I Motivation (Vague!)

Since we can talk about $C' \cap C''$ when $C' \neq C''$, we would like to put an algebraic structure on $C(s)$, i.e., a pairing $C(s) \times U(s) \to \mathbb{Z}$.

So we need to be able to define $[C] \cap [C]$.

We use the idea of a tubular neighbourhood from topology. More precisely, we can hope to "infinitesimally" deform $C$:

"infinitesimally moving"

This seems to come from choosing a section $s \in N_C/s$ and its vanishing locus is the number of fixed points.

We will now discuss infinitesimal deformations and relate it to the normal sheaf.

II Embedded Deformations

Defn. Let $X$ be a scheme over $K$, $Y \subseteq X$ a closed subscheme. A deformation of $Y$ over $D := \text{Spec } K[\varepsilon]/\varepsilon^2$ in $X$ is a

- closed subscheme $Y' \subseteq X'$, $Y = Y \times X$
- $Y'$ flat over $D$
- $Y' \times_D K = Y$
Affine version: \( X \longleftrightarrow A \text{ k-algebra} B \)
\[ Y \longleftrightarrow \text{an ideal } I \subseteq B \]

Embedded deformations \( \longleftrightarrow \) Ideals \( I' \subseteq B' := B[\varepsilon]/\varepsilon^2 \) such that
- \( B'/I' \) flat over \( K[\varepsilon]/\varepsilon^2 = D \)
- The image of \( I' \) in \( B = B'/\varepsilon B' \) is \( I \)
\[ (B'/I') \otimes_D K = B/I \]

By the local criterion of flatness, flatness of \( B'/I' \) over \( D \) is equivalent to the exactness of:
\[ 0 \longrightarrow B/I \xrightarrow{\varepsilon} B'/I' \longrightarrow B/I \longrightarrow 0 \]

Prop: With notation as above, to give \( I' \subseteq B' \) such that \( B'/I' \) is flat over \( D \) and the image of \( I' \) in \( B \) is \( I \) is equivalent to giving an element of \( \text{Hom}_B(I, B/I) \).

\[ 0 \longrightarrow 0 \longrightarrow 0 \xrightarrow{\varepsilon} 0 \xrightarrow{\varepsilon} I' \longrightarrow I \longrightarrow 0 \]

\[ 0 \longrightarrow 0 \longrightarrow B \xrightarrow{\varepsilon} B' \longrightarrow B \longrightarrow 0 \]

\[ 0 \longrightarrow B/I \xrightarrow{\varepsilon} B'/I' \longrightarrow B/I \longrightarrow 0 \xrightarrow{\varepsilon} 0 \xrightarrow{\varepsilon} 0 \]

By 9.9-lm, exactness at the bottom \( \Rightarrow \) exactness at the top.

1. Defining a map \( \varepsilon \in \text{Hom}_B(I, B/I) \) gives the diagram on the left:

\[ \begin{array}{c}
0 \\
I' \\
I
\end{array} \xrightarrow{\varepsilon} \begin{array}{c}
X \\
X'
\end{array} \xrightarrow{\varepsilon} \begin{array}{c}
B \\
B'
\end{array} \xrightarrow{\varepsilon} \begin{array}{c}
y_1 \longrightarrow x+y_1\varepsilon \\
y_2 \longrightarrow x+y_2\varepsilon
\end{array} \xrightarrow{\varepsilon} B/I \\
[X] = [y_2] \bigg]\text{ because } (x+y_1\varepsilon) - (x+y_2\varepsilon) \longrightarrow x-x = 0 \\
I' \longrightarrow I
\]

2. Conversely, given \( \varepsilon \in \text{Hom}_B(I, B/I) \) define
\[ I' = \{ x+\varepsilon y \mid x \in I, y \in B \} \text{ and image of } y \text{ in } B/I \text{ is } \{0\} \]
Notes:

- $e = 0$ corresponds to $I' = I \oplus \mathfrak{e} I$, which is the trivial deformation. $Y \times D \to X \times D$.
- Can generalize to schemes: Deformations of $Y$ over $D$ in $X$:

$$\text{Hom}_X(I, \mathcal{O}_Y) = H^0(X, \mathcal{Hom}_X(I, \mathcal{O}_Y))$$

A $B$-module hom $I \to B/I$ Factors as $I/I^2 \to B/I$. But $I/I^2$ is a $B/I$ module! Thus $\text{Hom}_B(I, B/I) = \text{Hom}_B(I/I^2, B/I)$

or

$$\text{Hom}_X(I, \mathcal{O}_Y) = \text{Hom}_Y(I/I^2, \mathcal{O}_Y).$$

Recall that $\text{Hom}_Y(I, \mathcal{O}_Y) = N_{Y/X}$ is the normal sheaf; as promised we have related these first order deformations to the normal sheaf.

