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Some solvable quantum groups

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§1. Introduction.

The recent outburst of interest in quantum groups [**D**] has focused largely on semi-simple Lie groups and algebras. The purpose of the present paper is to show that solvable Lie groups and algebras can also lead to interesting quantum groups, whose analysis can be more elementary than that for the semi-simple ones, so that in particular it may provide some useful guidence in studying the semi-simple ones.

At present, the development of quantum groups is taking place primarily at three levels. One of these is associated with universal enveloping algebras of Lie algebras [**D**] , while the second is associated with algebraic groups [M] . These are both purely algebraic. The third level is associated with the algebras of continuous or smooth functions on a Lie group [**W1**,W2,W3,W4,VS] . This involves analysis as well as algebra, and so is technically more difficult than the first two. As a result, it benefits most from the simplification afforded by the solvability of the groups we work with. For this reason we will concentrate our attention on this third level.

Actually, the most elementary version of the examples which we study was, in effect, discovered a quarter of a century ago by Kac and Palyutkin [**KP**], long before the term quantum group came into use. They worked in a von Neumann algebra setting, and provide little motivation for the specific, fairly complicated, formulas which they set down. At the time when their paper first appeared I studied their example, and found it quite tantalizing, though it remained very mysterious to me. I was thus quite delighted a couple of years ago to notice that at least at the Lie algebra level their example came very nicely within the newly developed framework of Poisson Lie groups [**D**]. It was thus natural for me to seek to apply to their example the theory of deformation quantization of manifolds in the C^*-algebra setting which I had just been developing [**R1**,R2,R3]. This is accomplished in the present paper. But accordingly, rather than work in the setting of von Neumann algebras used by Kac and Palyutkin, we will work in the more refined setting of C^*-algebras and smooth sub-algebras. In the process we will obtain formulas which seem simpler and better motivated than those of Kac and Palyutkin. This has permitted us to also see how to obtain some fairly straight-forward higher-dimensional generalizations of their example (which also help to clarify some of the issues involved). Nevertheless, the fact that the construction works out still seems quite magical to me, as does the fact that Kac and Palyutkin managed to find their original example.

At the time when I spoke at this conference on the material presented here, I was not aware of other work being done on quantum versions of non-semi-simple groups. In the intervening months I have learned of several new developments in this direction. Quantum versions of the "$ax + b$" group play a role in the work of Podles and Woronowicz on the quantum Lorentz group [**PW**] (and these are also being studied by Van Daele). I have also received preprints from Van Daele [**VD**] and from Szymdzak and Zakrzewski [**SZ**] in which, independently, they treat quantum versions of the Heisenberg group. In fact, the quantum groups they study are essentially the Pontryagin duals of the ones studied in the present paper, and so are as closely related to the example of Kac and Palyutkin as those considered here. We will leave to another time discussion of the precise relationship between the quantum groups considered here and those of this other work, as well as various topics such as the representation theory and differential calculus for our examples.

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§2. The groups.

The solvable groups which we will consider will have the following form. Let X and Z be finite dimensional real vector spaces. We will denote their vector space duals by X^* and Z^*. Let π and ρ be two representations of the additive group Z^* on the vector space X^*. As a space we will let G be $X^* \times X^* \times Z^*$, but equipped with the product defined by

$$(p, q, r)(p', q', r') = (\pi(r')p + p', \rho(r')q + q', r + r').$$

Thus G is a two-step solvable Lie group. We have worked here with the duals of vector spaces in order to accord somewhat with the setting of [**R3**].

To put matters in a C^*-algebraic setting, we consider the corresponding Hopf algebra [**I,V,BS**] . Let $A = C_\infty(G)$, the algebra of continuous complex-valued functions on G which vanish at infinity, with pointwise multiplication. We define a comultiplication Δ on A by

$$(\Delta f)(p, q, r, p', q', r') = f(\pi(r')p + p', \rho(r')q + q', r + r'),$$

for $f \in A$. Note that, as is to be expected, Δ does not have values in $A \otimes A$, but rather in the multiplier algebra thereof, which consists of the bounded continuous functions on $G \times G$. Of course, Δ is coassociative, but not cocommutative. The coidentity element then consists of evaluating functions at the identity element of G, and the antipodal map consists of composing functions with the taking of inverses in G.

