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Induced Representations of C^*-Algebras

MARC A. RIEFFEL*

Department of Mathematics, University of California, Berkeley, California 94720

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Induced representations were first introduced in 1898 by Frobenius [21] in the course of his development of the theory of representations of finite groups. They provide a general method for constructing representations of a group from representations of its subgroups. We refer the reader to [35] for a statement of Frobenius' definition of induced representations in more modern terminology, remarking only that the definition is in terms of a certain space of vector-valued functions on the group, and that the motivation for this definition appears at first to be somewhat obscure.

The next several decades saw the development of the theory of representations of compact groups, but the first definitions of induced representations for compact groups which we have found did not appear until 1938 in a paper of Nakayama [45], and 1940 in the treatise of Weil [65]. The definition given by Weil is the natural generalization to compact groups of the definition of Frobenius, in which the vector-valued functions in the space of an induced representation are now required to be square-integrable with respect to Haar measure on the group.

The first paper to treat infinite dimensional unitary representations of a noncompact, non-Abelian group was Wigner's paper [67] of 1939 on the representations of the inhomogeneous Lorentz group. Already in this paper there appear representations which are induced from certain special subgroups of the Lorentz group. However, it was Mackey who in 1949 first formulated the general definition of induced representations for locally compact groups [34, 35], again by an appropriate modification of Frobenius' definition so that the space of vector-valued functions of an induced representation becomes a Hilbert space. (At

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about the same time Mautner gave an equivalent definition [40, 41] for the special case in which the subgroup is compact.) In subsequent papers [36–38] Mackey extensively developed the theory of induced representations for locally compact groups.

Parallel to the development of the theory of representations of finite groups came the theory of representations of algebras (hypercomplex numbers). In 1897, Molien [75] tied these two theories together by introducing the group algebra of a finite group. (I thank T. Hawkins for bringing this work of Molien to my attention. A good part of the history of these developments can be found in Hawkin's paper [73].) This approach was resumed in 1929 by Noether [46], in the setting of modules over rings. Very shortly thereafter Levitzki [32] gave the first generalization of Frobenius' definition of induced representations to the setting of finite dimensional semisimple algebras, although, as G. W. Mackey has kindly pointed out to me, Weyl at about the same time gave a treatment of induced representations for finite groups (p. 335 of [66]) which was phrased in terms of their group algebras, so that the definition for algebras is also implicit in Weyl's treatment. Levitzki's treatment, like Weyl's, was based on the use of idempotents, and so did not generalize readily to more general algebras. (Nakayama's approach for compact groups [45] was also in terms of idempotents.) The first general (and functorial) definition of induced representations for arbitrary algebras to appear in the literature was given in 1955 by D. G. Higman [28]. His definition was the following: let A be an algebra, let B be a subalgebra of A, and let V be a B-module. Then the A-module obtained by inducing V up to A is just $A \otimes_B V$ (where A is viewed as an A–B-bimodule). This construction is just a change-of-rings operation familiar from other areas of algebra, and Higman indicated that it has as a special case the induced representations of Frobenius, thus providing a quite satisfactory motivation for Frobenius' original definition of induced representations.

Almost as soon as the theory of infinite-dimensional unitary representations of locally compact groups began to be developed, Segal [57] and Gelfand and Naimark [23] began the process of showing, by means of the group algebra, $L^1(G)$, of a locally compact group G, that much of this theory was a special case of the theory of $*$-representations of involutory Banach algebras and, in particular, of C^*-algebras. References to many of the subsequent developments in this direction can be found in [13].

The purpose of this paper is to develop a theory of induced representations for C^*-algebras, and to show that a substantial part of Mackey's theory of induced representations for locally compact groups is a special case of this theory for C^*-algebras. In particular, we show that this is true of Mackey's imprimitivity theorem, which gives an answer to the question "Which representations of a group G are induced from representations of a given subgroup H?"

Our definition of induced representations of C^*-algebras is roughly as follows. Let A be a C^*-algebra, let B be a sub-C^*-algebra of A, and let V be a Hermitian B-module, that is, the Hilbert space of a nondegenerate $*$-representation of B. In analogy with Higman's construction (as well as with the constructions in our earlier papers [51, 53]), we begin by forming the algebraic tensor product $A \otimes_B V$. We then ask how we can equip this A-module with an inner-product in such a way that we obtain a nondegenerate $*$-representation of A. An analysis of this question shows that in general there are many different ways of doing this, in contrast to Mackey's theory for locally compact groups where there seems to be essentially only one natural choice of inner-product. This difference is explained by the fact that in the case of a locally compact group, G, and a closed subgroup, H, an additional piece of structure is present, namely the restriction map from functions on G to functions on H. We will see that, roughly speaking, this map is a conditional expectation, where for a C^*-algebra A and a subalgebra B, a conditional expectation is a positive projection, P, of A onto B having the property that $P(ab) = P(a)b$ for all $a \in A$ and $b \in B$. Once a conditional expectation has been chosen, there is a canonical choice of a preinner-product on $A \otimes_B V$, whose value on elementary tensors is given by

$$\langle a_1 \otimes v_1 , a \otimes v \rangle = \langle P(a^* a_1) v_1 , v \rangle_V .$$

This definition, which is very similar to that used in the Gelfand–Naimark–Segal construction of a representation from a positive linear functional, is very closely related to a construction of Stinespring [59]. The Hilbert space obtained in this way is a Hermitian A-module, which we call the Hermitian A-module obtained by inducing V up to A via P. The possibility that the induced representations of Mackey might be a special case of a construction of the type described above was suggested to us by Blattner's elegant note [4] in which he shows that an alternate definition of Mackey's induced representations can be given in terms of lifting positive type measures from subgroups.

Actually, if H is a closed but not open subgroup of the locally compact group G, then the group C^*-algebra, $C^*(H)$, of H is not a subalgebra of $C^*(G)$, but rather acts as an algebra of right centralizers [29] on $C^*(G)$. Also, the natural candidate for a conditional expectation from $C^*(G)$ to $C^*(H)$ is not continuous (or everywhere defined). For these reasons we must generalize the definition of a conditional expectation and overcome a few technical obstacles if we are to include the general case of Mackey's construction as a special case of our construction for C^*-algebras. But we will see that this can be done without much difficulty, and leads to what is perhaps a more natural approach to induced representations even in the original case of Mackey's theory for locally compact groups. We remark that an important tool in carrying out this more general construction is a generalization to noncommutative C^*-algebras of the "C^*-modules" introduced by Kaplansky [30] for commutative C^*-algebras. (Essentially the same generalization has been introduced simultaneously, for different reasons, by Paschke [76] and Takahashi [78]. A sketch of Takahashi's work is given by Hofmann on p. 364 of [74].)

Once the definition of induced representations of C^*-algebras has been given, we turn to deriving some of the basic properties of these induced representations. Some of these properties, such as the theorem on induction in stages, follow quite trivially from the definition. On the other hand, a large part of this paper is devoted to formulating and proving the imprimitivity theorem for induced representations of C^*-algebras. Our treatment of the imprimitivity theorem has been strongly influenced by what have come to be known as "the Morita theorems" [42, 1, 10]. Roughly speaking, if B is a C^*-algebra acting as right centralizers on a C^*-algebra A, and if P is a generalized conditional expectation from A to B, then we use P to construct a certain C^*-algebra, C, of linear operators of A into itself. This construction can be carried out for any positive map between C^*-algebras, and so may be useful in other situations. We then define a two-sided ideal, E, in C which is an analog of the algebra of finite rank operators on a Hilbert space. (This is the algebra alluded to in [54], where we distilled some of the ideas of the present paper to give an elementary proof of the uniqueness of the Heisenberg commutation relations.) We then find that, in analogy with the Morita theorems, the category of Hermitian E-modules is equivalent to the category of Hermitian B-modules (the prototype of this equivalence being the equivalence between the category of Hermitian modules over the algebra of compact operators on a Hilbert space with the category of Hermitian modules over the one-dimensional C^*-algebra). In fact, we describe a fairly general construction of pairs of C^*-algebras having equivalent categories of Hermitian modules. The algebra E serves as a "system of imprimitivity" for the representations of A induced from B via P, and in the case of induced representations of locally compact groups, E turns out to be essentially the transformation group algebra which was first associated with the theory of induced representations by Glimm [24]. By using this association, we show that Mackey's imprimitivity theorem can be derived from the imprimitivity theorem which we prove for C^*-algebras, and we obtain in this way an approach to Mackey's imprimitivity theorem which we feel is conceptually simpler and better motivated than the proofs presently available (see [34, 38, 33, 4, 19]). Further results concerning Morita equivalence are contained in a paper now in preparation,[1] a very brief sketch of which appeared in [77].

At the time of writing this paper we believe that we see how to prove Mackey's infinitesimal Frobenius reciprocity theorem [37] in the setting of C^*-algebras, at least in what would correspond to the unimodular case, but we have not yet worked through all the details involving direct integrals. We leave this matter and others, such as the relation between the theory presented here and the theory presented in our earlier paper [53], to a later time.

There are other parts of Mackey's theory of induced representations which are somewhat special to groups, and so which one would not expect to generalize readily to C^*-algebras. For example, Mackey's subgroup theorem and intertwining number theorem (Theorems 12.1 and 13.1 of [36]) depend on the special way in which $C^*(H)$ is mapped into the right centralizer algebra of $C^*(G)$. Even in the case of finite dimensional semisimple algebras these theorems can only be formulated by means of somewhat cumbersome hypotheses concerning how B is embedded in A (see [56]). Presumably one could also use such hypotheses in the context of C^*-algebras, but we did not feel that it was worthwhile carrying this out until other situations are known in which such hypotheses would be satisfied. Similar comments apply to Mackey's tensor product theorem (Theorem 12.2 of [36]) which requires being able to form inner tensor products of representations, which in turn seems to require the presence of a Hopf algebra structure on the algebras involved (see [50]). More elusive is Mackey's normal

[1] See [80].

subgroup analysis [38], since no appropriate definition of a "normal" subalgebra of an algebra seems to be known. On the other hand, the treatment of induced representations which we give here can be used to simplify some of the proofs of Mackey's normal subgroup analysis in its original form. We will treat this matter at a later time.

Induced representations have been generalized in a number of ways to contexts which mix groups and $*$-algebras [68, 62, 19, 31, 9, 43]. We consider it very likely that our techniques and results will apply to these cases also, but we have not verified this in full detail. However, J. M. G. Fell tells us that he has verified that this is the case for his Banach $*$-algebraic bundles [19], and, in fact, Fell has given in [20] a sketch of many of the main results of the present paper in a formulation particularly suited for use with Banach $*$-algebraic bundles. Furthermore, in the present paper we do indicate how to treat projective representations by means of our techniques (see Example 4.21), and this should provide a good indication of how to apply our techniques and results to more general contexts. On the other hand, Mackey has given a generalization of his theory of induced representations to the context of virtual subgroups [39, 47], and at present it is not at all clear to us in what way our approach might be applicable to that context.

Conditional expectations on C^*-algebras have recently arisen in a number of other situations, both in the theory of C^*-algebras and in the applications of C^*-algebras to physics (see for example [15, 63] and the references contained therein, as well as p. 9 of [14]), and we hope that our theory may turn out to be useful in some of these situations also, although we have not investigated this matter. Since almost all such situations which have come to our attention involve the considerably simpler case of ordinary conditional expectations rather than our generalizations using algebras of right centralizers, we have considered it advisable to begin our exposition by treating first this special case, so as to make it readily accessible to those who do not wish to become entangled in the complications which are needed to treat the general case for groups. By treating this simpler case first we also provide motivation for the development of the more general case.

Accordingly, our exposition is organized in the following way. In Section 1 we give the definition of representations induced by means of an ordinary conditional expectation, and in Section 3 we formulate and prove the imprimitivity theorem for this special case. Section 2 is devoted to showing how to associate a C^*-algebra to a positive map between C^*-algebras and to developing the generalization to noncommutative C^*-algebras of Kaplansky's "C^*-modules." This material is needed for the imprimitivity theorem in both the special and general cases. In Section 4, motivated by the case of a closed subgroup which is not open, we develop the definition of generalized conditional expectations, and we also discuss briefly the representation of C^*-algebras on the noncommutative analog of Kaplansky's "C^*-modules." In Section 5 we define induced representations in the general setting and derive some of their basic properties. We also show there that Mackey's definition of induced representations occurs as a special case of the definition for C^*-algebras. In Section 6 we formulate and prove the imprimitivity theorem for the general case, and also discuss the connection mentioned above with the Morita theorems. Finally, in Section 7 we show how to derive Mackey's imprimitivity theorem from the imprimitivity theorem for C^*-algebras discussed in Section 6. Some of our main results have already been announced in [55].

In the course of conducting the research reported in this paper I had many enjoyable conversations with J. M. G. Fell, and I am deeply indebted to him for many very helpful suggestions, without which my progress would certainly have been slower and this paper would have had many more rough edges. I also thank R. V. Kadison and R. C. Busby for helpful comments. Most of this research was conducted while I was visiting at the University of Pennsylvania, and I would like to express my appreciation to the members of the Department of Mathematics there for their warm hospitality during my visit.

1. Representations Induced from Subalgebras

In this section we will consider the process of inducing representations from a subalgebra of a C^*-algebra. This process will have as a special case the process of inducing representations of a locally compact group from an *open* subgroup, and we will motivate the exposition in this section by considering this special case first.

We begin with a general comment about our terminology and notation, namely, that in most places we find it considerably more convenient to use the terminology and notation of modules rather than the (equivalent) terminology of representations. Accordingly, if G is a topological group, then by a *unitary G-module* we will mean a Hilbert space, W, on which G acts by means of a strongly continuous unitary representation (13.1 of [13]), and we will write xw for the action of an element x of G on a vector w in W. If A is a $*$-normed algebra [48] (in particular, a pre-C^*-algebra), then by a *Hermitian left A-module* (we will usually omit the word "left") we will mean a Hilbert space, W, on which A acts by means of a norm continuous nondegenerate $*$-representation by bounded operators (2.2 of [13]), and we will denote this action by aw for $a \in A$, $w \in W$. If instead, this action of A on W is by means of an antirepresentation, then we will speak of a *Hermitian right A-module.* If W and W' are Hermitian A-modules, then the Banach space of bounded A-module homomorphisms (frequently called intertwining operators) from W to W' will be denoted by $\mathrm{Hom}_A(W, W')$.

Let G be a locally compact group. We will denote by $L(G)$ the usual group algebra of G, that is, the $*$-normed algebra of all complex-valued functions on G which are integrable with respect to left Haar measure on G, with convolution as multiplication, and with the usual involution (see 13.2.2 of [13]). (In all that follows, we could just as well work with the dense $*$-subalgebra, $C_C(G)$, of $L(G)$ consisting of the continuous functions of compact support.) If R is a strongly continuous unitary representation of G on a Hilbert space W, then a $*$-representation, also denoted by R, of $L(G)$ can be defined by

$$R_f w = \int_G f(x)\, R_x w\, dx,$$

for all $f \in L(G)$, $w \in W$. Then by using all the unitary representations of G, a C^*-algebra norm can then be defined on $L(G)$ by

$$\| f \|_{C^*(G)} = \sup\{\| R_f \| \colon R \text{ is a unitary representation of } G\}.$$

The C^*-algebra obtained by completing $L(G)$ with respect to this norm is called the group C^*-algebra of G, denoted by $C^*(G)$ (see 13.9.1 of [13]). Then it is well known that the above process of using a unitary representation of G to define a representation of $L(G)$ (and then of $C^*(G)$) establishes a bijective correspondence between unitary G-modules and Hermitian $L(G)$ or $C^*(G)$-modules which preserves intertwining operators (so that the category of unitary G-modules is isomorphic to the category of Hermitian $L(G)$ or $C^*(G)$-modules).

Now let H be an open subgroup of G. Because H is open (so that Haar measure on H is the restriction to H of Haar measure on G), the group algebra, $L(H)$, of H can be viewed in an obvious way as a $*$-subalgebra of $L(G)$. We will show shortly that, in addition, $C^*(H)$

can be viewed as a subalgebra of $C^*(G)$. Then we see that the process of inducing representations of H up to G should be a process for inducing representations of $C^*(H)$ up to $C^*(G)$, that is, a process for constructing representations of a C^*-algebra in terms of representations of a subalgebra.

