Correction to Thursday's Lecture (Week 1)

Theorem Let x, y be integers such that xy is even and $x + y$ is even. Then, x and y are both even.

In class I used this example to describe how to use "without loss of generality". The first proof, which I did in class, was actually a proof by contradiction (I erroneously said that it was by contraposition). The second proof is a proof by contraposition.

Proof 1

Step 1 Let x, y be any fixed integers. Let p be the statement "xy is even and $x + y$ is even" and q the statement "x and y are both even". Assume that p is true. We need to show that q is true; equivalently, assuming p is true we want to show that $p \to q$ is true. We will prove this statement, by contradiction.

Step 2 In particular, assume $\neg(p \to q)$ is true. The important part, where the confusion arises, is that p is still true (it was a premise). Thus, $\neg(p \to q) \equiv p \land \neg q$ and p are true. This implies that $\neg q$ must be true. In words, this is just the statement, that both x and y are not even. Now the proof proceeds as was done in class: x is odd or y is odd (or both). Without loss of generality, assume that x is odd. Then, since p is true, $x + y$ and xy are even. The fact that xy is even implies y is even. Thus, $x + y$ is odd i.e. $\neg p$ is also true. Thus $q \to p \land \neg p$, which is what a contradiction is. ■

You might be wondering why I said let x, y be fixed integers in **Step 1**. This is because, the statement we really want to prove is $\forall x \forall y(p(x, y) \to q(x, y))$ where $p(x, y)$ is the function "xy is even and $x + y$ is even", $q(x, y)$ is the function, "x and y are both even" and the domain is the integers. To prove this, we need to show that $p(x, y) \to q(x, y)$ is true for every pair of integers x, y.

Proof 2 We want to prove $\neg q \to \neg p$. Since $\neg q$ is true, one of x or y is odd (or both). Without loss of generality, assume that x is odd.

1 If y is even, then $x + y$ is odd. Thus, $\neg p$ is true.

2 If y is odd, then xy is odd. Thus, $\neg p$ is true.

Since y is either even or odd, this completes the proof of $\neg q \to \neg p$. ■