We want to obtain a quantum group from the above Lie group. While no precise definition of exactly what is meant by a quantum group has been given, it is generally understood to be mean a Hopf algebra obtained from a Lie group by deforming its Hopf algebra to a non-commutative one. The way by which we will accomplish such a deformation here is by performing a quantum deformation in the sense of [**R3**] of the algebra A , or more precisely, of a dense subalgebra of it, to obtain a non-commutative algebra, all in such a way that the coproduct defined above remains compatible with the new product (and similarly for the coidentity and antipode). In general, it is far from clear how to do this. But in the present situation, we will see that if we proceed quite naively by just following along the lines of [**R3**], it can be arranged that, surprisingly, all works out.

As discussed in [**R1**,R2], an effective way to give the direction for a deformation quantization of a manifold is by specifying a Poisson structure. The existing results on quantum groups [**D**] make clear that in the case where the manifold is a Lie group, if one is to hope that the deformed structure will be compatible with the group structure, then one should require that the Poisson structure on the group make the group into a Poisson group, as defined in [**D**] . In particular, there will then correspond the dual Poisson group [**D**] . Rather than working out here the Poisson structure, which we do not explicitly need, we will consider directly the Poisson dual group, and leave to the reader to check that it does indeed give a Poisson group structure to G.

The space for the Poisson dual group, H, will be $X \times X \times Z$. To specify the group law, we will assume given a bilinear map, β, from X to Z. We will see later that we need to impose some conditions on β, but we defer discussing these conditions until they are needed. We take the group product on H to be given by

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z'+\beta(x,y')).$$

Thus H is a two-step nilpotent Heisenberg-type group. We choose a Lebesgue measure on H, which will be a Haar measure for H.

To make explicit the process of deformation, we should everywhere multiply β by a Planck's constant, exactly as done in [**R3**]. But since the making explicit of the deformation process is not essential for our present purposes, we will, for ease of notation, suppress the Planck's constant in all that follows, and leave it to the reader to imagine it back in if desired.

§3. The deformation.

Viewing G just as the dual vector space to the vector space H, we can choose a Plancherel measure on G for the given choice of Lebesgue measure on H. We let $\mathcal{F}$ denote the ordinary

Fourier transform from $\mathcal{L}^2(H)$ to $\mathcal{L}^2(G)$ given by

$$(\mathcal{F}\phi)(p,q,r) = \int \bar{e}(\tilde{x}\cdot p + \tilde{y}\cdot q + \tilde{z}\cdot r)\phi(\tilde{x},\tilde{y},\tilde{z}),$$

where e is the function $e(t) = exp(2\pi i t)$, where dots denote the pairing between a vector space and its dual, and where, when convenient, we will put tildes over variables of integration instead of writing dx, etc. The Fourier transform will then be a unitary operator, whose inverse is given by the same formula but with $\bar{e}$ replaced by e.

The left regular representation, L, of H on $\mathcal{L}^2(H)$ is given by

$$(L_{xyz}\phi)(x',y',z') = \phi(x'-x, y'-y, z'-z-\beta(x,y'-y)).$$

It will be useful to define $\beta_x(r) \in X^*$ by $y\cdot\beta_x(r) = \beta(x,y)\cdot r$. Then we can use the Fourier transform to move the left regular representation of H to $\mathcal{L}^2(G)$. A simple calculation shows that one obtains

$$(\mathcal{F}L_{xyz}\mathcal{F}^{-1}\eta)(p,q,r) = \bar{e}(x\cdot p + y\cdot q + z\cdot r)\eta(p, q+\beta_x(r), r),$$

for $\eta \in \mathcal{L}^2(G)$.