Suppose now that A is a C^*-algebra, B is a C^*-subalgebra of A, and that V is a Hermitian B-module. We would like to construct from V a Hermitian A-module. In analogy with the purely algebraic case [28] it is natural to view A as a left-A-right-B-bimodule and form the algebraic tensor product $A \otimes_B V$, which is a left A-module, and then try to find a preinner product on $A \otimes_B V$ for which the corresponding Hilbert space will be a Hermitian A-module. Now if A happens to have an identity element, 1, if we let $B = \mathbb{C}1$ where $\mathbb{C}$ denotes the complex numbers, and if V is the one-dimensional Hermitian B-module, then the left A-module $A \otimes_B V$ is naturally identified with A viewed as a left A-module, and then any state of A gives such a preinner product. This indicates that in general there will be no unique way of choosing such a preinner product on $A \otimes_B V$, which is in sharp contrast to what seems to be the case with the induced representations of Mackey.

However, as indicated in the introduction, there is another natural piece of structure relating $C^*(G)$ and $C^*(H)$ which we will show plays a crucial role in the definition of induced representations. This is the natural projection of $C^*(G)$ onto $C^*(H)$ corresponding to the obvious projection, P, of $L(G)$ on $L(H)$ consisting of restricting functions on G to functions on H.

LEMMA 1.1. *Let H be an open subgroup of the locally compact group G, and let P be the projection of $L(G)$ onto $L(H)$ consisting of restricting functions from G to H. Then P is positive in the sense that if $f \in L(G)$, then $P(f^* * f)$ is positive as an element of the C^*-algebra $C^*(H)$. Furthermore, if p is a positive type function on H, and if p is extended to a function, q, on G by defining it to have value 0 off of H, then q is a positive type function on G.*

Proof. This proposition and its proof are just a reformulation of a special case of the first theorem in Blattner's paper [4]. Given $f \in L(G)$, we must show that for every unitary H-module (hence Hermitian $L(H)$-module) V and every $v \in V$ we have $\langle P(f^* * f)v, v\rangle \geqslant 0$. Now let $\{x_i\}$ be a complete set of representatives for the left cosets of H in G. Then

$$\begin{aligned}\langle P(f^* * f)v, v\rangle &= \int_H \int_G \bar{f}(x^{-1})\, \Delta(x^{-1}) f(x^{-1}s)\, dx \langle sv, v\rangle\, ds \\ &= \int_H \sum_i \int_H \bar{f}(x_i^{-1}t^{-1})\, \Delta(x_i^{-1}t^{-1}) f(x_i^{-1}t^{-1}s)\, dt \langle sv, v\rangle\, ds \\ &= \sum_i \int_H \int_H \bar{f}(x_i^{-1}t^{-1})\, \Delta(x_i^{-1})\, \Delta(t^{-1}) f(x_i^{-1}s) \langle tsv, v\rangle\, ds\, dt \\ &= \sum_i \Delta(x_i^{-1}) \left\langle \int_H f(x_i^{-1}s)\, sv\, ds, \int_H f(x_i^{-1}t)\, tv\, dt \right\rangle \\ &\geqslant 0,\end{aligned}$$

as desired, where the last integrals can be taken as Bochner integrals, and all integrals are with respect to left Haar measures.

Now every positive type function on H is of the form $s \mapsto \langle sv, v\rangle$ for some cyclic vector v in some unitary H-module (see 13.4.5 of [13]), and so the above calculation can be interpreted as showing that if p is a positive type function on H and if it is extended to a function, q, on G by letting it have value 0 off H, then q is a positive type function on G. Q.E.D.

PROPOSITION 1.2. *The norm on $L(H)$ from $C^*(G)$ coincides with the norm from $C^*(H)$, so that $C^*(H)$ can be viewed as a subalgebra of $C^*(G)$. The projection P of $L(G)$ onto $L(H)$ is continuous with respect to the norm from $C^*(G)$, and so extends to a projection of $C^*(G)$ onto $C^*(H)$, which we also denote by P. Furthermore P is positive, in the sense that P carries positive elements of the C^*-algebra $C^*(G)$ to positive elements of $C^*(H)$. In addition, P satisfies the conditional expectation property*

$$P(ba) = bP(a) \quad \text{and} \quad P(ab) = P(a)b$$

for all $a \in C^*(G)$, $b \in C^*(H)$. Finally, P has norm one.

Proof. Let $f \in L(H)$. Now every unitary representation of G restricts to a unitary representation of H, and from this it follows that $\|f\|_{C^*(H)} \geqslant \|f\|_{C^*(G)}$. We must prove the opposite inequality. Now one of the equivalent definitions (2.7.1 of [13]) of the norm of f as an element of $C^*(H)$ is

$$\|f\|_{C^*(H)} = \sup\{p(f^* * f)^{1/2} : p \in S\},$$

where S is the collection of positive linear functionals of norm one (states) on $L(H)$. But the positive linear functionals on $L(H)$ correspond to the positive type functions on H, with the norm of the functional being equal to the value of the function at the identity element (13.4.5 of [13]). Furthermore, we have seen in Lemma 1.1 that every positive type function on H can be extended to a positive type function on G (taking, of course, the same value at the identity element). Thus every positive linear functional on $L(H)$ extends to a positive linear functional on $L(G)$ of the same norm. From this the desired inequality follows. Thus $C^*(H)$ can be viewed as a subalgebra of $C^*(G)$.

We remark next that because the modular function of H is just the restriction to H of the modular function of G, it is clear that $P(f^*) = P(f)^*$ for any $f \in L(G)$. We now show that P is continuous. Suppose now that f is a self-adjoint element of $L(G)$ (and so of $C^*(G)$). Then $P(f)$ is self-adjoint, and so there is a state of $C^*(H)$, that is, a positive type function p of norm one on H, such that $\| P(f)\|_{C^*(H)} = |p(P(f))|$. Then if we let q be the extension of p to G as in Lemma 1.1, we have

$$|p(P(f))| = \left| \int_H f(x)\, p(x)\, dx \right|$$

$$= \left| \int_G f(x)\, q(x)\, dx \right| = |q(f)|.$$

But $|q(f)| \leqslant \|f\|_{C^*(G)}$. It follows that $\| P(f)\|_{C^*(H)} \leqslant \|f\|_{C^*(G)}$. From this inequality for self-adjoint elements of $L(G)$, it is easily seen that P is continuous, and so extends to a projection of $C^*(G)$ onto $C^*(H)$. The positivity of P on $C^*(G)$ now follows by continuity from Lemma 1.1.

Now if $f \in L(G)$ and $g \in L(H)$, then

$$P(g * f)(s) = \int_G g(x) f(x^{-1}s)\, dx$$

$$= \int_H g(x) f(x^{-1}s)\, dx = (g * P(f))(s)$$

for all $s \in H$. Since the two sides are continuous functions of f and g with respect to the norm of $C^*(G)$, it follows that $P(ba) = bP(a)$ for all $b \in C^*(H)$ and $a \in C^*(G)$. The conditional expectation property for right multiplication by elements of $C^*(H)$ is verified in a similar way.

We seem to need the conditional expectation property to verify that P has norm one. Specifically, if a is any element of $C^*(G)$, then we have

$$\begin{aligned} \| P(a)\|^2 &= \| P(a) P(a^*)\| \\ &= \|(P(aP(a^*)) + P(P(a) a^*))/2 \| \\ &= \| P(aP(a^*) + P(a) a^*)/2 \|. \end{aligned}$$

Now the term inside the outermost parenthesis is self-adjoint, and it follows from what we saw above that P has norm one on self-adjoint elements. Thus the term after the last equality sign above is

$$\leqslant \| (aP(a^*) + P(a) a^*)/2 \| \leqslant \| a \| \| P(a)\|.$$

Cancelling $\| P(a)\|$ from both sides, we obtain the desired inequality. Q.E.D.

We remark that conversely Tomiyama has shown [64] that any projection of norm one of a C^*-algebra onto a C^*-subalgebra satisfies the conditional expectation property, at least if the algebras have identity elements.

We also remark that it is very tempting to invoke the Krein extension theorem (p. 227 of [48]) to give an incorrect proof that $C^*(H)$ can be viewed as a subalgebra of $C^*(G)$, as we did in the first version of this paper. I am grateful to J. M. G. Fell for bringing this error to my attention.

We mention that it is easily seen that P is definite on $L(G)$, that is, that $P(f^* * f) = 0$ only if $f = 0$. But P need no longer be definite when extended to $C^*(G)$. For example, let G be a nonamenable [26] discrete group, such as the free group on two generators, and let H be the open subgroup consisting of the identity element of G. Then the representation of G corresponding to the only positive type function on H is the left regular representation of G, and this representation is not faithful on $C^*(G)$ because G is not amenable (see Theorem 3.5.2 of [26]). It follows that if c is any element in the kernel of the left regular representation, then $P(c^*c) = 0$.

DEFINITION 1.3. If A is a C^*-algebra and if B is a C^*-subalgebra of A, then by a *conditional expectation* from A to B we mean a continuous positive projection of A onto B which satisfies the conditional expectation property

$$P(ba) = bP(a) \qquad \text{and} \qquad P(ab) = P(a)b$$

for $b \in B$ and $a \in A$.

We remark that if P is a positive map, then it is easily seen that $P(a^*) = (P(a))^*$ for all $a \in A$, and from this it follows easily that the conditional expectation property for right multiplication by elements of B is a consequence of that for left multiplication, and conversely.

In addition to the example of the natural conditional expectation associated above to a locally compact group and an open subgroup, we give several other examples.

EXAMPLE 1.4. Let A be a C^*-algebra with identity element 1, and let p be a state of A. Let $B = \mathbb{C}1$, and let $P(a) = p(a)1$.

EXAMPLE 1.5. Let A be a C^*-algebra and let G be a compact group which acts continuously as a group of automorphisms of A (as in [14, 15]). Let B be the subalgebra of elements which are invariant under the action of G, and let P be define by

$$P(a) = \int_G x(a)\, dx$$

(where the Haar measure on G is normalized so that G has measure one).

EXAMPLE 1.6. Let A be a C^*-algebra and let i be a self-adjoint idempotent in A. Let $B = iAi$, and define P by $P(a) = iai$.

As soon as we have specified a conditional expectation from a C^*-algebra A to a C^*-subalgebra B, then for any Hermitian B-module V there exists a canonical preinner-product on $A \otimes_B V$, which is defined in a way analogous to the way a state is used to define a preinner product on a C^*-algebra.

LEMMA 1.7. *Let A be a C^*-algebra, let B be a C^*-subalgebra of A, and let P be a conditional expectation from A to B. Then for any Hermitian B-module V the sesquilinear form on $A \otimes_B V$ whose value on elementary tensors is given by*

$$\langle a_1 \otimes v_1, a \otimes v \rangle = \langle P(a^* a_1) v_1, v \rangle_V$$

is a preinner-product.

Proof. Because P satisfies the conditional expectation property, it is easily seen that the indicated form is B-balanced in both entries and so is well-defined on $A \otimes_B V$. It is also easily seen to be conjugate symmetric. Thus, what we need to show is that it is nonnegative. We show first that it suffices to treat the case in which V is cyclic. Let an element $\sum a_i \otimes v_i$ (finite sum) of $A \otimes_B V$ be given. Since V is nondegenerate, we can find a finite collection $\{V_k\}$ of mutually orthogonal cyclic (closed) submodules of V such that $v_i \in \bigoplus V_k$ for all i. For each i, let $v_i = \sum_k u_{ik}$ where $u_{ik} \in V_k$ for each k. Then a routine calculation using the properties of P shows that

$$\left\langle \sum a_i \otimes v_i , \sum a_j \otimes v_j \right\rangle = \sum_k \left\langle \sum a_i \otimes u_{ik} , \sum a_j \otimes u_{jk} \right\rangle.$$

Thus it suffices to show that each term in the sum over k on the right is nonnegative. But each of these terms involves only vectors from a cyclic submodule of V.

Suppose now that V is cyclic with cyclic vector z, so that Bz is dense in V. Now

$$\left\langle \sum a_i \otimes v_i , \sum a_j \otimes v_j \right\rangle = \sum_{i,j} \langle P(a_j{}^* a_i) v_i , v_j \rangle,$$

and the right-hand side is clearly a continuous function of the v_i. Thus to show that this quantity is nonnegative it suffices to show that it is nonnegative whenever all the v_i are in the dense subset Bz. Accordingly, let $v_i = b_i z$ for each i. Then a routine calculation shows that the above quantity is equal to

$$\left\langle P\left(\left(\sum a_j b_j\right)^* \left(\sum a_i b_i\right)\right) z, z \right\rangle,$$

which is nonnegative because of the properties of P. Q.E.D.

THEOREM 1.8. *The representation of A on $A \otimes_B V$ is a continuous nondegenerate $*$-representation by bounded operators with respect to the preinner product defined above. Thus the Hilbert space obtained by completing the quotient of $A \otimes_B V$ with respect to the subspace of vectors of length zero is a Hermitian A-module.*

Proof. We show first that if $\sum a_i \otimes v_i$ is an element of $A \otimes_B V$ and if $a \in A$, then

$$\left\langle \sum aa_i \otimes v_i , \sum aa_j \otimes v_j \right\rangle \leqslant \| a \|^2 \left\langle \sum a_i \otimes v_i , \sum a_j \otimes v_j \right\rangle.$$

Now by the same arguments as in the proof of Lemma 1.7, it suffices to consider the case in which V is cyclic with cyclic vector z and $v_i = b_i z$ for each i for appropriate $b_i \in B$. Then if we let $c = \sum a_i b_i$, a routine calculation shows that the desired inequality above becomes

$$\langle P(c^*a^*ac)z, z\rangle \leqslant \| a \|^2 \langle P(c^*c)z, z\rangle.$$

But this inequality is an immediate consequence of the fact that $c^*a^*ac \leqslant \| a \|^2 c^*c$ as positive elements of A (see 1.6.8 of [13]). Thus the representation of A is by bounded operators and is continuous.

It is easily verified that the representation is a $*$-representation. To show that it is nondegenerate it suffices to show that if e_k is a bounded approximate identity for A (see 1.7 of [13]) then $e_k(\sum a_i \otimes v_i)$ converges to $\sum a_i \otimes v_i$ with respect to the preinner product. But this follows from a routine calculation and the continuity of P. Q.E.D.

We remark that the above construction is very closely related to a construction of Stinespring [59]. In fact the proof of Lemma 1.7 can be considered to be a generalization of the proof of complete positivity in Theorem 3 of [59].

DEFINITION 1.9. The Hermitian A-module obtained as above from $A \otimes_B V$ is called the *Hermitian A-module obtained by inducing V from B to A via P*. We will denote it by ${}^A_PV$, or by ${}^AV$ when there is no uncertainity about what conditional expectation is being used.

COROLLARY 1.10. *If $V = \bigoplus V_k$ (a possibly uncountable Hilbert space direct sum of Hermitian B-modules), then ${}^AV = \bigoplus {}^AV_k$.*

COROLLARY 1.11. *If V is cyclic with cyclic vector z, then the elements of the form $a \otimes z$, $a \in A$, are dense in ${}^AV$.*

These corollaries follow from the proof of Lemma 1.7.

EXAMPLE 1.12. If A, B, and P are as in Example 1.4, and if V is the one-dimensional B-module, then ${}^AV$ is easily seen to be just the usual representation of A obtained from the state p. Thus the Gelfand–Naimark–Segal construction of a representation from a state can be viewed as a special case of the above construction of induced representations. (For the case in which A does not have an identity element see Example 4.14.)

EXAMPLE 1.13. If P is the natural conditional expectation associated above with a locally compact group G and an open subgroup H (with $A = C^*(G)$ and $B = C^*(H)$), and if V is a unitary H-module, and so a Hermitian $C^*(H)$-module, then ${}^AV$ will be a unitary G-module. It is not difficult to verify that ${}^AV$ is (unitarily equivalent to) the representation of G induced from V according to Mackey's definition [35] of induced representations. We will give a verification of this fact in a more general context later (Theorem 5.12).

We now give an example to show that even if A, B, and V are finite dimensional, ${}^AV$ may easily differ from the usual algebraic induced module $A \otimes_B V$.