Turn $\{\text{Deformations of } Y \text{ over } D \text{ in } X \}$ into $\{H^0(Y, N_Y/X)\}$.

Examples (We will use $B[\mathfrak{e}]$ to denote $B[\mathfrak{e}]/\mathfrak{e}^2 \ldots$)

1. $B = \mathbb{K}[x]$  
   $I = (x^n)$  
   $\{\text{Deformations of } \text{Spec } \mathbb{K}[x]/(x^n) \text{ in } \mathbb{A}^1 \}$  
   \[\xymatrix{ & B[\mathfrak{e}]/(x^n + \mathfrak{e} a_1 x^{n-1} + \ldots + a_n) \ar[rr] & & B[\mathfrak{e}]/(x^n + \mathfrak{e} a_1 x^{n-1} + \ldots + a_n) \ar[rr] & & B[\mathfrak{e}]/(x^n + \mathfrak{e} a_1 x^{n-1} + \ldots + a_n) \ar[rr] & & \}
   \]
   \[\begin{array}{c} a_1, \ldots, a_n \in \mathbb{K}^n \end{array}\]

   Pf. By prop.
   we need a $\mathbb{K}[x]$-module map $(x^n) \to \mathbb{K}[x]/(x^n)$ completely determined by the image of $1$. Say $1 \mapsto a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$.

   Looking at the definition of $I'$ we see $I' = (x^n + \mathfrak{e} (a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}))$.

2. $B = \mathbb{K}[x,y]$  
   $I = (xy)$  
   $\{\text{Deformations of } \text{Spec } \mathbb{K}[x,y]/(xy) \text{ in } \mathbb{A}^2 \}$  
   \[\xymatrix{ & B[\mathfrak{e}]/(xy + \mathfrak{e} (a + xp(x) + y p(y))) \ar[rr] & & B[\mathfrak{e}]/(xy + \mathfrak{e} (a + xp(x) + y p(y))) \ar[rr] & & B[\mathfrak{e}]/(xy + \mathfrak{e} (a + xp(x) + y p(y))) \ar[rr] & & \}
   \]
   \[\begin{array}{c} a \in \mathbb{K}, p(x) \in \mathbb{K}[x], p(y) \in \mathbb{K}[y] \end{array}\]

   Pf. Same as above. $\forall \alpha \in \mathbb{K}[x,y]/(xy)$, $\alpha \equiv a + xp(x) + y p(y)$.
III. Interlude

There are many questions one can ask; here are some of them:
1) Can we deform $Y$ without caring about its embedding?
2) What about higher order deformations? Global deformations?
3) Can we deform other structures: schemes, complexes, etc.

Unfortunately, due to the lack of time, we will only focus on 1) and briefly touch upon 2).

III. V. A Global Deformation

Example: Choose $\lambda \in K = \bar{K}$ and $\lambda \neq 0, 1$. We have a family of affine elliptic curves

$$\{ x + \lambda \} = \{ y^2 = x(x-1)(x-(\lambda+1))^3 \} \text{ over } K[t]$$

We are going to work around $t = 0$, so ignore $t = \lambda, \lambda + 1$.

1) $x + \lambda$ is not trivial over any neighborhood of 0 i.e.

$$\{ x + \lambda \text{ is trivial over } K[t] \}$$

2) $x + \lambda$ is trivial over $K[t] \setminus \{0\}$ (Exercise)
IV  Deformations of Rings (Schemes, algebras will be finite type \( k \))

**Defn** \( X \) be a \( k \)-scheme. A deformation of \( X \) over \( k[t] \) is a scheme \( X' \), flat over \( k[t] \) together with a closed immersion \( i: X \to X' \) such that \( i^*_{\mathcal{O}_X} : X \to X' \) is an isomorphism.