We let $\mathcal{S}(G)$ denote the space of Schwartz functions on G. The inverse Fourier transform carries these to Schwartz functions on H, where they can be integrated against the left regular representation, in its form acting on $\mathcal{L}^2(G)$ found just above. A simple computation shows that the result, denoted by L_f for $f \in \mathcal{S}(G)$, is given by the iterated integral

$$(L_f\eta)(p,q,r) = \iint \bar{e}(\tilde{x}\cdot(p-\tilde{p}))f(\tilde{p},q,r)\eta(p, q+\beta_{\tilde{x}}(r), r).$$

The composition of operators then defines a product on $\mathcal{S}(G)$ which is easily seen to be defined by essentially the same formula. That is, if we denote this product by $*_\beta$ to show its dependence on β, then

$$(f *_\beta g)(p,q,r) = \iint \bar{e}(\tilde{x}\cdot(p-\tilde{p}))f(\tilde{p},q,r)g(p, q+\beta_{\tilde{x}}(r), r).$$

This is our deformed product on a dense subalgebra of A. With this product, $\mathcal{S}(G)$ is, of course, just isomorphic to $\mathcal{S}(H)$ with its usual convolution product from the group H. We place on $\mathcal{S}(G)$ the operator norm, so that its completion is isomorphic to the usual group C^*-algebra of H. When convenient, we will denote this C^*-algebra by B. It is clear that when $\beta \equiv 0$ we obtain exactly A.

In contrast to the situation in [**R3**], the corresponding adjoint operation on $\mathcal{S}(G)$ is not given just by complex conjugation, the reason being that in [**R3**] H was parametrized by the exponential map so that taking group inverses in H was the same as taking vector space inverses, which is not true here. For $\phi \in \mathcal{S}(H)$ the adjoint is well-known to be given by

$$\phi^*(x,y,z) = \bar{\phi}(-x,-y,-z+\beta(x,y)).$$

Taking Fourier transforms, the adjoint on $\mathcal{S}(G)$ is given by

$$f^*(p,q,r) = \iint \bar{e}(\tilde{x}\cdot(p-\tilde{p}))\bar{f}(\tilde{p},q+\beta_{\tilde{x}}(r),r).$$

This formula can be awkward to use, since it is only an iterated integral.

It is not reasonable to expect that Δ will be a homomorphism for $*_\beta$, but we will now see that, surprisingly, it will be if we require a relatively mild compatibility conditions relating π, ρ, and β. But first we must be more precise about how Δ is defined. If it is to be a homomorphism, then, given the identification of the product on $\mathcal{S}(G)$ with the convolution product on $\mathcal{S}(H)$, we see that Δ must be essentially the integrated form of a unitary representation of H on $\mathcal{L}^2(G\times G)$. Let us find this unitary representation. We will at first work purely formally, since some of the integrals which we will consider are not well-defined. Let ϕ_{xyz} denote the delta-function at $(x,y,z)\in H$. The Fourier transform of ϕ_{xyz} is, of course,

$$f_{xyz}(p,q,r) = \bar{e}(x\cdot p + y\cdot q + z\cdot r).$$

Then we would expect from what was said earlier, that

$$\Delta f_{xyz}(p,q,r,p',q',r',) = \bar{e}(x\cdot(\pi(r')p+p') + y\cdot(\rho(r')q+q') + z\cdot(r+r')).$$

For brevity, let us write Δf_{xyz} also for our expected $(L\otimes L)_{\Delta f_{xyz}}$. Then, working purely formally with Fourier transforms and the formulas given above, we find for $\xi \in \mathcal{L}^2(G\times G)$ that

$$(\Delta f_{xyz}\xi)(p,q,r,p',q',r') =$$
$$\bar{e}(x\cdot(\pi(r')p+p') + y\cdot(\rho(r')q+q') + z\cdot(r+r'))\times$$
$$\xi(p,q+\beta_{\pi(r')^t x}(r),r,p',q'+\beta_x(r'),r'),$$

where ^{t} denotes transpose. For each fixed (x,y,z) this latter expression clearly defines a unitary operator on $\mathcal{L}^2(G\times G)$. It is also clear that when $\beta \equiv 0$, we find exactly the expected pointwise multiplication by the function Δf_{xyz}. What is quite problematical, however, is whether, when $\beta \not\equiv 0$, we obtain a *homomorphism* from the group H to the group of unitary operators. To check this, we calculate (rigorously) that

$$(\Delta f_{(x,y,z)(x',y',z')}\xi)(p,q,r,p',q',r') =$$
$$\bar{e}((x+x')\cdot(\pi(r')p+p') + (y+y')\cdot(\rho(r')q+q') + (z+z'+\beta(x,y'))\cdot(r+r'))$$
$$\times\xi(p,q+\beta_{\pi(r')^t(x+x')}(r),r,p',q'+\beta_{(x+x')}(r'),r').$$