EXAMPLE 1.14. Let A be commutative and finite dimensional, so that A is just a direct sum of a finite number of copies of $\mathbb{C}$. Let p be any homomorphism of A onto $\mathbb{C}$, that is, any pure state of A. Let $B = \mathbb{C}1$ and let $P(a) = p(a)1$. Let V be the one-dimensional B-module. Then $A \otimes_B V$ is equivalent to the (algebraic) left regular representation of A, whereas, according to Example 1.12, ${}^AV$ is the module determined by the state p, which is one-dimensional.

We remark that in analogy with the situation for algebraic induced representations [28], as well as with the developments in [51, 53], one should be able to define another kind of induced representation by putting an inner product on a suitable subspace of $\mathrm{Hom}_B(A, V)$, but we do not know how to do this. If $A = C(X)$ for some compact Hausdorff space X, if $B = \mathbb{C}1$, and if V is the one-dimensional B-module, then this probably corresponds to putting an inner product on some subspace of the space $M(X)$ of measures on X (which is the dual of A and so is an A-module). We do not recall having seen any such construction considered in the literature.

In the case of representations of a C^*-subalgebra B which are obtained from a state of B, our construction of induced representations has a simple alternate description which is closely related to the first theorem of Blattner's paper [4].

PROPOSITION 1.15. *Let P be a conditional expectation from a C^*-algebra A to a subalgebra B, let p be a positive linear functional on B, and let V_p denote the corresponding Hermitian B-module. Let $q = p \circ P$, so that q is a positive linear functional on A, and let W_q denote the corresponding Hermitian A-module. Then ${}^A(V_p)$ is unitarily equivalent to W_q.*

Proof. Let z be the cyclic vector for V_p such that $p(b) = \langle bz, z\rangle$ for all $b \in B$ (which exists by 2.4.4 of [13]). Then the map $a \mapsto a \otimes z$ from A into $A \otimes_B V_p$ is A-linear and its range is a dense submanifold of $A \otimes_B V_p$. Thus it suffices to show that this map is an isometry with respect to the preinner product on A obtained from q and the preinner product on $A \otimes_B V_p$. But this is verified by a routine computation. Q.E.D.

The theorem on induction in stages has a quite trivial form in the present context.

THEOREM ON INDUCTION IN STAGES 1.16. *Let C be a C^*-subalgebra of the C^*-subalgebra B of the C^*-algebra A, let P be a conditional expectation from A onto B and let Q be a conditional expectation from B onto C. Then $R = Q \circ P$ is a conditional expectation from A onto C. If U is a Hermitian C-module, then ${}^A_P({}^B_Q U)$ is unitarily equivalent to ${}^A_R U$.*

Proof. The images of elements of $A \otimes_B (B \otimes_C U)$ form a dense submanifold of ${}^A_P({}^B_Q U)$. It thus suffices to show that the A-linear map which sends such an element, $\sum a_i \otimes (b_i \otimes u_i)$, to the element $\sum a_i b_i \otimes u_i$ of ${}^A_R U$ is an isometry and has dense range. The fact that it is an isometry is verified by a routine calculation. To show that the range is dense, it suffices to show that if $a \in A$ and $u \in U$ and if b runs through a bounded approximate identity for B, then $ab \otimes u$ will converge to $a \otimes u$ in the norm of ${}^A_R U$. But this also is verified by a routine computation and by invoking the continuity of Q. Q.E.D.

To see that the theorem on induction in stages for groups is a special case of this theorem (when the subgroups involved are open) it suffices to note that if K is an open subgroup of the open subgroup H of G, then the natural conditional expectation from $C^*(G)$ to $C^*(K)$ is just the composition of the natural conditional expectation from $C^*(G)$ to $C^*(H)$ with the natural conditional expectation from $C^*(H)$ to $C^*(K)$.

2. The C^*-Algebra of a Rigged Space

Let A be an algebra, and let B be a subalgebra of A. The most fundamental theorem about induced representations is the imprimitivity theorem, which answers the question, "Which representations of A are induced from representations of B?" In the present context in which A and B are C^*-algebras, and induced representations are constructed by means of a conditional expectation P, we must add "via P." The observation which indicates where to look for an answer to this question is the following. If V is a left B-module, then not only is $A \otimes_B V$ an A-module, but it is in fact a module in the obvious way over the (generally larger) algebra of all linear transformations of A into itself which commute with the right action of B on A. If V is a Hermitian B-module and if we wish to take into account the preinner product defined on $A \otimes_B V$ by means of P, then it is natural to expect to have to restrict attention to those linear transformations of A into itself which act as bounded operators on $A \otimes_B V$ for all Hermitian B-modules V. It turns out that this algebra of operators is in a natural way a C^*-algebra. In this section we will define and study this C^*-algebra, in preparation for treating the imprimitivity theorem in the next section. Actually, we will define this C^*-algebra in a slightly more general setting which is needed for the discussion of the imprimitivity theorem in the analog of the situation in which the subgroup need not be open, and which may also be useful in other settings. We will see that this leads to a generalization for noncommutative C^*-algebras of the "C^*-modules" which Kaplansky defined for the case of commutative C^*-algebras [30].

We begin by using P to define a B-valued preinner-product on A (conjugate linear in the first variable) by setting

$$\langle a, a_1 \rangle_B = P(a^* a_1).$$

More generally, we make the following definition, in which we let B be a pre-C^*-algebra (that is, not necessarily complete), because we will need this generality later.

Definition 2.1. Let B be a pre-C^*-algebra. By a *pre-B-Hilbert* space we mean a vector space X on which there is defined a *B-valued preinner product*, that is, a B-valued sesquilinear form, $\langle\ ,\ \rangle_B$, (here it does not matter which variable is conjugate linear) such that

(1) $\langle x, x\rangle_B \geqslant 0$ for all $x \in X$, in the sense that $\langle x, x\rangle_B$ is a positive element of B (that is, positive in the completion of B),

(2) $(\langle x, y\rangle_B)^* = \langle y, x\rangle_B$ for all $x, y \in X$.

If $\langle x, x\rangle_B = 0$ only when $x = 0$, we will say that $\langle\ ,\ \rangle_B$ is *definite*, and we will call it a *B-valued inner product.*

EXAMPLE 2.2. If P is any positive map from a C^*-algebra A to another C^*-algebra B [60], then a B-valued preinner product can be defined as above on A by setting $\langle a, a'\rangle_B = P(a^*a')$, so that A becomes a pre-B-Hilbert space. Presumably the C^*-algebra which will be associated in Proposition 2.5 with a pre-B-Hilbert space might in this case contain some information about P, but we have not investigated this question. (But see [76].)

We will give later a number of additional examples of pre-B-Hilbert spaces, in which in fact more structure is present. For the present we turn immediately to showing how to define the analog for a pre-B-Hilbert space of the algebra of all bounded operators on an ordinary Hilbert space. It is natural to require the operators involved to be bounded with respect to the B-valued preinner product in an appropriate sense. But in addition, we must include the assumption that an adjoint operator exists as part of our definition of a bounded operator, because in the present situation we cannot prove that adjoints automatically exist, since no completeness is assumed (but see [76]).

DEFINITION 2.3. Let X be a pre-B-Hilbert space. By a *bounded operator* on X we mean a linear operator, T, from X into itself such that

(1) There is a nonnegative constant, k_T, such that

$$\langle Tx, Tx\rangle_B \leqslant k_T{}^2\langle x, x\rangle_B$$

for all $x \in X$, where the inequality is with respect to the usual ordering of positive elements of B.

(2) There is a linear operator, T^*, from X into itself satisfying condition 1 above, such that

$$\langle Tx, y\rangle_B = \langle x, T^*y\rangle_B$$

for all $x, y \in X$. Such a T^* will be called an *adjoint* for T.

By the *norm* of T, denoted by $\| T\|$, we will mean the least constant

k_T for which the inequality in condition 1 holds. The collection of all bounded operators on X will be denoted by $L(X)$.

We remark that T^* will also be in $L(X)$ since it is easily seen to have T as an adjoint. Since the B-valued preinner product is not assumed to be definite, there can be nonzero bounded operators of norm zero, and a bounded operator can have many adjoints.

We would like to show that $L(X)$ is almost a C^*-algebra under the usual operations. Because no appropriate version of the Cauchy–Schwartz lemma seems to hold in a pre-B-Hilbert space (see, however, Proposition 2.9), we need the following device which permits us to pass to the scalar case.

LEMMA 2.4. *Let X be a pre-B-Hilbert space. For each state p of B the function $p(\langle \cdot, \cdot \rangle_B)$ is a scalar-valued preinner product on X. Let X_p denote the Hilbert space obtained in the usual way by taking the quotient of X by the subspace of vectors of length zero and completing. Then each element, T, of $L(X)$ acts on X_p as a bounded operator of norm no greater than $\| T \|$. If $\| T \|_p$ denotes the norm of T acting on X_p, then*

$$\| T \| = \sup\{\| T \|_p : p \text{ is a state of } B\}.$$

If T^* is an adjoint of T, then T^* acts as the adjoint of T on X_p.

Proof. For any $x \in X$, we have

$$p(\langle Tx, Tx \rangle_B) \leqslant p(\| T \|^2 \langle x, x \rangle_B) = \| T \|^2 p(\langle x, x \rangle_B),$$

which shows that T acts on X_p as a bounded operator, and that $\| T \|_p \leqslant \| T \|$. Let

$$M = \sup\{\| T \|_p : p \text{ is a state of } B\}.$$

Then $M \leqslant \| T \|$. But let $x \in X$ and let p be any state of B. Then

$$p(\langle Tx, Tx \rangle_B) \leqslant M^2 p(\langle x, x \rangle_B).$$

Since this is true for all states, it follows that

$$\langle Tx, Tx \rangle_B \leqslant M^2 \langle x, x \rangle_B .$$

Since this is true for all x, it follows that $\| T \| = M$. The last statement of the Lemma is verified in a routine manner. Q.E.D.

PROPOSITION 2.5. *If X is a pre-B-Hilbert space, then $L(X)$ is a normed algebra except for the fact that some nonzero elements may have norm zero. If $S, T \in L(X)$ and if S^* and T^* are adjoints for S and T, respectively, then $S^* + T^*$ and T^*S^* are adjoints of $S + T$ and ST, respectively. If $T \in L(X)$, and if T^* is any adjoint of T, then $\| T^* \| = \| T \|$, and $\| T^*T \| = \| T \|^2$. If J denotes the set of elements of $L(X)$ of norm zero, then J is a two-sided ideal in $L(X)$, and any two adjoints of an element of $L(X)$ differ by an element of J. Thus the quotient algebra $L(X)/J$ is a pre-C^*-algebra.*

Proof. Let $S, T \in L(X)$. We show first that $S + T$ satisfies condition 1 of Definition 2.3. Let $x \in X$ and let p be any state of B. Then from Lemma 2.4,

$$p(\langle (S+T)x, (S+T)x \rangle_B) = \|(S+T)x\|_p^2$$
$$\leqslant (\| S \|_p + \| T \|_p)^2 \| x \|_p^2 \leqslant (\| S \| + \| T \|)^2 p(\langle x, x \rangle_B).$$

Since this is true for all states p of B, it follows that

$$\langle (S+T)x, (S+T)x \rangle_B \leqslant (\| S \| + \| T \|)^2 \langle x, x \rangle_B .$$

Since this is true for all $x \in X$, it follows that condition 1 is satisfied and that

$$\| S + T \| \leqslant \| S \| + \| T \|.$$

Now if S^* and T^* are adjoints for S and T, respectively, then it follows from what we have just seen that $S^* + T^*$ satisfies condition 1. Furthermore, a routine calculation shows that $S^* + T^*$ is an adjoint for $S + T$. Thus $S + T \in L(X)$.

It is easily seen that $L(X)$ is closed under scalar multiplication, and that $\| \ \|$ is a seminorm on $L(X)$.

We now turn to the composition of operators. If $S, T \in L(X)$, then for any $x \in X$ we have

$$\langle STx, STx \rangle_B \leqslant \| S \|^2 \langle Tx, Tx \rangle_B \leqslant \| S \|^2 \| T \|^2 \langle x, x \rangle_B .$$

Thus ST satisfies condition 1 of Definition 2.2, and

$$\| ST \| \leqslant \| S \| \| T \|.$$

If S^* and T^* are adjoints of S and T, respectively, then it follows that T^*S^* satisfies condition 1, and a routine calculation shows that

T^*S^* is an adjoint for ST. Thus $ST \in L(X)$, and $L(X)$ is a normed algebra except that some elements may have norm zero.

Now if $T \in L(X)$ and if T^* is an adjoint for T, then by Lemma 2.4,

$$\| T^* \| = \sup\{\| T^* \|_p\} = \sup\{\| T \|_p\} = \| T \|,$$

and

$$\| TT^* \| = \sup\{\| TT^* \|_p\} = \sup\{\| T \|_p^2\} = \| T \|^2,$$

where the supremums are taken over all states p of B. Thus the norm satisfies the asserted equalities.

It is clear from the inequalities for the norm obtained earlier that J is a two-sided ideal in $L(X)$. If T^* and T' are two adjoints of an element T of $L(X)$, then $T^* - T'$ is an adjoint of the operator 0, and so from above we see that $\| T^* - T' \| = \| 0 \| = 0$, so that $T^* - T' \in J$. It follows that $L(X)/J$ is a pre-C^*-algebra. Q.E.D.

COROLLARY 2.6. *The representation of $L(X)$ on $\oplus \{X_p$: p is a state of $B\}$ which comes from the representation of $L(X)$ on the X_p (which were defined in Lemma* 2.3), *has J as its kernel, and so provides a faithful isometric $*$-representation of $L(X)/J$.*

COROLLARY 2.7. *The image in $L(X)/J$ of an element T of $L(X)$ will be a positive element of $L(X)/J$ if and only if $\langle Tx, x\rangle_B$ is a positive element of B for all $x \in X$.*

Because the positive maps which arise in connection with induced representations satisfy the conditional expectation property, the pre-B-Hilbert spaces they define possess additional structure. This additional structure is described by the following definition.

DEFINITION 2.8. Let B be a pre-C^*-algebra. By a *right B-rigged space*, we mean a right B-module X (in the algebraic sense) which is a pre-B-Hilbert space (with compatible multiplication by complex numbers on B and X), with preinner product conjugate linear in the first variable, such that

$$\langle x, yb\rangle_B = \langle x, y\rangle_B b$$

for all $x, y \in X$, and $b \in B$, which implies that

$$\langle xb, y\rangle_B = b^*\langle x, y\rangle_B ,$$

and such that the range of $\langle\ ,\ \rangle_B$ generates a dense subalgebra of B. *Left B-rigged spaces* are defined similarly except that we require that B act on the left of X, that the preinner product be conjugate linear in the second variable, and that

$$\langle bx, y\rangle_B = b\langle x, y\rangle_B\,.$$

This definition can be compared to Higman's definition of S-bilinear forms in Section 8 of [28]. We have chosen to make the preinner products on right B-rigged spaces conjugate linear in the first variable in order to simplify various formulas. It was D. Kastler who suggested such a possibility to us.

We remark that the range of $\langle\ ,\ \rangle_B$ is clearly closed under multiplication on either side by elements of B, and so the linear span of the range of $\langle\ ,\ \rangle_B$ will be a two-sided ideal in B (which is required to be dense).

The right B-rigged spaces seem to be the appropriate analog for the noncommutative case of the "C^*-modules" introduced by Kaplansky [30] for the case in which B is commutative.

For B-rigged spaces, there does exist a useful analog of the Cauchy–Schwartz inequality.

