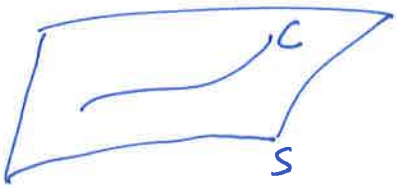
An (elementary?) introduction to Deformation Theory – Ritvik

I Motivation (Vague!)



$C \subseteq S$
curve surface

Since we can talk about $C' \cap C''$ when $C' \neq C''$, we would like to put an algebraic structure on $\mathcal{C}(S)$ i.e. a pairing $\mathcal{C}(S) \times \mathcal{C}(S) \longrightarrow \mathbb{Z}$.

So we need to be able to define $[C] \cap [C]$.

Use the idea of a tubular neighbourhood from topology. More precisely we can hope to "infinitesimally" deform C:

"infinitesimally moving"

# of fixed points of infinitesimal movement

This seems to come from choosing a section $s \in N_{C/S}$ and it's vanishing locus is the number of fixed points.

~~We will approach the question of infinitesimal deformations in~~

We will now discuss infinitesimal deformations and ~~numerically~~ relate it to the normal sheaf.

II Embedded Deformations

Defn Let X be a scheme over k, $Y \subseteq X$ a closed subscheme. A <u>deformation of Y over $D := \operatorname{Spec} k[\epsilon]/\epsilon^2$ in X</u> is a

- closed subscheme $Y' \subseteq X' := X \times D$
- Y' flat over D
- $Y' \times_D k = Y$

Affine version: $X \longleftrightarrow$ A k-algebra B

$Y \longleftrightarrow$ an ideal $I \subseteq B$

Embedded Deformations $\longleftrightarrow$ Ideals $I' \subseteq B' := B[\varepsilon]/\varepsilon^2$ such that

- B'/I' flat over $K[\varepsilon]/\varepsilon^2 = D$
- The image of I' in $B = B'/\varepsilon B'$ is I

$$\Updownarrow$$

$$(B'/I') \otimes_D K = B/I$$

By the local criterion of flatness, Flatness of B'/I' over D is equivalent to the exactness of: $0 \longrightarrow B/I \xrightarrow{\cdot\varepsilon} B'/I' \longrightarrow B/I \longrightarrow 0$

<u>Prop</u>: With notation as above, to give $I' \subseteq B'$ such that B'/I' is flat over D and the image of I' in B is I is equivalent to giving an element of $\mathrm{Hom}_B(I, B/I)$.

Pf) From the discussion before the proposition, we are searching for ideals I' such that, the diagram on the ~~right~~ left is exact at the bottom row.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r, "\varepsilon"] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "\varepsilon"] \arrow[d] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B/I \arrow[r, "\varepsilon"] \arrow[d] & B'/I' \arrow[r] \arrow[d] & B/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By ~~snake~~ 9-lemma, exactness at the bottom $\Rightarrow$ exactness at the top.

① Defining a map $\varphi \in \mathrm{Hom}_B(I, B/I)$ given the diagram on the ~~right~~ left:

$$\begin{tikzcd}
 & I' \arrow[r] \arrow[dl] \arrow[dr] & I \\
 & x' \arrow[r] & x \\
B \; y_1 \arrow[r] & x+y_1\varepsilon \arrow[r] & x+y_2\varepsilon \; B' \\
y_2 \arrow[urr] & & \\
[y_1] = [y_2] \; B/I & &
\end{tikzcd}$$

because $\rightarrow$ $(x+y_1\varepsilon) - (x+y_2\varepsilon) \longmapsto x - x = 0$

$I' \longrightarrow I$

② Conversely, given $\varphi \in \mathrm{Hom}_B(I, B/I)$ define

$I' = \{x + \varepsilon y \mid x \in I, y \in B$ and image of y in B/I is $\varphi(x)\}$

Notes:

- $\varepsilon = 0$ corresponds to $I' = I \oplus \varepsilon I$ which is the trivial deformation : $Y \times D \subseteq X \times D$.
- Can globalize to schemes: Deformations of Y over D in X

$$\Updownarrow$$

$$\mathrm{Hom}_X(\mathcal{I}, \mathcal{O}_Y) = H^0(X, \mathcal{H}om_X(\mathcal{I}, \mathcal{O}_Y))$$

- A B-module hom $I \longrightarrow B/I$ factors as $I/I^2 \longrightarrow B/I$. But I/I^2 is a B/I module! Thus $\mathrm{Hom}_B(I, B/I) = \mathrm{Hom}_{B/I}(I/I^2, B/I)$

or

$$\mathcal{H}om_X(\mathcal{I}, \mathcal{O}_Y) = \mathcal{H}om_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y).$$

Recall that $\mathcal{H}om_Y(\mathcal{I}, \mathcal{O}_Y) = \mathcal{N}_{Y/X}$ is the normal sheaf; as promised we have related these first order deformations to the normal sheaf.