Whereas

$$(\Delta f_{xyz})(\Delta f_{x'y'z'}\xi)(p,q,r,p',q',r') =$$
$$\bar{e}((x+x')\cdot(\pi(r')p+p')+(y+y')\cdot(\rho(r')q+q')+$$
$$\beta(\pi(r')^t x,\rho(r')^t y')\cdot r+\beta(x,y')\cdot r'+(z+z')\cdot(r+r'))$$
$$\times\xi(p,q+\beta_{\pi(r')^t(x+x')}(r),r,p',q'+\beta_{(x+x')}(r'),r').$$

We see that these two expressions will agree as long as

$$\beta(\pi(r')^t x,\rho(r')^t y)=\beta(x,y)$$

for all r', x, y. This will be our final form of the condition (though it also has an attractive infinitesimal form). If Z is one-dimensional and β is non-degenerate, it is easy to see that we must have $det(\pi(r))det(\rho(r))=1$. It will be convenient for us to require this condition in general. Thus we make:

DEFINITION: We will say that β, π and ρ are *compatible* if

$$\beta(\pi(r)^t x,\rho(r)^t y)=\beta(x,y)$$

and

$$det(\pi(r))det(\rho(r))=1$$

for all $x, y\in X$ and $r\in Z^*$.

We will assume from now on that β, π and ρ are compatible. Thus we obtain a unitary representation of H on $\mathcal{L}^2(G\times G)$, which is clearly strongly continuous. We will denote this unitary representation by ΔL_{xyz}. We take as our rigorous definition of Δ on $\mathcal{S}(G)$ that it be the integrated form of this unitary representation (via Fourier transform to $\mathcal{S}(H)$), that is:

DEFINITION: For $f\in\mathcal{S}(G)$ we define Δf (or ΔL_f, if more convenient) to be the operator defined by

$$\Delta f=\int(\mathcal{F}^{-1}f)(\tilde{x},\tilde{y},\tilde{z})\Delta L_{\tilde{x}\tilde{y}\tilde{z}}.$$

We need to verify that Δ carries $\mathcal{S}(G)$ into the multiplier algebra of the norm closure, $B\otimes B$, of $\mathcal{S}(G)\otimes\mathcal{S}(G)$ as operators on $\mathcal{L}^2(G\times G)$. (Since the operator algebras here are of type I, we do not need to wonder which operator tensor product to use.) For this it suffices to show that the above unitary representation is itself into the multiplier algebra, that is, that ΔL_{xyz} is a multiplier for each $(x,y,z)\in H$. Now $\mathcal{S}(G\times G)$, acting in the evident way, is clearly dense in $B\otimes B$, but quick inspection shows that we can not expect it to be carried into itself by ΔL_{xyz}, because π and ρ may have exponential growth. We deal with this difficulty, here and in several latter places, in the following way. We let $\mathcal{S}_{3c}(G\times G)$ denote the space of Schwartz functions which are of compact support in their

r and r' variables (with a similar meaning for $\mathcal{S}_{3c}(G)$, etc.). We will see that this space *is* carried into itself by ΔL_{xyz}. This space is clearly dense in $B \otimes B$, and so we will then be able to conclude that Δ maps into multipliers. Specifically, a straight-forward calculation shows that for $F \in \mathcal{S}_{3c}(G \times G)$ and ξ in $\mathcal{L}^2(G \times G)$ we have

$$(\Delta L_{xyz})((L \times L)_F \xi) = (L \times L)_J(\xi)$$

where J is the function given by

$$J(p,q,r,p',q',r') = \bar{e}(x \cdot (\pi(r')p + p') + y \cdot (\rho(r')q + q') + z \cdot (r + r'))$$
$$\times F(p, q + \rho(-r')\beta_x(r), r, p', q' + \beta_x(r'), r').$$