PROPOSITION 2.9. *If X is a (right) B-rigged space and if $x, y \in X$, then*

$$\langle x, y\rangle_B{}^*\langle x, y\rangle_B \leqslant \|\langle x, x\rangle_B\|\,\langle y, y\rangle_B\,,$$

from which it follows that

$$\|\langle x, y\rangle_B\| \leqslant \|\langle x, x\rangle_B\|^{1/2}\,\|\langle y, y\rangle_B\|^{1/2}.$$

Proof. Let p be any state of B. Then

$$p(\langle x, y\rangle_B{}^*\langle x, y\rangle_B) = p(\langle x\langle x, y\rangle_B\,, y\rangle_B),$$

which by the ordinary Cauchy–Schwartz inequality is

$$\begin{aligned}
&\leqslant (p(\langle x\langle x, y\rangle_B\,, x\langle x, y\rangle_B\rangle_B))^{1/2}(p(\langle y, y\rangle_B))^{1/2}\\
&= (p(\langle x, y\rangle_B{}^*\langle x, x\rangle_B\langle x, y\rangle_B))^{1/2}(p(\langle y, y\rangle_B))^{1/2}\\
&\leqslant \|\langle x, x\rangle_B\|^{1/2}(p(\langle x, y\rangle_B{}^*\langle x, y\rangle_B))^{1/2}(p(\langle y, y\rangle_B))^{1/2}
\end{aligned}$$

(from the fact, see 1.6.8 of [13], that if c is a positive element of a

C^*-algebra B, then $b^*cb \leqslant \| c \| b^*b$ for all $b \in B$). Cancelling and squaring, we obtain

$$p(\langle x, y\rangle_B{}^*\langle x, y\rangle_B) \leqslant \|\langle x, x\rangle_B\| \, p(\langle y, y\rangle_B).$$

Since this is true for all states, the desired inequality follows. Q.E.D.

It is natural to try to define a norm, or at least a seminorm, on a B-rigged space by setting

$$\| x \|_B = \|\langle x, x\rangle_B\|^{1/2}.$$

As with ordinary inner-product spaces, the only slight difficulty is in showing that the triangle inequality holds, and for this one uses Proposition 2.9 in the same way as one uses the ordinary Cauchy–Schwartz inequality to prove the ordinary triangle inequality.

PROPOSITION 2.10. *If X is a B-rigged space and if $\| \;\|_B$ is defined as above, then*

$$\| x + y \|_B \leqslant \| x \|_B + \| y \|_B$$

for all $x, y \in X$. Thus $\| \;\|_B$ is a seminorm on X.

One can continue and consider such matters as the completion of a B-rigged space, but we will not do so here since this is not necessary for our discussion of induced representations.

DEFINITION 2.11. If X is a B-rigged space, then by a *bounded operator* on X we mean an operator satisfying Definition 2.3 but which also commutes with the action of B on X. We will denote the collection of bounded operators on X again by $L(X)$.

We remark that it is easily seen that if the pre-B-inner product on a B-rigged space is definite, then the bounded operators on X in the sense of Definition 2.3 will automatically commute with the action of B.

PROPOSITION 2.12. *If X is a B-rigged space, then $L(X)$ is almost a C^*-algebra as in Proposition 2.5, and the quotient by the two-sided ideal, J, of operators of norm zero will again be a pre-C^*-algebra.*

The proof is routine.

EXAMPLE 2.13. As indicated above, if P is a conditional expectation from a C^*-algebra A onto a subalgebra B, then A becomes a B-rigged space when B acts by right multiplication on A and $\langle a, a'\rangle_B = P(a^*a')$.

EXAMPLE 2.14. Let A be a C^*-algebra, let $B = A$ and let P be the identity map of A onto itself, so that A becomes an A-rigged space with A-inner product $\langle a, a'\rangle_A = a^*a'$. We determine $L(A)$. The elements of $L(A)$ must commute with the right action of A on itself. If A has an identity element, then it is easily seen that this implies that the elements of $L(A)$ are exactly the operators of left multiplication by elements of A, so that we can write $L(A) = A$. However, if A does not have an identity element, let $T \in L(X)$ and let T^* be the adjoint of T. Define an operator T' on A by $T'(a) = (T^*a^*)^*$. Then it is easily verified that the pair (T, T') is a double centralizer [29, 8] of A. We obtain in this way an injective $*$-homomorphism of $L(A)$ into the algebra, $M(A)$, of double centralizers of A, which Busby [8] has shown is a C^*-algebra. We claim that in this way $L(A)$ becomes identified with $M(A)$. Now A is a two-sided ideal in $M(A)$, and if $c \in M(A)$, then

$$\langle ca, ca\rangle = a^*c^*ca \leqslant \| c \|^2 a^*a,$$

for $a \in A$, so that each element c of $M(A)$ acts by left multiplication as an element of $L(A)$ of norm no greater than $\| c \|$. Furthermore, if $T \in L(A)$ corresponds to $c \in M(A)$, then by definition

$$\begin{aligned}\| c \|^2 &= \sup \| ca \|^2 = \sup \| a^*c^*ca \| \\ &= \sup \| \langle ca, ca\rangle_A \| = \sup \| T \|^2 \| a^*a \| = \| T \|^2,\end{aligned}$$

where the supremums are taken over the set of $a \in A$ for which $\| a \| \leqslant 1$. Thus the homomorphism of $L(A)$ into $M(A)$ is isometric and bijective.

3. THE IMPRIMITIVITY THEOREM FOR REPRESENTATIONS INDUCED FROM SUBALGEBRAS

We began the considerations of the last section by looking for operators on the C^*-algebra A which provide bounded operators on every Hermitian A-module which is induced from the subalgebra B via the conditional expectation P. We show that $L(A)$ consists of such operators.

PROPOSITION 3.1. *Let V be a Hermitian B-module. For any $T \in L(A)$ define a linear action of T on $A \otimes_B V$ by $T(a \otimes v) = (Ta) \otimes v$. Then this action is continuous with respect to the preinner product on $A \otimes_B V$ defined by means of P, and so T acts as a bounded operator on the induced module ${}^A V$. The norm of this action of T on ${}^A V$ is no greater than $\| T \|$. Thus the elements of the ideal J of operators of norm zero in $L(A)$ all act as the zero operator on ${}^A V$. If T^* is an adjoint for T, then T^* acts as the adjoint of T on ${}^A V$. The identity operator on A (which is always in $L(A)$) acts as the identity operator on ${}^A V$. Thus ${}^A V$ becomes a Hermitian module over the pre-C^*-algebra $C = L(A)/J$.*

Proof. We begin by noting that the action of T on $A \otimes_B V$ is well-defined because of the fact that T is assumed to commute with the action of B on A. We wish to show that T acts continuously on $A \otimes_B V$. Let $V = \oplus V_k$ be a decomposition of V into mutually orthogonal cyclic subspaces. Then ${}^A V = \oplus {}^A V_k$ by Corollary 1.10, and it is clear that T carries each $A \otimes_B V_k$ into itself. Hence it suffices to treat the case in which V is a cyclic module.

Let z be a cyclic vector for V. Then by the same continuity argument as was used in the second paragraph of the proof of Lemma 1.7 it suffices to consider elements of $A \otimes_B V$ of the form $a \otimes z$. But

$$\begin{aligned} &\langle T(a \otimes z), T(a \otimes z) \rangle = \langle \langle Ta, Ta \rangle_B z, z \rangle \\ &\leqslant \langle \| T \|^2 \langle a, a \rangle_B z, z \rangle = \| T \|^2 \langle a \otimes z, a \otimes z \rangle. \end{aligned}$$

Thus T acts as an operator of norm no greater than $\| T \|$ on $A \otimes_B V$, and so also on ${}^A V$. The remaining assertions of the Proposition are verified by routine calculations. Q.E.D.

Thus a necessary condition for a Hermitian A-module to be induced from B via P is that it admit an action of $C = L(A)/J$ under which it is a Hermitian C-module. In view of this, it is natural to examine which operators occur in C, and also how the action of C on an induced module is related to the action of A.

PROPOSITION 3.2. *Let P be a conditional expectation from A onto B. For each $a \in A$ let L_a denote the operator of left multiplication on A by a. Then $L_a \in L(A)$. In fact, $\| L_a \| \leqslant \| a \|$, and L_{a^*} is an adjoint for L_a. Thus we have a continuous $*$-homomorphism of A into C. If V is a Hermitian B-module, then the action of A on ${}^A V$ from Theorem 1.8 is* *the same as that which comes by Proposition* 3.1 *from the image of A in C. In particular,*

$$a(Tw) = (L_a T)w$$

for all $a \in A$, $T \in C$ and $w \in {}^A V$.

Proof. If $c \in A$, then

$$\langle L_a c, L_a c \rangle_B = P(c^* a^* ac) \leqslant \| a \|^2 P(c^* c) = \| a \|^2 \langle c, c \rangle_B .$$

Thus condition 1 of Definition 2.3 holds and $\| L_a \| \leqslant \| a \|$. Routine calculations provide the verifications for the remaining assertions. Q.E.D.

However, the operators described in Proposition 3.2 are not the only operators in $L(A)$. Essential for the imprimitivity theorem is the following fact.

PROPOSITION 3.3. *If P is a conditional expectation from A onto B, and if P is viewed as an operator on A, then $P \in L(A)$.*

Proof. This amounts to showing that if P is a conditional expectation, then P is a locally completely positive map in the terminology of Theorem 7.4 of [60]. That is:

LEMMA 3.4. *If P is a conditional expectation, then*

$$P(a^*) P(a) \leqslant P(a^* a).$$

Proof. If A has an identity element, 1, which is preserved by P, then this is just the special case of Proposition 2.9 in which $x = a$ and $y = 1$. If A does not have an identity element, then one can easily construct a proof by more or less imitating the proof of Proposition 2.9. Q.E.D.

We return to the proof of Proposition 3.3. If $a \in A$, then, by Lemma 3.4

$$\langle Pa, Pa \rangle_B = P(P(a^*) P(a)) = P(a^*) P(a) \leqslant P(a^* a) = \langle a, a \rangle_B ,$$

so that P satisfies condition 1 of Definition 2.3. Also, P is its own adjoint, because

$$\langle Pa, c \rangle_B = P(P(a)^* c) = P(a^*) P(c) = P(a^* P(c)) = \langle a, Pc \rangle_B$$

for $a, c \in A$. Thus $P \in L(A)$. Q.E.D.

As we saw earlier, a necessary condition for a Hermitian A-module W to be induced from B via P is that the action of A on W be extendible to an action of all of C such that W becomes a Hermitian C-module. But this condition is not in general sufficient. To obtain a condition which is also sufficient, we note that P, viewed as an element of C, cannot act trivially on any induced module ${}^AV$, for if $b \in B$ and $v \in V$ are chosen so that $bv \neq 0$, then $P(b \otimes v) = b \otimes v$ and

$$\| b \otimes v \|^2 = \langle P(b^*b)v, v \rangle = \langle bv, bv \rangle \neq 0.$$

We would like to obtain a subalgebra of C, or $L(A)$, which reflects this property of P.

PROPOSITION 3.5. *Let E denote the linear span of the set of elements of $L(A)$ of the form L_aPL_c for $a, c \in A$. Then E is a two-sided ideal in $L(A)$, and every element of E has an adjoint in E. Thus the image of E in C is a two-sided $*$-ideal of C, and in particular is a pre-C^*-algebra, which we will also denote by E.*

Proof. The fact that E is an ideal follows from easily verified relations

$$TL_aPL_c = L_{Ta}PL_c, \qquad \text{and} \qquad L_aPL_cT = L_aPL_{(T^*(c^*))^*}$$

for any $T \in L(A)$ and $a, c \in A$. An adjoint for L_aPL_c is easily seen to be $L_{c^*}PL_{a^*}$. Q.E.D.

DEFINITION 3.6. We will call E the *imprimitivity algebra* of the conditional expectation P, for reasons which will soon be evident.

We remark that the algebra E will be given an interesting interpretation near the beginning of Section 6.

PROPOSITION 3.7. *Let V be a Hermitian B-module, so that ${}^AV$ is a Hermitian C or $L(A)$-module, and so an E-module. Then ${}^AV$ is non-degenerate as an E-module, and is thus a Hermitian E-module.*

Proof. It suffices to show that each elementary tensor $a \otimes v$ can be approximated in ${}^AV$ by elements in the ranges of suitable elements of E. Choose as these suitable elements ones of the form $L_a P L_b(b \otimes v)$ for $b \in B$. Then

$$\begin{aligned}\| a \otimes v - L_a P L_b(b \otimes v)\|^2 &= \|(a - ab^2) \otimes v\|^2 \\ &= \langle P((a - ab^2)^*(a - ab^2))v, v\rangle \\ &= \langle (P(a^*a) - (b^2)^* P(a^*a) - P(a^*a)\, b^2 \\ &\quad + (b^2)^* P(a^*a)\, b^2)v, v\rangle,\end{aligned}$$

which converges to zero as b runs through a bounded approximate identity for B.

THE IMPRIMITIVITY THEOREM 3.8 (for representations induced from subalgebras). *Let A, B, P, C, and E be as above. Then a Hermitian A-module W is induced from a Hermitian B-module via P if and only if W can be made into a Hermitian E-module in such a way that*

$$(*) \qquad a(Tw) = (L_a T)w$$

for all $a \in A$, $T \in E$, and $w \in W$. If W can be made into such an E-module, then this action of E can be extended to an action of C on W so that W becomes a Hermitian C-module in such a way that $(*)$ holds for all $T \in C$. In particular, P will act as a projection on W, and to obtain a Hermitian B-module from which W is induced it suffices to take $V = P(W)$.

Proof. We have already seen in Propositions 3.2 and 3.7 that if $W = {}^A V$ for some Hermitian B-module V, then W admits an action of E of the indicated form. To prove the converse, we begin with a general result about extending the action of an ideal on a module to an action of the whole algebra. This result generalizes 2.10.4 of [13], and is closely related to Lemma 1.7 of [24]. We recall from [51] that if A is a Banach algebra, then by a *Banach A-module*, we mean a Banach space W which is a left A-module such that $\| aw \| \leqslant \| a \| \| w \|$ for all $a \in A$ and $w \in W$. The Banach A-module W is said to be *essential* if AW is dense in W.

PROPOSITION 3.9. *Let C be a Banach algebra, let I be a closed left ideal in C which has an approximate identity of norm one for itself, and let W be an essential Banach I-module. Then the action of I on W can be extended in one and only one way to an action of C on W such that W becomes an essential Banach C-module. If C is a C^*-algebra and I is a*

$*$-ideal of C (and hence a two-sided ideal and a C^*-subalgebra), and if W is a Hermitian I-module, then, as a C-module under the extended action, W will be a Hermitian C-module.

Proof. By the Hewitt–Cohen factorization theorem (see [52] for references and discussion) every element w of W can be written in the form iw' for some $i \in I$ and $w' \in W$. We see then that if there is any extension to C of the action of I on W, it must be given by $cw = (ci)w'$. We would like to take this as our definition of the action of C on W, but first we must show that this definition does not depend on the factorization of w. To this end let e_k be an approximate identity of norm one for I. Then

$$(ci)\,w' = \lim(ce_k i)\,w' = \lim(ce_k)w,$$

which clearly does not depend on the factorization of w. The proof of the rest of the Proposition is accomplished by routine calculations. Q.E.D.

We return to the proof of Theorem 3.8. By taking the completions of E and C we can apply Proposition 3.9 to W, so that the action of E extends to an action of C making W into a Hermitian C-module. Then W is an A-module in two ways, one being the original action, and the second being the action via the map $a \mapsto L_a$ of A into C. The hypothesis $(*)$ of Theorem 3.8 says that these two actions coincide on the vectors of the form Tw for $T \in E$ and $w \in W$. But this set of vectors is dense, and so the actions are the same. Thus the extension of the action of E to all of C is also an extension of the action of A to all of C.

Since W is now a Hermitian C-module, and P is a self-adjoint idempotent in C, P acts as a self-adjoint projection on W. Let $V = PW$. Then V is stable under the action of B, for L_b and P commute as elements of C for any $b \in B$, since

$$(L_b P)(a) = bP(a) = P(ba) = (PL_b)(a)$$

for any $a \in A$, and consequently,

$$L_b V = L_b PW = PL_b W \subseteq V.$$

Furthermore, V is nondegenerate as a B-module. To see this let e_k be a bounded approximate identity for A. Then

$$L_{P(e_k)} P(a) = P(e_k)\,P(a) = P(e_k P(a)) = PL_{e_k} P(a)$$

for all $a \in A$, so that $L_{P(e_k)}P = PL_{e_k}P$. Then for any $w \in W$, we have

$$L_{P(e_k)}Pw = PL_{e_k}Pw = P(e_kPw),$$

which converges to $PPw = Pw$. Since $P(e_k) \in B$, it follows that Pw is in the closure of BV. Thus V is a Hermitian B-module.