Thm $\{$ Deformations of Y over D in $X \} \xleftrightarrow{1:1} H^0(Y, \mathcal{N}_{Y/X})$.

<u>Examples</u> (We will use $B[\varepsilon]$ to denote $B[\varepsilon]/\varepsilon^2 \ldots$)

① $B = K[x]$
$I = (x^n)$

$$\{\text{Deformations of } \operatorname{Spec} K[x]/(x^n) \text{ in } \mathbb{A}^1_K\} \longleftrightarrow \left\{ B[\varepsilon]/(x^n + a_1\varepsilon x^{n-1} + \ldots + a_n\varepsilon) : (a_1, \ldots, a_n) \in K^n \right\}$$

"n^{th} order point"

Pf By prop. we need a $K[x]$-module map $(x^n) \longrightarrow K[x]/(x^n)$. Completely determined by the image of 1. Say $1 \longmapsto a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$.
Looking at the definition of I' we see $I' = (x^n + \varepsilon(a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}))$ ∎

② $B = K[x,y]$
$I = (xy)$

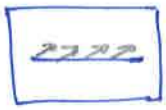
$$\{\text{Deformations of } + \text{ in } \mathbb{A}^2_K\} \longleftrightarrow \left\{ B[\varepsilon]/(xy + \varepsilon(a + xp(x) + yq(y))) : a \in K,\ p(x) \in K[x],\ q(y) \in K[y] \right\}$$

Pf Same as above: $\forall \alpha \in K[x,y]/(xy)$, $\alpha \equiv a + xp(x) + yp(y)$ ∎

3 $B = K[x,y]$ $\quad$ $\{\text{Deformations of } \mathbb{A}^1_k \text{ in } \mathbb{A}^2_k\} = \{K[y][\varepsilon]\}$
$I = (x)$ $\quad$ $= \mathbb{A}^1_k \times D$

PF $\quad (x) \to K[x,y]/(x) \rightsquigarrow K[x,y][\varepsilon]/(x+\varepsilon p(y)) = K[y][\varepsilon]$
$\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad K[y]$

So an embedded deformation of $\mathbb{A}^1_k$ is just moving the line along a normal direction



Note: This generalizes as we have essentially computed the tangent space to the Grassmanian.

III Interlude

There are many questions one can ask; here are some of them:

1) Can we deform Y without caring about its embedding
2) What about higher order deformations? Global deformations?
3) Can we deform other structures: Sheaves, complexes etc.

Unfortunately, due to the lack of time, we will only focus on 1) and briefly touch upon 2).

III.V A global deformation

Example Choose $\lambda \in k = \bar{k}$ and $\lambda \neq 0, 1$. We have a family of affine elliptic curves $\{X_t\}_t = \{y^2 = x(x-1)(x-(\lambda+t))\}_t$ over $K[t]$ (we are going to work around $t=0$, so ignore $t=-\lambda, -\lambda+1$)

1 $\{X_t\}_t$ is not trivial over any neighbourhood of 0 i.e.
$\{X_t\}_{t\in U} \not\cong \{y^2 = x(x-1)(x-\lambda)\} \times U$: The j-invariant varies in the Fibers of this family.

2 $\{X_t\}_t$ is trivial over $K[\varepsilon]$ ∇_0 (Exercise)

Very interesting.

IV Deformations of Rings (Schemes, algebras will be finite type / k)

Defn X be a k-scheme. A deformation of X over $k[\varepsilon]$ is a scheme X', flat over $k[\varepsilon]$ together with a closed immersion $i: X \hookrightarrow X'$ such that $i \times_A k: X \longrightarrow X' \times k$ is an isomorphism.

- Two deformations (X_1', i_1) and (X_2', i_2) are equivalent if there's an isomorphism $F: X_1' \longrightarrow X_2'$ ~~compatible~~ such that $i_2 = F \circ i_1$

$$\begin{tikzcd} X_1' \arrow[rr] \arrow[dr] & & X_2' \arrow[dl] \\ & k[\varepsilon] & \end{tikzcd}$$

Affine: • ~~A map of~~ B' flat over $k[\varepsilon]$, a map $i: B' \longrightarrow B$ such that $B' \otimes_D k \xrightarrow{\sim} B$

- (B', i') and (B'', i'') are equivalent if $\exists\, \alpha: B' \xrightarrow{\sim} B''$ such that $B \simeq B' \otimes_D k \xrightarrow{\sim} B'' \otimes_D k \simeq B$.