It is easily seen that J is again in $\mathcal{S}_{3c}(G \times G)$. It follows that ΔL_{xyz} is a left multiplier. Because of the self-adjointness of the situation, ΔL_{xyz} is then actually a multiplier, as desired. We remark that the above formula provides a convenient way of seeing that our Δ actually satisfies the more stringent requirement made by Vallin [V] that for $a, b \in A$ one have $(\Delta a)(1 \otimes b) \in A \otimes A$, where 1 denotes here the identity element of the multiplier algebra of A. For if $f, g \in \mathcal{S}_{3c}(G)$, then from the above formula one finds that

$$(\Delta g)((L \otimes L)_{1 \otimes f}) = (L \otimes L)_K,$$

where

$$K(p,q,r,p',q',r') =$$
$$\int \grave{g}(\tilde{x}, \rho(r')q + q', r + r',)\bar{e}(\tilde{x} \cdot (\pi(r')p + p'))f(p', q' + \beta_{\tilde{x}}(r'), r'),$$

where $\grave{g}$ denotes Fourier transform in the first variable only. Clearly K is well-defined, and has compact support in r and r', and a bit more argument shows that it is, in fact, in $\mathcal{S}_{3c}(G \times G)$.

It is clear that Δ as originally defined at the level of functions, is coassociative. It is thus natural to expect that it will be also at the level of operators. However, our passage from functions to operators for Δ was purely formal, and it seems difficult to rigorously convert the coassociativity at the function level to that at the operator level. Thus we will instead prove the coassociativity at the operator level directly. We do this in a time-honored manner, used in particular by Kac and Palyutkin [KP], and beautifully systematized very recently by Baaj and Skandalis [BS]. That is, we find first an especially well-behaved unitary operator, U, on $\mathcal{L}^2(G \times G)$ such that

$$\Delta L_f = U(L_f \otimes I)U^*.$$

One advantage of this is that it enables us to obtain an explicit expression for $(\Delta \otimes I)\Delta L_f$, which might otherwise be hard to find.

We will find U in two steps, first by taking care of π and ρ, and then by taking care of β. For the first of these steps it is natural, in view of the form of Δ, to consider the unitary operator W defined by

$$(W\xi)(p,q,r,p',q',r') = \xi(\pi(r')p, \rho(r')q, r, p', q', r').$$

Then a simple computation shows that

$$(W^*(\Delta L_{xyz})W\xi)(p,q,r,p',q',r') =$$

$$\bar{e}(x\cdot(p+p') + y\cdot(q+q') + z\cdot(r+r'))\xi(p, q+\beta_x(r), r, p', q'+\beta_x(r'), r'),$$

(where we have used the fact that π, ρ, and β are compatible). That is, we see that W^* conjugates ΔL_f to exactly the comultiplication, say $\Delta^o L_f$, for the situation in which π and ρ are the identity representations. But Δ^o is, via Fourier transform, just the usual comultiplication for the group algebra of the group H. (In fact, instead of obtaining our quantum group by deforming the pointwise multiplication on $\mathcal{S}(G)$, we could obtain it, via the Fourier transform, by deforming the comultiplication on the group algebra of H, by introducing π and ρ , decorated with a deformation parameter.) Now it is well-known how to define the corresponding unitary operator in the group case. Namely, for a group, say K, and elements $u, v \in K$, one defines the operator V by

$$(V\eta)(u,v) = \eta(u, u^{-1}v),$$

for η a function on $K \times K$. So we must take the Fourier transform of this for the case in which K is H. But formulas will ultimately work out more simply if we actually only take Fourier transform in the second and third variables, that is, if we convert everything to the Hilbert space $\mathcal{L}^2(M)$, where $M = X \times X^* \times Z^*$. If we denote this partial Fourier transform still by $\mathcal{F}$, simple calculation shows that

$$(\mathcal{F}V\mathcal{F}^{-1}\xi)(x,q,r,x',q',r') =$$

$$\xi(x, q+q', r+r', x'-x, q'+\beta_x(r'), r').$$

We will denote this operator again by V. We must in the same way move the representation L to $\mathcal{L}^2(M)$. Denoting it there again by L, we find that

$$(L_{xyz}\eta)(u,q,r) = \bar{e}(y\cdot q + z\cdot r)\eta(u-x, q+\beta_x(r), r).$$