We would like to show that ${}^AV$ is naturally unitarily equivalent to W. To this end define a map, F, from $A \otimes_B V$ to W whose value on elementary tensors is given by

$$F(a \otimes v) = av$$

for $a \in A$ and $v \in V$ (recalling that $V \subseteq W$). It is clear that F is an A-module homomorphism. We show that F preserves inner-products. If $a, a' \in A$, and $v, v' \in V$, then

$$\begin{aligned}\langle F(a' \otimes v'), F(a \otimes v)\rangle &= \langle a'v', av\rangle \\ &= \langle L_{a'}Pv', L_aPv\rangle = \langle PL_{a^*a'}Pv', v\rangle.\end{aligned}$$

But for any $c \in A$, we have

$$PL_{a^*a'}P(c) = P(a^*a'P(c)) = P(a^*a')\,P(c) = L_{P(a^*a')}P(c),$$

so that $PL_{a^*a'}P = L_{P(a^*a')}P$. Thus the inner product above is

$$= \langle L_{P(a^*a')}Pv', v\rangle = \langle P(a^*a')\,v', v\rangle.$$

By the bilinearity of the inner products it follows that they are preserved by F. Thus F drops to an isometric A-homomorphism (also denoted by F) of ${}^AV$ into W.

It remains to show that F is surjective. We begin by remarking that V is in the range of F, for, given an element bv in the dense subspace BV of V, we have $F(b \otimes v) = bv$. Suppose now that w is a vector in W which is orthogonal to the range of F. Then, since V is contained in the range of F and $Pw \in V$, we have

$$\| Pw \|^2 = \langle Pw, Pw\rangle = \langle Pw, w\rangle = 0,$$

so that $Pw = 0$. Now the range of F is A-invariant, and so is its orthogonal complement. Thus $P(aw) = 0$ for all $a \in A$, and so $L_cPL_aw = 0$ for all c, $a \in A$. It follows that $Ew = 0$. Since W was assumed to be nondegenerate as an E-module, it follows that $w = 0$.

Thus the range of F is all of W, and ${}^A V$ and W are unitarily equivalent. Q.E.D.

We remark that in general there are Hermitian C-modules (and so A-modules) which are not induced from B. For, taking C and E to be completed, C/E will be a C^*-algebra (if, as is usually the case, it is $\neq \{0\}$), and it is clear from Proposition 3.7 that a Hermitian C-module which arises in the obvious way from a Hermitian C/E-module cannot be induced from B.

If G is a locally compact group, H is an open subgroup, and P is defined as in Section 1, then it is not difficult to show that the above theorem has as a consequence the imprimitivity theorem of Mackey [34, 36] for this case. We will show this in detail for the general case in Section 7. We remark that P as an operator on W will correspond to the projection of Mackey's system of imprimitivity which corresponds to the coset of G/H containing the identity element of G. More generally, the left translation operators L_x for $x \in G$ will all be in C, and $L_x P L_{x^{-1}}$ will be the projection of Mackey's system of imprimitivity which corresponds to the coset of G/H containing x.

It is easily seen that the inducing process is in fact a functor. (This is discussed in Section 5.) The following theorem generalizes for the case of open subgroups Theorem 6.4 of Mackey's paper [38], which was discussed further by Blattner [3]. It is also closely related to the Morita theorems mentioned in the introduction. We will treat the general case in Section 6.

THEOREM 3.10. *The functor $V \mapsto {}^A V$, in which ${}^A V$ is viewed as a Hermitian E-module, is an equivalence of the category of Hermitian B-modules with the category of Hermitian E-modules. The inverse of this functor is the functor $W \mapsto P(W)$, where W is a Hermitian E-module which is viewed as a C-module by Lemma* 3.9 *so that $P(W)$ can be defined.*

Proof. The proof of Theorem 3.8 shows that ${}^A(P(W))$ is naturally isomorphic to W. In the other direction, if V is a Hermitian B-module, then a dense subset of $P({}^A V)$ consists of $P(A \otimes_B V) = B \otimes_B V$. On this subset define a natural map into V by $b \otimes v \mapsto bv$. This map is clearly B-linear and has dense range. A routine calculation verifies that it preserves inner products. We leave the details concerning morphisms to the reader (or see Theorem 5.3). Q.E.D.

4. Hermitian B-rigged A-modules

In this section we begin to develop the theory of a more general type of induced representation for C^*-algebras, which will have as a special case Mackey's induced representations when the subgroup need not be open [36]. We motivate the exposition in this section by considering this special case first.

Let G be a locally compact group, and let H be a closed subgroup of G. All integrals over G or H will be with respect to left Haar measures. We will let Δ and δ denote the modular functions of G and H, respectively.

The first obstacle which we encounter in trying to generalize our earlier results to this situation, is that the group algebra, $L(H)$, of H is no longer a subalgebra of the group algebra, $L(G)$, of G. But this is not a serious difficulty, for $L(H)$ and $L(G)$ can both be viewed as subalgebras of the measure algebra, $M(G)$, of G, and since $L(G)$ is in fact a two-sided ideal in $M(G)$, $L(H)$ acts by convolution on both the right and left of $L(G)$ as double centralizers [29, 8]. We took this point of view in our earlier paper [51]. For later reference, we state here the formula for the right action of $L(H)$ on $L(G)$. If $f \in L(G)$ and $\phi \in L(H)$, then

$$f * \phi(x) = \int_H f(xt^{-1})\, \Delta(t^{-1})\, \phi(t)\, dt$$

for all $x \in G$.

To put matters in perspective, we will show now that we can extend this arrangement to the group C^*-algebras of G and H, although this will turn out not to be very important. Notice first that if R is any unitary representation of G, then its integrated form is a $*$-representation of $M(G)$, and in particular

$$(*) \qquad \| R_{f*\phi} \| = \| R_f R_\phi \| \leqslant \| R_f \| \, \| R_\phi \|$$

for all $f \in L(G)$ and $\phi \in L(H)$. Since the restriction of R to H is a unitary representation of H whose integrated form is the same as the representation coming from $M(G)$, and since $(*)$ is true for all unitary representations R of G, it follows that

$$\| f * \phi \|_{C^*(G)} \leqslant \| f \|_{C^*(G)} \, \| \phi \|_{C^*(H)} \,.$$

The same result holds for $\phi * f$. We obtain in this way the following result.

PROPOSITION 4.1. *The action of $L(H)$ as double centralizers on $L(G)$ extends to a bounded $*$-homomorphism of $C^*(H)$ into the double centralizer algebra of $C^*(G)$.*

It does not seem to be known whether this homomorphism is injective. It will be injective if and only if every unitary representation of H is weakly contained in the restriction to H of some unitary representation of G [18]. J. M. G. Fell has pointed out to us that the example that he gave in which this appeared to fail (p. 445 of [18]) depended on the completeness of the classification of the irreducible representations of $SL(3, \mathbb{C})$ given in [22], and there is now some doubt that this classification is complete [58]. But this is not of importance to us here.

The next obstacle which we encounter is the fact that the restriction map, P, from $L(G)$ to $L(H)$ is not well-defined (on the equivalence classes of which $L(G)$ consists), since H may be a null set for Haar measure on G. However, P is well-defined on the dense subalgebra $C_c(G)$ consisting of the continuous complex-valued functions on G of compact support, and on this domain its range is contained in $C_c(H)$ (in fact, is all of $C_c(H)$). We note that $C_c(G)$ is carried into itself by the left and right actions of $C_c(H)$ on $C^*(G)$, so that the pre-C^*-algebra $C_c(H)$ acts as double centralizers on the pre-C^*-algebra $C_c(G)$.

In analogy with Proposition 1.2, we can now inquire as to whether P is a positive map between these pre-C^*-algebras. But it is easily seen that P will not even preserve the involution on these algebras if Δ and δ do not coincide on H. However, as suggested by Theorem 1 of [4], this is easily rectified by suitably adjusting P.

Notation 4.2. Let γ denote the continuous homomorphism of H into the multiplicative group of positive real numbers defined by

$$\gamma(s) = (\Delta(s)/\delta(s))^{1/2}$$

for all $s \in H$. Let P denote the linear map from $C_c(G)$ to $C_c(H)$ whose value on $f \in C_c(G)$ is given by

$$P(f)(s) = \gamma(s) f(s)$$

for all $s \in H$.

Then a routine calculation shows that:

PROPOSITION 4.3. *If $f \in C_c(G)$, then*

$$P(f^*) = (P(f))^*.$$

A slight reformulation of Theorem 1 of Blattner's paper [4] then asserts that this redefined P is indeed a positive map from $C_c(G)$ to $C_c(H)$. We include a proof here, both for completeness and because at a later point we will need most of the calculations which it contains. The main difference between the proof given here and the proof which we gave in Proposition 1.2 for the special case in which H was open, is that instead of choosing coset representatives, we must now use a Bruhat approximate cross section for G over G/H, that is, a nonnegative bounded continuous function, b, on G whose support has compact intersection with the saturant, CH, of any compact subset C of G, and is such that $\int_H b(xt)\,dt = 1$ for all $x \in G$ (see p. 103 of [7] or Proposition 8 on p. 51 of [6]).

THEOREM 4.4 (Blattner). *If $f \in C_c(G)$, then $P(f^* * f)$ is a positive element of $C^*(H)$. In fact, if V is a unitary H-module, if $u, v \in V$, and if $f, g \in C_c(G)$, then*

$$\langle P(g^* * f)u, v\rangle = \int_G b(x)\langle P(f_x)u, P(g_x)v\rangle\, dx,$$

where by definition $h_x(y) = h(xy)$ for $h \in C_c(G)$ and $x, y \in G$ (that is, $h_x = x^{-1}h$). More generally, if μ is any positive type Radon measure on H, then

$$\int_G P(f^* * f)(x)\, d\mu(x) \geqslant 0$$

for all $f \in C_c(G)$, so that P composed with μ is a positive type Radon measure on G.

Proof. Let μ be a complex Radon measure on H and let $f, g \in C_c(G)$. Then

$$\begin{aligned}
\int_H P(g^* * f)\, d\mu &= \int_H \gamma(s) \int_G \bar{g}(x) f(xs)\, dx\, d\mu(s) \\
&= \int_H \gamma(s) \int_G \bar{g}(x) f(xs) \int_H b(xt)\, dt\, dx\, d\mu(s) \\
&= \int_H \gamma(s) \int_H \int_G \bar{g}(xt^{-1}) f(xt^{-1}s)\, b(x)\, \Delta(t^{-1})\, dx\, dt\, d\mu(s) \\
&= \int_G b(x) \int_H \int_H \gamma(t^{-1})\, \bar{g}(xt^{-1})\, \delta(t^{-1})\, \gamma(t^{-1}s)\, f(xt^{-1}s)\, dt\, d\mu(s)\, dx \\
&= \int_G b(x) \int_H [(P(g_x))^* * P(f_x)](s)\, d\mu(s)\, dx.
\end{aligned}$$

It is easily seen that the support of the inner integral as a function of x is contained in

$$(\text{support of } g)H \cap (\text{support of } f)H,$$

which by assumption meets the support of b in a compact set, so that the outer integral is well defined. The earlier integrals are justified in a similar manner.

Now if μ is taken to be of positive type and $g = f$, then the last assertion of the Theorem follows from the above computation. (Note that the composition of P with μ is a Radon measure since it is clearly continuous for the inductive limit topology.) On the other hand if V is a unitary H-module, if $u, v \in V$, and if μ is taken to be the Radon measure $\langle su, v\rangle\, ds$, then we obtain the rest of the Theorem. Q.E.D.

The next obstacle which we encounter is the fact that because of the way in which we modified P in Notation 4.2, P no longer has the conditional expectation property (except when Δ and δ coincide on H). However, we can regain this property for the action of $C_c(H)$ on the right of $C_c(G)$ by redefining this action. In the process we lose the double centralizer property (retaining only the right centralizer property [29]), but this turns out not to matter. This device for handling the nonunimodular case was suggested to us by J. M. G. Fell.

Notation 4.5. For $f \in C_c(G)$ and $\phi \in C_c(H)$, we define $f \cdot \phi \in C_c(G)$ by

$$f \cdot \phi = f * (\gamma\phi),$$

where $\gamma\phi$ denotes the pointwise product of γ and ϕ.

Routine calculations then show that:

PROPOSITION 4.6. *Under the action* $(f, \phi) \mapsto f \cdot \phi$ *the algebra* $C_c(H)$ *acts as an algebra of right centralizers on* $C_c(G)$, *that is,* $C_c(G)$ *is a right* $C_c(H)$*-module such that*

$$(f * g) \cdot \phi = f * (g \cdot \phi)$$

for all $f, g \in C_c(G)$ *and* $\phi \in C_c(H)$. *Furthermore,* P *has the conditional expectation property with respect to this action, that is,*

$$P(f \cdot \phi) = P(f) * \phi$$

for all $f \in C_c(G)$ *and* $\phi \in C_c(H)$.

COROLLARY 4.7. *If we define a $C_c(H)$-valued inner-product on $C_c(G)$ by*

$$\langle f, g\rangle_{C_c(H)} = P(f^* * g),$$

then $C_c(G)$ becomes a right $C_c(H)$-rigged space (see Definition 2.8) with respect to the action established in Notation 4.5.

We did not say "preinner product" above because a routine calculation shows that:

PROPOSITION 4.8. *If $f \in C_c(G)$ and if $f \neq 0$ then $P(f^* * f) > 0$, that is, P is faithful. Thus the inner product defined above is definite.*

Of course, we are implicitly using here the fact that $C_c(H)$ is faithfully represented on $L^2(H)$.

The next obstacle which we encounter is the fact that in general P will not be continuous with respect to the norms on the pre-C^*-algebras $C_c(G)$ and $C_c(H)$. In the case in which H was an open subgroup of G the continuity of P was closely related to the fact that the left action of $C^*(G)$ on itself is by bounded operators with respect to the $C^*(H)$-inner product. However, in the present situation, P has a partial continuity property which is quite adequate for our purposes, namely, P is relatively bounded in a sense generalizing the definition of relatively bounded positive linear functionals given on p. 214 of [48].

PROPOSITION 4.9. *The map P is relatively bounded, that is, for any $g \in C_c(G)$ the positive map $f \mapsto P(g^* * f * g)$ is continuous with respect to the norms on the pre-C^*-algebras $C_c(G)$ and $C_c(H)$.*

This Proposition is a consequence of the following more general result which we will need somewhat later.

PROPOSITION 4.10. *If m is an element of $M_c(G)$, the space of finite Radon measures of compact support on G, (so that $m * f \in C_c(G)$ whenever $f \in C_c(G)$), then*

$$\langle m * f, m * f\rangle_{C_c(H)} \leqslant (\| m \|_{C^*(M(G))})^2 \langle f, f\rangle_{C_c(H)}$$

for all $f \in C_c(G)$. (Here $C^*(M(G))$ denotes the C^*-algebra obtained in the usual way from the integrated forms of all the unitary representations of G.) Furthermore,

$$\langle m * f, g\rangle_{C_c(H)} = \langle f, m^* * g\rangle_{C_c(H)},$$

for all $f, g \in C_c(G)$ and all $m \in M_c(G)$. Thus each element of $M_c(G)$, acting by convolution on the left, is a bounded operator on the $C_c(H)$-rigged space $C_c(G)$.

Proof. From Theorem 4.4, we know that $P(f^* * n^* * n * f)$ is a positive element of $C^*(H)$. Let p be a state of $C^*(H)$. Then p corresponds to a positive type function on H, which we can view as a positive type Radon measure on H. Then, also by Theorem 4.4, $p \circ P$ is a positive type measure on G, and so determines a unitary representation of G on the Hilbert space obtained by equipping $C_c(G)$ with the inner product $\langle\ ,\ \rangle_{p \circ P}$ (see 13.7.9 of [13]). The integrated form of this representation is still given by convolution. Thus

$$p(P(f^* * m^* * m * f)) = \langle m * f, m * f\rangle_{p \circ P}$$
$$\leqslant \| m \|^2_{C^*(M(G))} \langle f, f\rangle_{p \circ P} = (\| m \|_{C^*(M(G))})^2 \, p(P(f^* * f)).$$

Since this is true for all states p of $C^*(H)$, we obtain the first statement of the proposition, and we see that the operator of convolution on the left by m satisfies condition 1 of Definition 2.3. The statement concerning the adjoint of convolution by m is verified by a routine calculation. Finally, it is clear that the left action of m on $C_c(G)$ commutes with the right action of $C_c(H)$ on $C_c(G)$. Q.E.D.