As you might expect two non-isomorphic embedded deformations can become isomorphic! To see this, one can first use Nakayama's lemma to prove the following:

Lemma Let B' and B'' be flat $k[\varepsilon]$ algebras (f.g.) and let $\alpha: B' \longrightarrow B''$ be a $k[\varepsilon]$ morphism such that the closed fibers $\alpha \otimes k: B' \otimes_D k \longrightarrow B'' \otimes_D k$ are isomorphic. Then α is an isomorphism!

Example (char k | n...) The deformation of $\operatorname{Spec} k[x]/(x^n) \hookrightarrow \mathbb{A}^1$ corresponding to $B[\varepsilon]/(x^n)$ and $B[\varepsilon]/(x-a\varepsilon)^n = B[\varepsilon]/(x^n - na\varepsilon x^{n-1})$ are isomorphic for any a.

Define $B[\varepsilon]/(x^n) \longrightarrow B[\varepsilon]/(x-a\varepsilon)^n$ by $\begin{array}{l} x \longmapsto x - a\varepsilon \\ \varepsilon \longmapsto \varepsilon \end{array}$

descends to $B/(x^n) \xrightarrow{\sim} B/(x-0)^n$ on the closed fiber.

We will now outline the classification of deformations. The main idea is to "embed" our deformation and study the isomorphism of embedded deformations.

Thm Let B be a k-algebra and $S = k[x_1, \dots, x_r]$ with a surjection $S \twoheadrightarrow B$ i.e. $B = S/I$. Then consider the conormal sequence of $k \to S \twoheadrightarrow B$

$$I/I^2 \to B \otimes \Omega_{S/k} \to \Omega_{B/k} \to 0.$$ Dualizing we have,

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_B(\Omega_{B/k}, B) \arrow[r] & \mathrm{Hom}_B(B \otimes \Omega_{S/k}, B) \arrow[r] & \mathrm{Hom}_B(I/I^2, B) \arrow[d] & \\ & & & T^1_{B/k} \arrow[r] & 0. \end{tikzcd}$$

Then, $\{$First-order deformations upto isom$\} = T^1_{B/k}$.

Pf) Since $\mathrm{Hom}_B(I/I^2, B)$ classified embedded deformations, a quotient of it should classify deformations.

Say (B', i') and (B'', i'') are isomorphic as deformations. Then their embedded deformations are isomorphic. By the classification, say $\varphi' : I/I^2 \to B'$ and $\varphi'' : I/I^2 \to B''$ were the corresponding morphisms.

① $\varphi - \varphi' : I/I^2 \to B' \simeq B''$ takes $g \mapsto \sum a_i \frac{\partial g}{\partial x_i}$. In other words an element of $\mathrm{Hom}_B(B \otimes \Omega_{S/k}, B)$

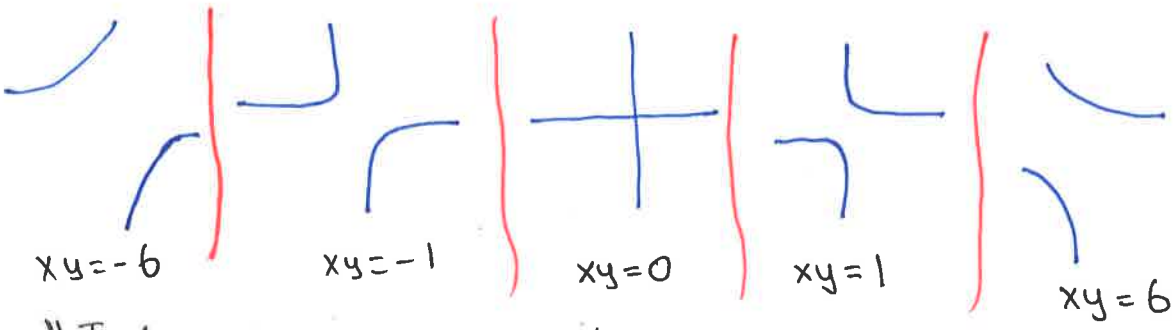
② Converse also holds ∎

Example $S = k[x,y]$
$B = k[x,y]/(xy)$. The space of deformations of B is just $\{S[\varepsilon]/(xy + a\varepsilon) : a \in k\}$.