Then a simple calculation shows that

$$(V(L_{xyz}\otimes I)V^*\xi)(u,q,r,u',q',r') =$$

$$\bar{e}(y \cdot (q+q') + z \cdot (r+r'))\xi(u-x, q+\beta_x(r), r, u'-x, q'+\beta_x(r'), r').$$

(If one wishes, one can also easily check by direct calculation that the latter is indeed $\Delta^o L_x yz$, as moved by Fourier transform to $\mathcal{L}^2(M \times M)$.) From all the above it follows that if we set $U = WV$, we will have

$$\Delta L_{xyz} = U(L_{xyz} \otimes I)U^*.$$

Now a simple calculation shows that

$$(U\xi)(x, q, r, x', q', r') =$$

$$det(\pi(-r'))\xi(\pi(-r')^t x, \rho(r')q + q', r + r', x' - \pi(-r')^t x, q' + \rho(r')\beta_x(r'), r').$$

With the help of U we can find explicit expressions for $(\Delta \otimes I)(\Delta L_{xyz})$ and $(I \otimes \Delta)(\Delta L_{xyz})$. For this purpose we use the standard notation that for an operator T on $\mathcal{H}\otimes\mathcal{H}$, where $\mathcal{H}$ is some Hilbert space, $T_{12} = T \otimes I$, $T_{23} = I \otimes T$, and $T_{13} = (I \otimes S)(T \otimes I)(I \otimes S)$ on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, where S is the flip operator defined by $S(\eta \otimes \zeta) = \zeta \otimes \eta$. Then for any operator T in the multiplier algebra of $B \otimes B$ we have

$$(\Delta \otimes I)T = U_{12}T_{13}U_{12}^*,$$

$$(I \otimes \Delta)T = U_{23}T_{12}U_{23}^*.$$

Thus to verify coassociativity, it suffices to show that

$$U_{12}(\Delta L_{xyz})_{13}U_{12}^* = U_{23}(\Delta L_{xyz})_{12}U_{23}^*.$$

But a straight-forward, if a bit tedious, calculation (using the compatibility condition), shows that for $\xi \in \mathcal{L}^2(M \times M \times M)$ one has

$$(U_{12}(\Delta L_{xyz})_{13}U_{12}^*\xi)(u, q, r, u', q', r', u'', q'', r'') =$$

$$\bar{e}(y \cdot (\rho(r'+r'')q + \rho(r'')q' + q'') + z \cdot (r + r' + r''))\times$$

$$\xi(u - \pi(r'+r'')^t x, q + \rho(-r'-r'')\beta_x(r), r,$$

$$u' - \pi(r'')^t x, q' + \rho(-r'')\beta_x(r'), r', u'' - x, q'' + \beta_x(r''), r'') =$$

$$(U_{23}(\Delta L_{xyz})_{12}U_{23}^*\xi)(u, q, r, u', q', r', u'', q'', r'').$$

(Another appraoch is to verify the pentagonal relation of [**BS**], but the above approach has the mild advantage of giving an explicit formula for $(\Delta \otimes I)(\Delta L_{xyz})$.) This concludes the proof that Δ is a comultiplication.

§4. Counit, antipode, Haar measure.