Proof of Proposition 4.9. If $f, g \in C_c(G)$, then $P(g^* * f * g) = \langle g, f * g\rangle_{C_c(H)}$. An application of the generalized Cauchy–Schwartz inequality of Proposition 2.9 together with Proposition 4.10 then shows that

$$\| P(g^* * f * g)\|_{C^*(H)} \leqslant \| f \|_{C^*(G)} \| P(g^* * g)\|_{C^*(H)},$$

which gives the desired continuity. Q.E.D.

The final property of P which is of importance to us is the fact that the space of products of elements of $C_c(G)$ is dense in $C_c(G)$ with respect to the $C_c(H)$-inner product on $C_c(G)$, that is, with respect to the corresponding norm defined just before Proposition 2.10. In fact, a considerably stronger property holds, namely:

PROPOSITION 4.11. *There is a bounded self-adjoint approximate identity, e_k, for the pre-C^*-algebra $C_c(G)$ such that*

$$P((f - e_k * f)^* * (f - e_k * f))$$

converges to zero in $C^*(H)$ for all $f \in C_c(G)$.

Proof. Let e_k be an approximate identity of norm one for $L(G)$ of the usual type, consisting of nonnegative self-adjoint functions in $C_c(G)$ whose supports shrink down to the identity element of G. Then for $f \in C_c(G)$ it is easily seen that $(f - e_k * f)^* * (f - e_k * f)$ converges to 0 in the inductive limit topology. Since P is clearly continuous for the inductive limit topologies on $C_c(G)$ and $C_c(H)$, and since the inductive limit topology on $C_c(H)$ is stronger than the L^1-norm topology on $L(H)$, which in turn is stronger than the C^*-norm topology on $C_c(H)$, the desired property follows. Q.E.D.

Motivated by the above considerations, we make the following definition in analogy with Definition 1.3.

DEFINITION 4.12. Let A and B be pre-C^*-algebras, with B acting as an algebra of right centralizers on A. By a *generalized conditional expectation* from A to B we mean a linear map P from A to B satisfying

(1) P is self-adjoint, that is, $P(a^*) = P(a)^*$ for $a \in A$,

(2) P is positive, that is, $P(a^*a) \geqslant 0$ for $a \in A$,

(3) P satisfies the conditional expectation property, that is, $P(ab) = P(a)b$ for $a \in A$ and $b \in B$,

(4) P is relatively bounded, that is, for all $c \in A$ the map $a \mapsto P(c^*ac)$ is bounded,

(5) A^2 is P-dense in A, that is, for every $a \in A$ and every $\epsilon > 0$ there is a $c \in A^2$ (the linear span of products of elements of A) such that

$$\| P((a - c)^*(a - c))\|_B < \epsilon,$$

(6) The range of P generates B.

We will say that P is *faithful* if $P(a^*a) = 0$ only when $a = 0$. We will say that P is *smooth* if there exists a bounded self-adjoint approximate identity, e_k, for A such that

$$P((a - e_k a)^*(a - e_k a))$$

converges to zero for all $a \in A$.

If H is a closed subgroup of a locally compact group G, if $A = C_c(G)$ and $B = C_c(H)$, and if P and the action of B on A are defined as in Notations 4.2 and 4.5 above, then we have seen that P is a smooth faithful generalized conditional expectation from A to B.

EXAMPLE 4.13. Let A be a pre-C^*-algebra without unit and let p be a relatively bounded positive linear functional on A (see p. 214 of [48]). Let $B = \mathbb{C}$ acting in the obvious way on A, and define P by $P(a) = p(a)$. Then P will satisfy conditions 2, 3, 4, and 6 of Definition 4.12 (see Example 4.16). Thus if p is also Hermitian ($p(a^*) = (p(a))^-$) and A^2 is p-dense in A, then P will be a generalized conditional expectation.

EXAMPLE 4.14. Let A be a pre-C^*-algebra and let c be any positive element in the center of A (or more generally, in the center of the double centralizer algebra [8] of A). Let $B = Ac$ and define P by $P(a) = ac$ for $a \in A$. Then P is a generalized conditional expectation from A to B.

EXAMPLE 4.15. Let X be a locally compact space and let m be a positive Borel measure on X. Let $A = C_c(X \times X)$, viewed as kernels of integral operators on $L^2(X, m)$, so that the multiplication in A is composition of kernels and A is a pre-C^*-algebra of operators on $L^2(X, m)$. Let $B = C_c(X)$ with pointwise multiplication and sup-norm, and let B act on the right of A by

$$(Kf)(x, y) = K(x, y) f(y) \qquad x, y \in X$$

for $K \in A$ and $f \in B$. Let P be defined by

$$P(K)(x) = K(x, x).$$

We were led to this example by conversations with M. Sirugue.

We give next an example in which all the conditions except condition 5 hold.

EXAMPLE 4.16. Let A be the pre-C^*-algebra of complex-valued polynomials which vanish at 0, viewed as functions on $[0, 1]$. Let $B = \mathbb{C}$ acting in the obvious way on A, and define P by $P(f) = f''(0)$ (where f'' denotes the second derivative of f). Then P is easily seen to satisfy conditions 1–4 and 6 of Definition 4.12, but it has value zero on all products of three or more elements of A, and so does not satisfy condition 5 when a is the polynomial $f(x) = x$. (To obtain an example which satisfies conditions 2–5 but not 1, one can set $P(f) = (-1)^{1/2} f'(0)$.)

PROPOSITION 4.17. *Let A and B be pre-C^*-algebras with B acting as right centralizers on A, and let P be a generalized conditional expectation from A to B. If we define a pre-B-inner product on A by*

$$\langle a, a' \rangle_B = P(a^*a'),$$

then A becomes a right B-rigged space.

Proof. This is clear from properties 1, 2, 3, and 6 of Definition 4.12. Q.E.D.

We can thus form the algebra $L(A)$ as in Section 2, and also its quotient algebra $C = L(A)/J$. It is natural to ask whether the operator, L_a, of left multiplication on A by an element $a \in A$, is in $L(A)$, in analogy with Proposition 3.2. We will see now that the relative boundedness of P is exactly what is needed to ensure that this is the case. This result is just a generalization of Theorem 4.5.2 of [48], in which it is shown that a relatively bounded positive linear functional is "admissible." In fact, this theorem provides the heart of the proof of our assertion.

PROPOSITION 4.18. *Let A, B, and P be as above, so that A is a right B-rigged space. Then for all $a \in A$ the operator L_a is in $L(A)$. In fact, L_{a^*} is an adjoint for L_a and $\| L_a \| \leqslant \| a \|$. Thus we have a continuous $*$-homomorphism of A into $L(A)$ and $C = L(A)/J$.*

Proof. Let p be any state of B. Then it is easily seen that $p \circ P$ is a relatively bounded positive linear functional on A. Thus Theorem 4.5.2 of [48] is applicable, and we conclude that

$$(p \circ P)(c^*dc) \leqslant \nu(d)(p \circ P)(c^*c),$$

for all $c, d \in A$, where $\nu(d) = \lim \| d^n \|^{1/n}$. Since this is true for all states, and since $\nu(a^*a) = \| a \|^2$, we conclude that if we set $d = a^*a$ we obtain

$$P(c^*a^*ac) \leqslant \| a \|^2 P(c^*c).$$

But this is just condition 1 of Definition 2.3, as well as the fact that $\| L_a \| \leqslant \| a \|$. A routine calculation shows that L_{a^*} is an adjoint for L_a. Q.E.D.

This leads us to make the following definition.

DEFINITION 4.19. Let A and B be pre-C^*-algebras. By a *left* *pre-Hermitian B-rigged A-module* we mean a right B-rigged space X which is a left A-module by means of a continuous $*$-homomorphism of A into $L(X)$ (so that a^* goes to some adjoint of the image of a), which is nondegenerate in the sense that AX is dense in X with respect to the B-seminorm on X (defined just before Proposition 2.10). If the B-preinner product on X is definite and if X is complete with respect to the B-norm, then we will call X a *Hermitian B-rigged A-module*. Right pre-Hermitian left B-rigged A-modules are defined similarly. If $B = \mathbb{C}$ (the complex numbers), we will say simply *pre-Hermitian A-module* rather than pre-Hermitian $\mathbb{C}$-rigged A-module.

EXAMPLE 4.20. Let P be a generalized conditional expectation from A to B. Then A becomes a pre-Hermitian B-rigged A-module. This is clear from Proposition 4.17 together with the observation that condition 5 in Definition 4.12 provides the required nondegeneracy.

EXAMPLE 4.21. In this example, we indicate one way of handling projective representations by our techniques. This method involves the twisted group-algebras associated with projective representations, and so may be applicable to certain more general situations in which induced representations have been defined [9, 31]. However, in the nonseparable case there are difficulties with twisted group algebras. For example, it does not seem to be known in the nonseparable case whether or not the projective representations for a given cocycle are in bijective correspondence with the nondegenerate $*$-representations of the corresponding twisted group algebra. For this reason we will indicate at the end of Section 7 another method for handling projective representations which does work smoothly in the nonseparable case, although this other method is probably not applicable to contexts such as those found in [9, 31].

Let G be a locally compact group, and let c be a normalized Baire 2-cocycle (i.e., multiplier) with values in T, the group of complex numbers of modulus one (for the definition of Baire 2-cocycles see [49]). We can then define the c-twisted measure algebra, $M(G, c)$, of G, which is the space of finite Radon measures on G with twisted convolution and involution given by

$$m *_c n(f) = \int f(xy)\, c(x, y)\, dm(x)\, dn(y),$$

$$m^*(f) = \left(\int \bar{f}(x^{-1})\, \bar{c}(x, x^{-1})\, dm(x)\right)^{-},$$

for $m, n \in M(G)$ and $f \in C_c(G)$. The c-twisted group algebra, $L(G, c)$, of G is then taken to be the two-sided ideal of $M(G, c)$ consisting of the measures which are absolutely continuous with respect to Haar measure on G.

If H is a closed subgroup of G, then c restricts to a normalized Baire 2-cocycle on H, and the c-twisted group algebra, $L(H, c)$, of H can be identified with a subalgebra of $M(G, c)$. In particular, it acts as double centralizers on $L(G, c)$.

We can define a c-projective representation of G to be a continuous $*$-representation of $L(G, c)$, and similarly for H. In the separable case anyway, these correspond to ordinary c-projective representations of G and H for which Mackey has shown how to define induced representations [38]. We can form the C^*-completions of $L(G, c)$ and $L(H, c)$, and so obtain the c-twisted C^*-algebras $C^*(G, c)$ and $C^*(H, c)$.

To put this situation within our general framework, it is natural to try to define a generalized conditional expectation from a dense subalgebra of $C^*(G, c)$ to a dense subalgebra of $C^*(H, c)$. Unfortunately, $C_c(G)$ need no longer be a subalgebra of $L(G, c)$, since c is not in general continuous. Accordingly, we try to find a slightly larger submanifold of $C^*(G)$ which will be a subalgebra. Let $K(G, c)$ denote the collection of (equivalence classes of) bounded measurable functions of compact support on G. Then it is easily seen that $K(G, c)$ is a dense $*$-subalgebra of $C^*(G, c)$. We can also form $K(H, c)$, and it is easily seen that the action of $K(H, c)$ on $L(G, c)$ carries $K(G, c)$ into itself. Thus we have two pre-C^*-algebras, with one acting as double centralizers on the other.

However, if we try to define a generalized conditional expectation by restricting functions in $K(G, c)$ to H, we find that this process need not be well-defined because H can be a null set with respect to Haar measure on G. But the situation can be salvaged by considering instead the $K(H, c)$-valued inner product which we would have liked to define in terms of such a generalized conditional expectation. Specifically, if $f, g \in K(G, c)$, let

$$\langle g, f\rangle_{K(H,c)}(s) = \gamma(s) \int_G g^*(sy) f(y^{-1})\, c(sy, y^{-1})\, dy.$$

Then it is clear that if we change f or g on a null set, this function does not change at all, and so it determines a well-defined element of $K(H, c)$.

If one redefines the right action of $K(H, c)$ on $K(G, c)$ by $f \cdot \phi = f *_c \gamma\phi$ for $f \in K(G, c)$ and $\phi \in K(H, c)$, then it is not difficult to imitate the development in the first part of this section to show that $K(G, c)$ becomes a right $K(H, c)$-rigged space. Furthermore, at least in the separable case where $K(G, c)$ can be shown to have an approximate identity, $K(G, c)$ becomes a $K(H, c)$-rigged $K(G, c)$-module. The application of the results of the next section to this example will then yield the induced projective representations of Mackey [38].

We remark that a possible alternate approach to these results might be to use the results of [49] to construct the corresponding Banach $*$-algebraic bundle of Fell [19], and then to try to show that it has enough continuous cross sections. If this latter can be shown, then one can again define a generalized conditional expectation (whose domain will consist of these continuous cross sections).

EXAMPLE 4.22. Let B and D be pre-C^*-algebras with B acting as right centralizers on D, and let P be a generalized conditional expectation from D to B. Let A be a $*$-subalgebra of D for which AD is dense in D with respect to the pre-B-inner product on D. Then D becomes a pre-Hermitian B-rigged A-module.

The following example is closely related to the one above.

EXAMPLE 4.23. Let G be a locally compact group, and let H and K be closed subgroups of G. Let P be defined from $C_c(G)$ to $C_c(H)$ as before, so that $C_c(G)$ is a $C_c(H)$-rigged space. Let $A = C_c(K)$, acting by convolution on the left of $C_c(G)$. Then from Proposition 4.10 and arguments similar to those in the proof of Proposition 4.11, it follows that $C_c(G)$ is a pre-Hermitian $C_C(H)$-rigged $C_c(K)$-module. This example is closely related to Mackey's subgroup theorem (Theorem 12.1 of [36]).

The following example is closely related to the one above.

EXAMPLE 4.24. The generalized induced representations of Moscovici [43] can be handled by our techniques, though as in Example 4.21, no generalized conditional expectation can in general be defined, and we must instead begin directly with a B-inner product. To be specific, if we use Moscovici's notation except that we write Z instead of X for his locally compact space, we define a $C_c(H)$-inner product on $C_c(Z)$ by

$$\langle g, f\rangle_{C_c(H)}(s) = (\Delta(s)/\Delta_H(s))^{1/2} \int_Z \bar{g}(z) f(zs)\, d\mu(z).$$

Then one can imitate the development in the earlier part of this section to show that $C_c(Z)$ becomes a pre-Hermitian $C_c(H)$-rigged $C_c(G)$-module. The application of the results of the next section to this example then yield the generalized induced representations of Moscovici. These can be used to generalize ordinary induced representations to the situation in which rather than H being a subgroup of G, one has instead a homomorphism of H into G satisfying suitable hypotheses.

EXAMPLE 4.25. Let A be a pre-C^*-algebra and let L be a left ideal in A. Let B be the subalgebra L^*L of A, and define a B-valued inner product on L by

$$\langle a, c\rangle_B = a^*c.$$

Then it is easily seen that L becomes a pre-Hermitian B-rigged A-module. This example is a generalization of the pre-Hermitian B-rigged A-module coming from the conditional expectation in Example 1.6, as can be seen by letting $L = Ai$ in that example. We will see in Example 6.8 that this situation is related to results in Godement's paper [25]. Furthermore, if $L = I$ is a two-sided ideal in A, so that $B = L$, then it is easily verified that when the results of the next section are applied to this example, the induced representations so obtained coincide with the representations of A defined in Proposition 3.9.

EXAMPLE 4.26. Let D be a C^*-algebra, and let W and W' be Hermitian D-modules. Let $X = \mathrm{Hom}_D(W, W')$, and let B be the two-sided ideal of $\mathrm{End}_D(W) = \mathrm{Hom}_D(W, W)$ spanned by operators of the form T^*S for $S, T \in X$, with B acting on the right of X by composition of operators. Define a B-inner product on X by

$$\langle S, T\rangle_B = S^*T$$

for $S, T \in X$. Then X becomes a B-rigged space. Let $A = \mathrm{End}_D(W')$, acting on the left of X by composition of operators. Then X becomes a pre-Hermitian B-rigged A-module.