What's the map $I/I^2 \to B \otimes \Omega_{S/k}$? Well if $f(x,y) = xy$, it's just

$$\begin{aligned} f(x,y)\,g(x,y) + I^2 &\to \frac{\partial}{\partial x}(fg + I) \oplus \frac{\partial}{\partial y}(fg + I) \\ &= (f_x g + f g_x + I) \oplus (f_y g + f g_y + I) \\ &= (yg + I) \oplus (xg + I) \end{aligned}$$

Thus we are only left with this after we quotient



"Truly a first order deformation"

Example $S = K[x,y]$

$B = K[x,y]/(y^2-x^3)$

Deformations $= \left\{ S[\varepsilon]/(y^2-x^3+\varepsilon(ax+b)) : a,b \in K \right\}$

2-dimensional space

So we also have linear forms in X.

The map $I/I^2 \to \Omega_{S/K} \otimes B$ is $fg + I^2 \to (3x^2 g + I) \otimes (2yg + I)$

where $f(x,y) = y^2 - x^3$

Cor If $X = \operatorname{Spec} B$ is a ~~affine~~ ~~scheme~~ non-singular affine scheme, then it has no non-trivial deformations.

Pf If X is nonsingular we have an exact sequence $0 \to I/I^2 \to B \otimes \Omega_{S/K} \to \Omega_{B/K} \to 0$ and $\Omega_{B/K}$ is projective! Thus applying $\operatorname{Hom}(-,B)$ is exact i.e. we have a surjection $\operatorname{Hom}_B(B \otimes \Omega_{S/K}, B) \to \operatorname{Hom}_B(I/I^2, B)$; thus $T^1_{B/K} = 0$ ∎

Cor Let X be a non-singular variety over K. Then the deformations of X over $K[\varepsilon]$ is in 1-1 correspondence with the elements of the group $H^1(X, T_X)$.

Pf Let X' be a deformation and $\mathcal{U} = \{U_i\}$ an affine covering of X. On each U_i, the induced deformation U_i' is trivial. Thus we have an isomorphism $\varphi_i : U_i \times_K D \xrightarrow{\sim} U_i'$. Thus on $U_{ij} = U_i \cap U_j$ we have an automorphism $\psi_{ij} = \varphi_j^{-1} \circ \varphi_i$. One checks that ψ_{ij} corresponds to an element $\theta_{ij} \in H^0(U_{ij}, T_X)$. The $\{\theta_{ij}\}$ satisfy the cocycle relation giving us an element of $\check{H}^1(\mathcal{U}, T_X)$. One checks that this is independent of choice of φ_i and thus deformations correspond to $\check{H}^1(\mathcal{U}, T_X) = H^1(X, T_X)$. Now check the converse ∎

V - Obstructions

To get a better understanding of our family of deformations, we would like to lift the deformation over $K[\varepsilon]$ to higher order ~~artin~~ artinian rings; for example $K[x]/(x^n)$. I'll leave it to you to define what this means.

Anyway, you start with a deformation of X over $K[\varepsilon]$ and you want to lift it to $K[x]/(x^n)$. In other words lifting an element of $H^1(X, T_X)$. By working on affine cover one ~~can~~ ~~see~~ ~~that~~

will obtain a cocycle of automorphisms and thus an element of $H^2(X, T_X)$.

<u>Cor</u> If X is a non-singular curve, deformations are unobstructed.

~~Note: One can make analogous statements about non-singular varieties~~

VI - Moduli Problems

Here's some setup for the next talk:

Vague definition: The Hilbert scheme H parameterizes closed subschemes of $\mathbb{P}^n_k$ with the same Hilbert polynomial. In other words

$$\begin{array}{c} \{X_t\}_{t\in T} \\ \downarrow \\ T \end{array} \text{ Flat families} \quad \longleftrightarrow \quad \{\text{Morphisms } T \longrightarrow H\}$$

More precisely, the functor $\mathrm{Hilb} : (\mathrm{Sch})^{op} \to \mathrm{Sets}$ that maps

$$S \longmapsto \{Y \subseteq \mathbb{P}^n_S : Y \text{ flat over } S\}$$

is representable by a projective scheme, which we denoted by H (above).

* What's the tangent space to H at a point $[Y]$?

~~Well, the tangent space~~

Well, a tangent vector is a morphism

$$\begin{tikzcd} \operatorname{Spec} k[\varepsilon] \arrow[r] \arrow[dr] & H \arrow[d] \\ & k \end{tikzcd}$$

with the closed point $(\varepsilon) \longmapsto [Y]$.

By the equivalence (universal property) above, this just a flat family

$$\begin{array}{c} \{Y_t\}_t \\ \downarrow \\ \operatorname{Spec} k[\varepsilon] \end{array}$$

Also known as an infinitesimal deformation of Y in $\mathbb{P}^n_S$!

with $Y_0 \simeq Y$