We recall that a counit for a Hopf C^*-algebra A is a C^*-homomorphism, ϵ, of A into C such that, when extended to multiplier algebras, we have

$$(\epsilon \otimes \iota)\Delta = \iota = (\iota \otimes \epsilon)\Delta,$$

where ι denotes the identity automorphism of A. For our present situation we, of course, set

$$\epsilon(f) = f(0,0,0)$$

for $f \in \mathcal{S}(G)$. Under Fourier transform this corresponds to taking the Haar measure on H, which gives exactly the trivial representation of H, and so is continuous for the norm on $\mathcal{S}(G)$, since H is amenable so that the norm from the left-regular representation which we have, in effect, been using, is equal to the full group algebra norm. To verify the above equalities, it suffices to check that they give the same results on a dense set of elements of $B \otimes B$. As dense set we take

$\mathcal{S}_{3c}(G \times G)$ as defined in the previous section, and use the formula obtained there for how ΔL_{xyz} composes with $F \in \mathcal{S}_{3c}(G \times G)$. Taking its integrated form, we obtain, for $f \in \mathcal{S}(G)$,

$$((\Delta f)F)(p,q,r,p',q',r') =$$
$$\iint \bar{e}(\tilde{x} \cdot (\pi(r')p + p' - \tilde{p}))f(\tilde{p}, \rho(r')q + q', r + r')$$
$$F(p, q + \rho(-r')\beta_{\tilde{x}}(r), r, p', q' + \beta_{\tilde{x}}(r'), r').$$

Thus

$$(\epsilon \otimes \iota)((\Delta f)F)(p',q',r') =$$
$$\iint \bar{e}(\tilde{x} \cdot (p' - \tilde{p})f(\tilde{p}, q', r')F(0,0,0,p',q' + \beta_{\tilde{x}}(r'), r').$$

But $\epsilon \otimes \iota$ is a homomorphism, and so the above must also be equal to $((\epsilon \otimes \iota)(\Delta f))((\epsilon \otimes \iota)F)$, while

$$((\epsilon \otimes \iota)F)(p,q,r,p',q',r') = F(0,0,0,p',q',r').$$

Comparing the two expressions, and using the faithfulness of the representation involved, we see that we must have

$$(\epsilon \otimes \iota)(\Delta f) = f.$$

The other equality is verified in a similar way. Thus we have shown that ϵ is indeed a counit.

Just as the comultiplication and counit for our quantum group are just those from the group G, so too will this be true for the antipode and Haar integral. Now Kac ([K]) requires an antipode to be a conjugate linear automorphism, while more recent authors [V, BJ] take it to be a linear anti-automorphism. These can be related just by the *-operation. As suggested by the work of Kac and Palyutkin [KP], and with their notation, we set

$$f^+(p,q,r) = \bar{f}(-\pi(-r)p, -\rho(-r)q, -r),$$

for $f \in \mathcal{S}_{3c}(G)$. Then, using Vallin's notation [V], we set

$$j(f) = (f^*)^+.$$

It is easily checked that $^+$ is an (anti-linear) *-automorphism which is norm continuous. It follows that j is a continuous linear anti-automorphism. Routine calculation shows that $(f^*)^+ = (f^+)^*$, and also that, at the function level $\Delta(f^+) = \sigma((\Delta(f))^+$, where σ is the flip on $B \otimes B$. But we must verify the latter at the operator level, and it will be convenient to defer this until we have obtained some additional formulas.

The definition of the antipode in [K] and [V] is strongly related to the Haar integral. As indicated above, we take as Haar integral for our quantum group just the ordinary Haar measure on G, and denote it by m. It corresponds to evaluation at the identity element of H on $\mathcal{S}(H)$, and as such is well-known to define a faithful trace (unbounded) on B, whose GNS representation is just the left regular representation which we have been using. Note that for $\in \mathcal{S}(G)$ we have

$$m(f^+) = (m(f))^- = m(f^*),$$

so that

$$m(j(f)) = m(f).$$

Taking this and the trace property into account, the left-handed version of the defining relation of Kac (4.6 of [K]) is

$$(m \otimes m)((\Delta f)(g \otimes h)) = (m \otimes m)((\Delta h)(j(g) \otimes f)),$$

