type, $c_1(X) > 0$. Thus we obtain,

$$-\frac{c_2}{c_1^2} \leq -\frac{(\omega_X, \mathcal{L})}{c_1^2} + \frac{(\mathcal{L}, \mathcal{L})}{c_1^2}.$$

Apply the Hodge index theorem [Exc IV. 1.9] to obtain $\mathcal{L}^2 \omega_X^2 \leq (\omega_X, \mathcal{L})^2$.

$$\text{Since } \omega_X^2 = c_1(\omega_X)^2 \Rightarrow \frac{\mathcal{L}^2}{c_1^2} \leq \left[\frac{(\omega_X, \mathcal{L})}{c_1^2} \right]^2.$$

② let $M \subseteq T_{X/k}$ be the annihilator of \mathcal{L} . Then one can check that M is a foliation. Let $f: X \rightarrow Y$ be the induced height one morphism.

Since M need to be a bundle, let Z be the singular locus of Y .

Then we have $f^* \omega_Y = \omega_X \otimes M^{1-p}$ outside of Z .

He gives criterion for M to be a foliation.

③ i) If $(\omega_X, f^*\omega_Y) < 0 \Rightarrow$ any resolution of Y is birationally ruled
unique line bundle extension to X .

ii) Assume $(\omega_X, f^*\omega_Y) \geq 0$

$$\begin{aligned} \Rightarrow (\omega_X, \omega_X \otimes (\mathbb{L} \otimes \omega_X^{-1})^{\otimes p}) &= (\omega_X, \omega_X^p \otimes \mathbb{L}^{1+p}) \\ &= p(\omega_X, \omega_X) + (p-1)(\omega_X, \mathbb{L}) \\ &= p c_1^2 + (p-1)(\omega_X, \mathbb{L}) \geq 0 \end{aligned}$$

$$\Rightarrow \frac{(\omega_X, \mathbb{L})}{c_1^2} \leq \frac{p}{p-1}$$

Thus,

$$\begin{aligned} -\frac{c_2}{c_1^2} &\leq -\frac{(\omega_X, \mathbb{L})}{c_1^2} + \frac{\mathbb{L}^2}{c_1^2} \\ &\leq \left[\frac{(\omega_X, \mathbb{L})}{c_1^2} \right]^2 - \frac{(\omega_X, \mathbb{L})}{c_1^2} \\ &\leq \frac{p}{(p-1)^2} \end{aligned}$$

Maximizing $x - x^2$ for
 $0 \leq x \leq \frac{p}{p-1}$
because \mathbb{L} is effective

Eckhard ends by noting the following theorem is proved in his next paper:

Thm Let X be a minimal surface of general type over an alg. closed field of char p . Then $H^i(X, \omega_X^i) = 0$ for $i > 0$ except, possibly, when $i=1$, $p=2$, $X(\mathbb{O}_X)=1$ and there exists a dominant inseparable map of degree 2, $Y \rightarrow X$ where Y is rational or a $K3$ -surface. Furthermore, $\text{Im } K$ considered as a linear system is very ample for $m > 5$.

Finally enough, this is the only "Theorem" in the paper!

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