- Two deformations \( (X', i_1) \) and \( (X_2', i_2) \) are equivalent if there's an isomorphism \( F: X_1' \to X_2' \) such that \( i_2 = F \circ i_1 \).

**Affine:** \( B' \) flat over \( k[t] \), a map \( i: B' \to B \) such that \( B' \otimes_k B \cong B \).

- \( (B', i') \) and \( (B'', i'') \) are equivalent if \( \exists \alpha: B' \to B'' \) such that \( B = B' \otimes_k B'' \cong B'' \otimes_k B' \).

As you might expect, two non-isomorphic embedded deformations can become isomorphic! To see this, one can first use Nakayama's lemma to prove the following:

**Lemma** Let \( B' \) and \( B'' \) be flat \( k[t] \) algebras (F.g.) and let \( \phi: B' \to B'' \) be a \( k[t] \) morphism such that the closed fibers \( \phi \otimes_k: B' \otimes_k \to B'' \) are isomorphic. Then \( \phi \) is an isomorphism!

**Example** The deformation of \( \text{Spec} \ k[x]/(x^n) \to \mathbb{A}^1 \) corresponding to \( B[t]/(x^n) \) and \( B[t]/(x-ae)^n = B[t]/(x-nae x^{n-1}) \) are isomorphic for any \( a \).

Define \( B[t]/(x^n) \to B[t]/(x-ae)^n \) by \( x \mapsto x-ae \) \( e \mapsto e \).

It descends to \( B/(x^n) \to B/(x-0)^n \) on the closed fibers.

We will now outline the classification of deformations. The main idea is to "erude" our deformation and study the isomorphism of embedded deformations.
Let $B$ be a $k$-algebra and $S = K[x_1, \ldots, x_r]$ with a surjection $S \rightarrow B$. i.e. $B = S/I$. Then consider the canonical sequence of $k \rightarrow S \rightarrow B$

$I/I^2 \rightarrow B \otimes_{S/I} S/I \rightarrow \mathcal{O}(B) \rightarrow 0$. Evaluating we have,

$0 \rightarrow \text{Hom}_B(\mathcal{O}(B), B) \rightarrow \text{Hom}_B(B \otimes_{S/I} S/I, B) \rightarrow \text{Hom}_B(I/I^2, B)\
\downarrow T_{B/I}^{1}\
\mathcal{O} \rightarrow 0$. 

Then, if first-order deformations up to isomorphism $\mathcal{D} = T_{B/I}^{1}$.

**Proof:** Since $\text{Hom}_B(I/I^2, B)$ classified embedded deformations, it should classify deformations.

Say $(B', i')$ and $(B'', i'')$ are isomorphic as deformations. Then their embedded deformations are isomorphic. By the classification, say $\xi' : I/I^2 \rightarrow B'$ and $\xi'' : I/I^2 \rightarrow B''$ were the corresponding morphisms.

1. $\xi' - \xi'' : I/I^2 \rightarrow B' \otimes B''$ takes $g \mapsto \sum q_i \frac{2g}{2x_i}$. In other words an element of $\text{Hom}_B(B \otimes_{S/I} S/I, B)$

2. converse also holds.

**Example**

$S = K[x, y]$

$B = K[x, y]/(xy)$. The space of deformations in just $\text{Hom}_B(B \otimes_{S/I}[xy], \mathcal{O} \otimes_{S/I} \mathcal{O})$.

What is the map $I/I^2 \rightarrow B \otimes_{S/I}[xy]$? Well if $f(x, y) = xy$, it is just

$f(x, y) q(x, y) + I^2 \rightarrow \frac{\partial}{\partial x} (f g + I) \otimes \frac{\partial}{\partial y} (fg + I)$

$= (f \partial y + f \partial x I) \otimes (fg + I)$

$= (xy + I) \otimes (xy + I)$

Thus we are only left with the affine Hilbert quotient.