(where we suppress operator notation). This is the property which we will actually verify. But before doing that, let us relate it to the corresponding relation of Vallin (1.2.1v of [V]). Let us transform the above relation as follows, using the trace property and the relation between m and j. The right-hand side becomes

$$(m \otimes m)(j \otimes j)((\Delta h)(j(g) \otimes f)) = (m \otimes m)(g \otimes j(f))(j \otimes j)(\Delta h)) =$$

$$(m \otimes m)((g \otimes 1)(1 \otimes j(f))(j \otimes j)(\Delta h)),$$

while the left-hand side becomes

$$(m \otimes m)((g \otimes 1)(1 \otimes h)\Delta f).$$

Since this is true for all g , and since m is faithful, it follows that

$$(1 \otimes m)((1 \otimes h)\Delta f) = (1 \otimes m)((1 \otimes j(f))(j \otimes j)\Delta h) =$$

$$(1 \otimes m)(j \otimes j)(\Delta h(1 \otimes f)) = j(1 \otimes m)(\Delta h(1 \otimes f)).$$

This is exactly Vallin's condition, given the assumptions we have made.

We will now verify the condition above of Kac. For this purpose it is again easier to work on M. So we must transform our definitions to that setting. Taking appropriate Fourier transforms, we find that for $\phi \in \mathcal{S}_{3c}(M)$,

$$\phi^*(x, q, r) = \bar{\phi}(-x, q + \beta_x(r), r),$$

as is familiar from transformation group algebras. Also,

$$\phi^+(x, q, r) = det(\pi(r))\bar{\phi}(\pi(x)^t x, -\rho(-r)q, -r),$$

so that

$$(j(\phi))(x, q, r) = det(\pi(r))\phi(-\pi(r)^t x, -\rho(-r)(q + \beta_x(r)), -r).$$

Transforming the formula from near the beginning of this section for how Δ maps into multipliers, we find that for $\phi \in \mathcal{S}_{3c}(M)$ and $\Phi \in \mathcal{S}_{3c}(M \times M)$ we have

$$((\Delta\phi)\Phi)(x, q, r, x', q', r') =$$

$$\int \phi(\tilde{x}, \rho(r')q + q', r + r')\Phi(x - \pi(r')^t \tilde{x}, q + \rho(-r')\beta_{\tilde{x}}(r), r, x' - \tilde{x}, q' + \beta_{\tilde{x}}(r'), r').$$

Suppose now that $\phi, \psi, \theta \in \mathcal{S}_{3c}(M)$. Then a tedious but straight-forword computation shows that indeed

$$(m \otimes m)((\Delta\phi)(\theta \otimes \psi)) = (m \otimes m)((\Delta\psi)((j\theta) \otimes \phi)),$$

with both sides able to be brought to the form

$$\int det(\rho(-r'))\phi(x, q, r)\theta(-\pi(r')^t x, \rho(-r')(q - q' + \beta_x(r)), r - r')\psi(-x, q', r'),$$

where all variables are integrated.

Finally, as promised earlier, we must verify that

$$(j \otimes j)(\Delta\phi) = \sigma(\Delta(j\phi)),$$

or equivalently, and more comveniently for us,

$$(\Delta\phi)^+ = \sigma(\Delta(\phi^+)).$$

The main obstacle to doing this is finding a convenient expression for $(\Delta\phi)^+$, and for this we use the same device as used by Kac and Palyutkin [PK] , namely we express $^+$ as conjugation by an operator. Accordingly, define the operator T on $\mathcal{L}^2(M)$ by

$$(T\eta)(x,q,r) = det(\pi(r))\bar{\eta}(\pi(r)^t x, -\rho(-r)q, -r).$$

Then T is a conjugate linear "unitary" operator, with $T^2 = I$. It is easily verified that for $\phi \in \mathcal{S}_{3c}(M)$,

$$T\phi T = \phi^+.$$

Accordingly, the extension of $^+$ to the multiplier algebra of $B \otimes B$ will be given by conjugation by $T \otimes T$. Thus to verify the desired identity, we must show that for $\xi \in \mathcal{L}^2(M \times M)$,

$$(T \otimes T)(\Delta\phi)(T \otimes T)\xi = \sigma((\Delta(\phi^+))(\sigma\xi).$$

This is verified by a straight-forward calculation, once it is checked that $(\Delta\phi)\xi$ is calculated by the same formula as given above for $(\Delta\phi)\Phi$ but with Φ replaced by ξ.

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