We conclude this section by indicating some alternate formulations of the conditions of Definition 4.19. To begin with, the continuity condition on the $*$-homomorphism from A into $L(X)$ can be reformulated in a manner familiar from the case of ordinary representations. This alternate formulation was suggested to us by J. M. G. Fell.

PROPOSITION 4.27. *Let A and B be pre-C^*-algebras, and let X be a* *right B-rigged space. Suppose that X is a left A-module in such a way that*

$$\langle ax, y\rangle_B = \langle x, a^*y\rangle_B,$$

for all $x, y \in X$, and $a \in A$. Then the following two conditions are equivalent:

(1) $\langle ax, ax\rangle_B \leqslant \| a \|^2 \langle x, x\rangle_B$ *for all $a \in A$ and $x \in X$.*

(2) *For all $x \in X$ the linear map $a \mapsto \langle x, ax\rangle_B$ from A into B is continuous.*

Proof. To show that condition 1 implies condition 2, we apply the generalized Cauchy–Schwartz inequality of Proposition 2.9 to condition 1 to find that

$$\| \langle x, ax\rangle_B \| \leqslant \| a \| \, \| \langle x, x\rangle_B \|$$

for all $a \in A$ and $x \in X$.

Suppose conversely that condition 2 holds. Let $\tilde{A}$ denote the pre-C^*-algebra obtained by adjoining an identity element, 1, to A (see 1.3.8 of [13]), with the obvious action of $\tilde{A}$ on X. Then for any $x \in X$, the map $a \mapsto \langle x, ax\rangle_B$ from $\tilde{A}$ into B will also be continuous, and this map is easily seen to be positive in the sense that $\langle x, a^*ax\rangle_B \geqslant 0$ for all $a \in \tilde{A}$ and $x \in X$. Then it extends to a continuous positive map of the completion of $\tilde{A}$ into the completion of B. But in the completion every positive element has a positive square root. It follows that $\langle x, ax\rangle_B \geqslant 0$ for any positive element $a \in \tilde{A}$. Now $\| a^*a \| 1 - a^*a$ is a positive element of $\tilde{A}$, and so

$$\langle x, (\| a^*a \| 1 - a^*a)x\rangle_B \geqslant 0,$$

which is condition 1. Q.E.D.

We show next that the nondegeneracy condition of Definition 4.19 can be given a different formulation.

PROPOSITION 4.28. *Let A and B be pre-C^*-algebras, let X be a B-rigged space, and let there be given a continuous $*$-homomorphism of A into $L(X)$, so that X is a left A-module. Then the following two conditions are equivalent*

(1) *AX is dense in X with respect to the B-norm on X, so that X is a pre-Hermitian B-rigged A-module.*

(2) *For any bounded self-adjoint approximate identity,* e_k, *for* A *and for any* $x \in X$ *the net* $e_k x$ *converges to* x *in the* B*-norm on* X.

Proof. It is clear that condition 2 implies condition 1. The proof that condition 1 implies condition 2 is the same as the proof of Proposition 3.4 in [51].

5. Induced Representations—The General Case

In this section we show how to use pre-Hermitian B-rigged A-modules to induce representations of B to representations of A, and we derive a few properties of these induced representations. We then show that in the case of the pre-Hermitian B-rigged A-module associated with a locally compact group and a closed subgroup in Section 4, the corresponding induced representations are equivalent to the induced representations of Mackey [35] as generalized to the possibly non-separable case by Blattner [2].

Representations induced by means of a pre-Hermitian B-rigged A-module are defined in analogy with those defined in Theorem 1.8.

Theorem 5.1. *Let* A *and* B *be pre-*C^**-algebras, and let* X *be a pre-Hermitian* B*-rigged* A*-module. For any Hermitian* B*-module,* V, *define a sesquilinear form on* $X \otimes_B V$ *whose value on elementary tensors is given by*

$$\langle x \otimes v, x' \otimes v' \rangle = \langle \langle x', x \rangle_B v, v' \rangle.$$

Then this form is a preinner product. Furthermore, the action of $L(X)$ *on* $X \otimes_B V$ *which is defined on elementary tensors by*

$$T(x \otimes v) = (Tx) \otimes v$$

for $T \in L(X)$, $x \in X$, $v \in V$ *is an action by bounded operators of norm no greater than* $\| T \|$ *with respect to this preinner product, and any adjoint of* T *will act as an adjoint of* T *on* $X \otimes_B V$. *In this way we obtain a continuous nondegenerate* $*$*-representation of the quotient* C^**-algebra,* $C = L(X)/J$, *on the corresponding Hilbert space,* ${}^A V$. *When restricted to (the image in* $L(X)$ *of)* A, *this representation is still nondegenerate, so that* ${}^A V$ *becomes a Hermitian* A*-module.*

Proof. Except for obvious changes of notation, the proof that the indicated sesquilinear form is a preinner product is the same as the proof of Lemma 1.7, and the proof of the statement concerning the action of $L(X)$ is the same as the proof of Proposition 3.1. Thus what remains to be shown is that as an A-module ${}^AV$ is nondegenerate. The proof of this fact turns, of course, on the fact that X is assumed to be a nondegenerate A-module. Let an elementary tensor $x \otimes v$ in ${}^AV$ and an $\epsilon > 0$ be given. By assumption, we can find $y = \sum a_i x_i$ in X such that $\| x - y \|_B < \epsilon / \| v \|$. Then a routine calculation shows that

$$\left\| x \otimes v - \sum a_i(x_i \otimes v) \right\| < \epsilon.$$

Thus every elementary tensor, and so every element of ${}^AV$, is in the closure of $A({}^AV)$. Finally, ${}^AV$ is not the zero-dimensional A-module, for if it were, we would have $\langle \langle x', x \rangle_B v, v' \rangle = 0$ for all $x, x' \in X$ and $v, v' \in V$, which is impossible since the range of $\langle\ ,\ \rangle_B$ is assumed to span a dense subspace of B. Q.E.D.

DEFINITION 5.2. The Hermitian A-module ${}^AV$ is called the *Hermitian A-module obtained by inducing V from B to A via X*. If there is a possibility of confusion about which pre-Hermitian B-rigged A-module is being used, we will write ${}^A_XV$ instead of ${}^AV$.

THEOREM 5.3. *Let A and B be pre-C^*-algebras, let X be a pre-Hermitian B-rigged A-module, and let I_X denote the identity map of X onto itself. Let V and V′ be Hermitian B-modules, and let $T \in \mathrm{Hom}_B(V, V')$. Then the A-linear map*

$$I_X \otimes T\colon\ X \otimes_B V \to X \otimes_B V'$$

is bounded with respect to the preinner products on these two spaces, and so determines a bounded A-homomorphism ${}^AT$ of ${}^AV$ into ${}^AV'$. The correspondence $V \mapsto {}^AV$, $T \mapsto {}^AT$ is a functor from the category of Hermitian B-modules to the category of Hermitian A-modules.

Proof. Let $t \in X \otimes_B V$. We need to show that

$$\| T \|^2 \langle t, t \rangle \geqslant \langle (I_X \otimes T)t, (I_X \otimes T)t \rangle$$

that is, that

$$\langle [I_X \otimes (\| T \|^2 - T^*T)]t, t \rangle \geqslant 0.$$

But if $S = (\| T \|^2 - T^*T)^{1/2}$, this becomes

$$\langle (I_X \otimes S)t, (I_X \otimes S)t \rangle,$$

which is clearly $\geqslant 0$. The verification of the remaining assertions is routine. Q.E.D.

As with Corollary 1.10, the proof of Theorem 5.1 shows that:

COROLLARY 5.4. *If* $V = \oplus V_k$, *then* ${}^A V = \oplus {}^A V_k$.

A similar assertion can be made about direct integrals along the lines of Theorem 10.1 of [36].

In the case of representations of B which are obtained from a state of B, the construction of induced representations has a simple alternate description (related to Theorem 1 of [4]) generalizing Proposition 1.15. The proof is the same as the proof of Proposition 1.15 except for obvious changes of notation.

COROLLARY 5.5. *Let X be a pre-Hermitian B-rigged A-module, let p be a state of B, and let V_p denote the corresponding Hermitian B-module. Define an ordinary preinner product,* $\langle\ ,\ \rangle_p$, *on X by*

$$\langle x, y \rangle_p = p(\langle y, x \rangle_B),$$

and let X_p denote the corresponding Hilbert space. Under the action of A on X, X_p becomes a Hermitian A-module, which is unitarily equivalent to ${}^A(V_p)$.

For later use, we now make precise some notions related to unitary equivalence of modules, some of which we have already used implicitly in some simple cases.

DEFINITION 5.6. Let X and Y be pre-Hermitian B-rigged A-modules. Then X is said to be *preequivalent* to Y if there is an A–B-linear map R from X to Y which preserves the pre-B-inner products on the two spaces, that is,

$$\langle Rx, Rx' \rangle_B = \langle x, x' \rangle_B$$

for all $x, x' \in X$, and which has dense range in Y with respect to the B-seminorm on Y.

We remark that the above relation is not an equivalence relation. The following lemma has a routine proof.

LEMMA 5.7. *Let X and Y be pre-Hermitian A-modules (so $B = \mathbb{C}$), and let R be an A-linear map from X to Y under which X is preequivalent to Y. Then R defines a unitary equivalence between the corresponding Hermitian A-modules obtained from X and Y by taking the quotients by the subspaces of vectors of length zero and completing.*

PROPOSITION 5.8. *If X and Y are pre-Hermitian B-rigged A-modules which are preequivalent, and if V is a Hermitian B-module, then ${}^A_X V$ and ${}^A_Y V$ are unitarily equivalent Hermitian A-modules. In fact, the functors $V \mapsto {}^A_X V$ and $V \mapsto {}^A_Y V$ are equivalent.*

Proof. This follows from the fact that the pre-Hermitian A-modules $X \otimes_B V$ and $Y \otimes_B V$ will then be preequivalent, so that Lemma 5.7 is applicable. Q.E.D.

For representations induced by means of pre-Hermitian rigged modules, the theorem on induction in stages has a quite simple form.

THEOREM ON INDUCTION IN STAGES 5.9. *Let A, B and C be pre-C^*-algebras, let X be a pre-Hermitian B-rigged A-module, and let Y be a pre-Hermitian C-rigged B-module. Then $Z = X \otimes_B Y$, with the obvious actions of A and C and with the pre-C-inner product defined on elementary tensors by*

$$\langle x \otimes y, x' \otimes y' \rangle_C = \langle \langle x', x \rangle_B y, y' \rangle_C ,$$

is a pre-Hermitian C-rigged A-module. If U is any Hermitian C-module, then ${}^A_Z U$ is unitarily equivalent to ${}^A_X({}^B_Y U)$. In fact, the functor $U \mapsto {}^A_Z U$ is equivalent to the composition of functors $U \mapsto {}^A_X({}^B_Y U)$.

Proof. The fact that the indicated pre-C-inner product is non-negative is seen by applying states of C so that the arguments in the proof of Lemma 1.7 are applicable. The fact that the A-module Z is nondegenerate is verified by the same arguments as were used towards the end of the proof of Theorem 5.1. The rest of the proof that Z is a pre-Hermitian C-rigged A-module follows from routine calculations.

Now if U is a Hermitian C-module, then the A-linear map from $X \otimes_B (Y \otimes_C U)$ to $Z \otimes_C U$ which on elementary tensors is given by

$$x \otimes (y \otimes u) \mapsto (x \otimes y) \otimes u$$

is easily seen to preserve the preinner products on these spaces and have dense range, so that Lemma 5.7 is applicable. Q.E.D.

Unfortunately, the theorem on induction in stages for generalized conditional expectations, which is closer to the theorem for the case of locally compact groups, does not work quite as smoothly as that for pre-Hermitian rigged modules, the reason being that the composition of two generalized conditional expectations need no longer be a generalized conditional expectation. This is shown by the following example.

EXAMPLE 5.10. Let A be the algebra of complex polynomials which vanish at 0, viewed as a dense subalgebra of the C^*-algebra of continuous functions on $[-1, 1]$ which vanish at 0, let B be the principle ideal in A generated by the polynomial t^3. Let P be defined by $P(f)(t) = t^2 f(t)$ for all $f \in A$ (a special case of Example 4.15). Let $C = \mathbb{C}$ acting in the obvious way on A and B, and let $Q: B \to C$ be defined by $Q(f)(t) = f^{(4)}(0)$ for all $f \in B$, where $f^{(4)}$ is the fourth derivative of f. Then it is easily verified that Q is a generalized conditional expectation, but that $Q \circ P$ is not, because it is basically Example 4.16 and so is degenerate.

We have no reason to believe that the composition of generalized conditional expectations must satisfy conditions 3 or 4 of Definition 4.12 either, but we do not have examples where these fail.

We do not know how to strengthen the definition of generalized conditional expectation so as to avoid pathologies of the above kind and yet still have it apply to the case of locally compact groups. One possibility would be to require that a generalized conditional expectation be lower semicontinuous in analogy with the condition used in [11]. This is the same as requiring that P have the Lebesgue dominated convergence property, namely, that if a_n is a sequence of positive elements of A, if a is an element of A such that $a \geqslant a_n$ for all n, and if a_n converges to zero in norm, then $P(a_n)$ converges to zero in norm. The composition of lower semicontinuous generalized conditional expectations will automatically satisfy condition 5 of Definition 4.12 as can be seen by using the arguments employed in the proof of Lemma 2.1 of [11]. (This does not take care of conditions 3 and 4, however.) But we have been unable to determine whether the generalized conditional expectation associated with a locally compact group and a closed subgroup must be lower semicontinuous. More basically, we have been unable to determine whether any positive type Radon measure on a locally compact group G defines a lower semicontinuous linear functional on the pre-C^*-algebra $C_c(G)$.

However, if we simply assume that the composition of the generalized conditional expectations involved is again a generalized conditional expectation and is suitably well behaved, then we do again have a theorem on induction in stages.

THEOREM ON INDUCTION IN STAGES, 2ND VERSION 5.11. *Let A, B and C be pre-C^*-algebras, with B acting as right centralizers on A, and with C acting as right centralizers on both A and B in such a way that $(ab)c = a(bc)$ for all $a \in A$, $b \in B$, $c \in C$. Let P be a generalized conditional expectation from A to B and let Q be a generalized conditional expectation from B to C. Suppose that $R = Q \circ P$ is a generalized conditional expectation from A to C, and that AB is dense in A with respect to the C-seminorm on A coming from R (which will be the case if Q is smooth). Then for any Hermitian C-module U, the induced Hermitian A-modules ${}^AU$ and ${}^A({}^BU)$ are unitarily equivalent. In fact, the functors $U \mapsto {}^AU$ and $U \mapsto {}^A({}^BU)$ are naturally equivalent.*

Proof. As in Theorem 1.16 and Theorem 5.9, a routine calculation shows that the A-linear map from $A \otimes_B (B \otimes_C U)$ to $A \otimes_C U$ defined on elementary tensors by

$$a \otimes (b \otimes u) \mapsto ab \otimes u$$

preserves the preinner products on these two spaces. However, when we try to imitate the proof of Theorem 1.16 to show that this map has dense range we find that, since Q is no longer assumed continuous, we need to know that AB is R-dense in A (which is easily seen to be true if Q is smooth). (However, we do not know of an example in which AB fails to be R-dense in A.) We can now apply Lemma 5.7. Q.E.D.

It is easily verified that if K is a closed subgroup of the closed subgroup H of G, then the generalized conditional expectation from $C_c(G)$ to $C_c(K)$ is the composition of those from $C_c(G)$ to $C_c(H)$ and from $C_c(H)$ to $C_c(K)$ (which are smooth). From this it will follow easily that the theorem on induction in stages for Mackey's induced representations is a special case of the above theorem, once we have shown that the induced representations of Mackey are a special case of the induced representations defined in this section.