$x y = 6$

"Truly a first order deformation"
Example \[ S = K[x, y] \]
\[ B = \frac{K[x, y]}{(y^2 - x^3)} \]

The map \( \mathcal{I}_Y^2 \to \mathcal{O}_Y \) is \( f_g + \mathcal{I}_Y^2 \to (2x^3 + y) \otimes (2x^3 + y) \)

where \( f(x, y) = y^2 - x^3 \)

2-dimensional space \( \mathcal{Y} \) so we also have linear forms in \( x \).

For \( X = \text{Spec} B \) in an non-singular affine scheme, then it has no non-trivial deformation.

If \( X \) is non-singular we have an exact sequence \( 0 \to \mathcal{I}_X^2 \to B \otimes \mathcal{O}_X \to \mathcal{O}_X \to 0 \)
and \( \mathcal{O}_X \) is projective. Thus applying \( \text{Hom}(\mathcal{O}_X, \mathcal{O}_X) \) is exact i.e. we have a surjection \( \text{Hom}_B(B \otimes \mathcal{O}_X, B) \to \text{Hom}_B(\mathcal{I}_X^2, B) \).

Thus \( T^1 \mathcal{O}_X = 0 \).

Let \( X \) be a non-singular variety over \( k \). Then the deformations of \( X \) over \( k[t] \) are in 1-1 correspondence with the elements of the group \( H^1(X, T_X) \).

If \( X' \) be a deformation and \( U' = \left\{ U_i \right\} \) an affine covering of \( X \). On each \( U_i \),
the induced deformation \( U_i \) is trivial. Thus we have an isomorphism \( \psi_i : U_i \otimes \mathcal{O}_X \to U_i \).
Thus for \( U_{ij} = U_i \cap U_j \) we have an automorphism \( \psi_{ij} = \frac{U_{ij} \otimes \mathcal{O}_X}{U_{ij} \otimes \mathcal{O}_{X}} \).
One checks that \( \psi_{ij} \) corresponds to an element \( \Theta_{12} \in H^0(U_{ij}, T_X) \).
The \( \Theta_{12} \) satisfy the cocycle relation giving us an element of \( H^1(U_{ij}, T_X) \).
One checks that this is independent of choice of \( \mathcal{O}_j \) and thus deformations correspond to \( H^1(U_1, T_X) = H^1(X, T_X) \).
Now check the converse.

V - Obstructions

To get a better understanding of our family of deformations, we would like to lift the deformation over \( k[t] \) to higher order Artinian rings i.e. for example \( k[x]/(x) \). I'll leave it to you to define what this means.

Anyway, you start with a deformation of \( X \) over \( k[t] \) and you want to lift it to \( k[x]/(x) \). In other words lifting an element of \( H^1(X, T_X) \). By working on an affine cover one...
will obtain a couple of automorphisms and thus an element of \( H^2(X, T_X) \).

**If** \( X \) is a non-singular curve, deformations are unobstructed.

**VI - Moduli Problems**

Here's some setup for the next talk:

**Vague definition:** The Hilbert scheme \( \operatorname{Hilb} \) parameterizes closed subschemes of \( \mathbb{P}^n_k \) with the same Hilbert polynomial. In other words

\[
\left\{ X \in \mathbb{P}^n_k \right\}_{\text{Flat families}} \quad \longleftrightarrow \quad \left\{ \text{Morphisms } T \to \operatorname{Hilb} \right\}
\]

More precisely, the function \( \operatorname{Hilb} : (\text{Sch})^{op} \to \text{Sets} \) that maps

\[
S \mapsto \left\{ Y \in \mathbb{P}^n_S : Y \text{ flat over } S \right\}
\]

is representable by a projective scheme, which we denote by \( \operatorname{H} \) (above).

**What's the tangent space to \( \operatorname{H} \) at a point \([Y]\)?**

Well, a tangent vector is a morphism \( \text{Spec } k[[e]] \to \operatorname{H} \) with the closed point \((e) \mapsto [Y]\).

By the equivalence (universal property) above, this just a flat family

\[
\left\{ X \in \mathbb{P}^n_S \right\}_{\text{Flat families}} \quad \text{Spec } k[[e]] \quad \text{with } Y_0 = Y
\]

Also known as an **infinitesimal deformation** \( \xi \) of \( Y \) in \( \mathbb{P}^n_S \).