Turning to another property of induced representations, we remark that Theorem 3.2 of [27] concerning the existence of cyclic vectors can be generalized to the present setting if (as there) appropriate hypotheses are made concerning separability. To prove the analogue of Lemma 3.3 of [27] one needs to assume that the generalized conditional expectation involved is smooth. See also [69, 70], although the proof in [69] is incorrect because it depends on some incorrect assertions in [16] as was pointed out in [72].

We conclude this section by showing that for the case of the generalized conditional expectation P associated in the previous section with a locally compact group G and a closed subgroup H, the induced representations defined above coincide with Blattner's generalization [2] of the induced representations of Mackey [36] to the possibly nonseparable case.

Let V be a Hermitian $C_c(H)$-module. According to our definition, the corresponding induced representation (which we will denote by ${}^G V$) has as its space the Hilbert space obtained by equipping $C_c(G) \otimes_{C_c(H)} V$ (where the right action of $C_c(H)$ on $C_c(G)$ is that defined in Notation 4.5) with the preinner product which is defined on elementary tensors by

$$\langle f \otimes v, g \otimes v' \rangle = \langle P(g^* * f)v, v' \rangle.$$

We would like to realize this space as a space of continuous V-valued functions on G. A strong clue as to how this may be done is contained in the formula of Theorem 4.4. Define a bilinear map, π, of $C_c(G) \times V$ into the space of continuous V-valued functions on G by

$$\pi(f, v)(x) = P(f_x)v = \int_H \gamma(s) f(xs)\, sv\, ds.$$

This is essentially the map used by Blattner at the top of p. 82 of [2], and is a variant of a map first introduced by Weil in the case of compact groups (p. 83 of [65]) and by Mackey in the general case (in §3 of [36]). For further comments on this map see p. 484 of [51]. In Lemma 2 of [2] Blattner shows that the linear span of the range of this map is dense in the space of the induced representation which he defines there, which we will denote here by U^V. (He used right Haar measures instead of left Haar measures, so in what follows we have changed his definitions accordingly.)

THEOREM 5.12. *The bilinear map π defined above is $C_c(H)$-balanced, and so lifts to a linear map, which we also denote by π, from $C_c(G) \otimes_{C_c(H)} V$ onto a dense submanifold of U^V. This linear map π preserves the preinner* *products on these two spaces, and intertwines the action of G on these two spaces. Thus π defines a unitary equivalence from ${}^GV$ to U^V.*

Proof. The fact that π is $C_c(H)$-balanced is verified by a routine calculation (using Notation 4.5). If $f, g \in C_c(G)$ and $v, v' \in V$, then from Theorem 4.4 we see that

$$\langle f \otimes v, g \otimes v' \rangle = \int_G b(x) \langle \pi(f \otimes v)(x), \pi(g \otimes v')(x) \rangle \, dx.$$

By the sesquilinearity of both sides, it follows that the same result holds not just for elementary tensors, but for arbitrary tensors also.

Suppose now that we are given an arbitrary tensor $\sum f_i \otimes v_i$ and that $F \in U^V$ is defined by $F(x) = \pi(\sum f_i \otimes v_i)(x)$. Then according to Blattner's definition, the norm of $F \in U^V$ is determined as follows: for any $g \in C_c(G)$ define $\tau(g) \in C_c(G/H)$ by

$$\tau(g)(\dot{x}) = \int_H g(xs) \, ds,$$

so that τ is a linear map of $C_c(G)$ onto $C_c(H)$ [6]. Let the continuous real-valued function $x \to \|F(x)\|^2$ define a finite positive Radon measure, μ_F, on G/H whose value on any function of the form $\tau(g)$ in $C_c(G/H)$ is given by

$$(*) \qquad \int_{G/H} \tau(g)(\dot{x}) \, d\mu_F(\dot{x}) = \int_G g(x) \|F(x)\|^2 \, dx.$$

Then by definition $\|F\|^2 = \mu_F(G/H)$. Now the formula $(*)$ is easily seen to be true also for appropriate non-negative functions g on G which need not have compact support, but for which $\tau(g)$ is everywhere finite. But b is such a function, and in fact by its definition $\tau(b) \equiv 1$. Thus

$$\begin{aligned} \left\langle \pi\left(\sum f_i \otimes v_i\right), \pi\left(\sum f_j \otimes v_j\right)\right\rangle &= \|F\|^2 \\ &= \int_{G/H} d\mu_F \\ &= \int_G b(x) \langle F(x), F(x) \rangle \, dx \\ &= \left\langle \sum f_i \otimes v_i, \sum f_j \otimes v_j \right\rangle. \end{aligned}$$

By polarization it follows that π preserves preinner products. (If one prefers not to use functions which are not in $C_c(G)$, then b can be truncated so that $b \in C_c(G)$ and yet $\int_H b(xs)\, ds = 1$ for all x in the union of the supports of the f_i, much as Blattner does in his proof of Theorem 1 of [2]). The fact that π intertwines the action of G follows from the easily verified identity

$$x(\pi(f \otimes v)) = \pi(xf \otimes v)$$

for all $x \in G, f \in C_c(G), v \in V$. Q.E.D.

6. Imprimitivity Bimodules

In this section we will formulate and prove the imprimitivity theorem for the induced representations of C^*-algebras which were defined in the previous section. In the process of doing this we will be led to consider the question of when two C^*-algebras have equivalent categories of Hermitian modules, in analogy with the Morita theorems ([42], Chapter 2 of [1], [10]), and we will obtain a general method for constructing pairs of C^*-algebras whose categories of Hermitian modules are equivalent (Theorem 6.23). We doubt that all pairs of C^*-algebras whose categories of Hermitian modules are equivalent arise by means of this construction, but we are hopeful that an appropriate generalization of this construction will yield all such pairs. We plan to investigate this matter at a later time.[2]

We begin by considering a pre-C^*-algebra B and a B-rigged space X. Until further notice, we will assume that the pre-B-inner product on X is definite, so that it is a B-inner product (which corresponds to assuming that the generalized conditional expectations to which we apply these results are faithful, as is the case for those coming from a locally compact group and a closed subgroup). We make this assumption only for the convenience of having $L(X)$ contain no nonzero operators of norm zero, so that it is already a pre-C^*-algebra. The results of this section could be carried through without this assumption if we changed the definition of a pre-C^*-algebra to permit nonzero elements of norm zero (and nonuniqueness of adjoints), but we do not feel that going through the complications introduced by such a change is warrented until specific examples arise for which such a generalization would have interesting consequences.

[2] See [80].

If A is a pre-C^*-algebra and if X is in addition a pre-Hermitian B-rigged A-module, and if V is a Hermitian B-module, then ${}^AV$ will be not only a Hermitian A-module, but in fact a Hermitian $L(X)$-module. This gives a necessary condition for a Hermitian A-module to be induced from B via X, but this condition is not in general sufficient, as was seen already in Section 3 for the special case of representations induced via an ordinary conditional expectation. We thus need to look for an analog of the imprimitivity algebra E of Definition 3.6. For this purpose, the action of A on X is irrelevant, so we revert now to assuming only that X is a B-rigged space.

Now $L(X)$ is the natural analog for B-rigged spaces of the algebra of all bounded operators on an ordinary Hilbert space. In view of this, it is natural to look for the analog of the two-sided ideal of compact operators. Now the ideal of compact operators is generated by the operators of rank one, each of which is in turn determined by a pair of vectors in the Hilbert space. But in the present setting, there is an obvious analog of these operators of rank one. For any $x, y \in X$ we let $T_{(x,y)}$ be the operator on X defined by

$$T_{(x,y)}z = x\langle y, z\rangle_B \tag{6.1}$$

for all $z \in X$. The only thing which needs to be verified is that $T_{(x,y)}$ is in fact in $L(X)$.

LEMMA 6.2. *For any $x, y \in X$ the operator $T_{(x,y)}$ is in $L(X)$. In fact,*

$$\| T_{(x,y)} \| \leqslant \| x \|_B \| y \|_B$$

(*where these are the norms defined before Proposition* 2.10) *and the adjoint of $T_{(x,y)}$ is $T_{(y,x)}$.*

Proof. For any $z \in X$, we have

$$\begin{aligned}\langle T_{(x,y)}z, T_{(x,y)}z\rangle_B &= \langle x\langle y, z\rangle_B, x\langle y, z\rangle_B\rangle_B \\ &= \langle y, z\rangle_B^* \langle x, x\rangle_B \langle y, z\rangle_B \\ &\leqslant \|\langle x, x\rangle_B\| \langle y, z\rangle_B^* \langle y, z\rangle_B .\end{aligned}$$

Applying the generalized Cauchy–Schwartz inequality of Proposition 2.9 we see that this is

$$\leqslant \|\langle x, x\rangle_B\| \, \|\langle y, y\rangle_B\| \, \langle z, z\rangle_B,$$

which shows that condition 1 of Definition 2.3 and the statement concerning norms are satisfied. The statement concerning the adjoint of $T_{(x,y)}$ is verified by a routine computation, as is the fact that $T_{(x,y)}$ commutes with the right action of B. Q.E.D.

PROPOSITION 6.3. *Let E denote the linear span of the set of operators in $L(X)$ of the form $T_{(x,y)}$, $x, y \in X$. Then E is a two-sided ideal in $L(X)$. In fact,*

$$ST_{(x,y)} = T_{(Sx,y)} \quad \text{and} \quad T_{(x,y)}S = T_{(x,S^*y)}$$

for all $S \in L(X)$ and $x, y \in X$.

Proof. This fact, which is familiar from the case of ordinary Hilbert spaces, and of which Proposition 3.5 is a special case, is verified by routine computations. Q.E.D.

DEFINITION 6.4. We will call E the *imprimitivity algebra* of the B-rigged space X.

We remark that $T_{(x,y)}$ is an operator of B-rank one in the obvious imprecise sense, but that not all operators of B-rank one are of this form (for example y could instead be taken from the completion of X). Also, strictly speaking, E is more the analog of the two-sided ideal of finite rank operators rather than of compact operators. But little would be gained by taking the closure of E in $L(X)$ since $L(X)$ need not be complete.

It is easily seen that the algebra E defined in Definition 3.6 is a special case of the algebra E defined above.

Now X is of course a left E-module. It is profitable for us to make the roles of E and B symmetric by defining an E-valued preinner product on X, so that X becomes a left E-rigged space.

PROPOSITION 6.5. *For any $x, y \in X$ let*

$$\langle x, y\rangle_E = T_{(x,y)} .$$

Then $\langle\ ,\ \rangle_E$ is an E-inner product (conjugate linear in the second variable), with respect to which X becomes a left E-rigged space. The E and B inner products on X are related by the associativity relation

$$\langle x, y\rangle_E z = x\langle y, z\rangle_B .$$

Proof. We must show that $\langle x, x\rangle_E$ is a positive element of $L(X)$ for all $x \in X$. Now for any $y \in X$, a routine computation shows that

$$\langle\langle x, x\rangle_E y, y\rangle_B = \langle x, y\rangle_B^* \langle x, y\rangle_B ,$$

which is a positive element of B. It follows then from Corollary 2.7 that $\langle x, x\rangle_E$ is a positive element of E. Furthermore, because we have assumed that the B-inner product is definite, it follows that $\langle\ ,\ \rangle_E$ is also, for if $\langle x, x\rangle_E = 0$, then, letting $y = x$ in the identity just above, we find that $\langle x, x\rangle_B = 0$, so that $x = 0$.

The fact that $\langle x, y\rangle_E^* = \langle y, x\rangle_E$ for all $x, y \in X$ follows from the last statement of Lemma 6.2, while the fact that X is E-rigged follows from the last part of Proposition 6.3. The associativity relation is just 6.1 in different notation. Q.E.D.

We show next that the norms on E and B are closely related to the inner products on X.

PROPOSITION 6.6. *For any $e \in E$, $b \in B$ and $x \in X$ we have*

$$\langle ex, ex\rangle_B \leqslant \| e \|^2 \langle x, x\rangle_B ,$$

$$\langle xb, xb\rangle_E \leqslant \| b \|^2 \langle x, x\rangle_E .$$

Proof. The first of these inequalities just comes from the definition of the norm on E. The second inequality is verified by first making a routine calculation showing that for any $y \in X$, we have

$$\begin{aligned}\langle\langle xb, xb\rangle_E y, y\rangle_B &= \langle x, y\rangle_B^* bb^* \langle x, y\rangle_B \\ &\leqslant \| b \|^2 \langle\langle x, x\rangle_E y, y\rangle_B .\end{aligned}$$

We can now apply Corollary 2.7. Q.E.D.

EXAMPLE 6.7. Let X be the B-rigged space of Example 4.25, so that $X = L$ is a left ideal in the pre-C^*-algebra A, and $B = L^*L$. Then it is easily verified that the imprimitivity algebra E of X is the two-sided ideal LL^* of A, and that

$$\langle a, c\rangle_E = ac^*$$

for all $a, c \in L$. We remark that B is what J. M. G. Fell calls a block subalgebra, that is, it has the property that $bab' \in B$ for all $b, b' \in B$ and $a \in A$. When the comment of Kaplansky near the top of page 9 of

[79] is viewed in the context of this example, and Theorem 6.23 is applied, one obtains the fact that homogeneous C^*-algebras are Morita equivalent to their centers in the sense appropriate for C^*-algebras.

EXAMPLE 6.8. J. M. G. Fell has pointed out to us the following interesting special case of Example 6.7. If i is a self-adjoint idempotent in the double centralizer algebra of A, then $L = Ai$ is a left ideal in A. In particular, let G be a locally compact group and let K be a compact subgroup of G. Let $A = C_c(G)$ and let i be a minimal central idempotent in $C_c(K)$ corresponding to an irreducible unitary representation, R, of K. Then the irreducible representations of $E = AiA$ are in bijective correspondence with the irreducible representations of G whose restriction to K contains R. But one consequence of the main theorem of this section (Theorem 6.23) will be that the representations of AiA are in bijective correspondence with the representations of $B = iAi$. Now the subalgebra iAi is an important tool in Section 1 of Godement's paper [25] (see particularly the projection which Godement defines near the bottom of p. 504, which is the conditional expectation from which the B-inner product arises). As J. M. G. Fell has pointed out to us, the correspondence indicated above can be used to clarify a few of the results in Godement's paper.

EXAMPLE 6.9. Let X be the B-rigged space of Example 4.26. Then it is easily verified that the imprimitivity algebra E of X is the two-sided ideal of $A = \mathrm{End}_D(W')$ spanned by the operators of the form ST^* for $S, T \in X$, and that

$$\langle S, T\rangle_E = ST^*$$

for $S, T \in X$.

Let us return to the situation in which X is some B-rigged space, and let E be its imprimitivity algebra.

We now wish to study further the relations between B, X, and E which follow from the properties derived above. For this purpose we can forget how E was defined, and instead study an abstract system possessing the properties which interest us. For this purpose, it is no longer necessary to assume that the inner products are definite, and so we will now drop that assumption. This will be useful to us shortly.

DEFINITION 6.10. Let E and B be pre-C^*-algebras. By an E–B-

imprimitivity bimodule, we mean a left-E-right-B-bimodule, X, which is equipped with an E-valued and a B-valued preinner product with respect to which X is a left E-rigged space and a right B-rigged space, such that

(1) $\langle x, y\rangle_E z = x\langle y, z\rangle_B$ for all $x, y, z \in X$;

(2) $\langle ex, ex\rangle_B \leqslant \| e \|^2 \langle x, x\rangle_B$ for all $x \in X$ and $e \in E$;

(3) $\langle xb, xb\rangle_E \leqslant \| b \|^2 \langle x, x\rangle_E$ for all $x \in X$ and $b \in B$.

This definition should be compared with the definition of the bimodules occuring in the Morita theorems, which Bass [1] calls "equivalence data." In particular, the associativity relation (1) above should be compared with the relation at the top of p. 62 of [1].

The earlier part of this section can now be reformulated as saying that if we start with a B-rigged space X and form its imprimitivity algebra E with corresponding inner product, then X becomes an E–B-imprimitivity bimodule.

We give an example to show that conditions 2 and 3 of Definition 6.7 are independent.

EXAMPLE 6.11. Let X be the algebra of polynomials with complex coefficients. Let E be X viewed as a dense subalgebra of the C^*-algebra of continuous functions on the interval $[0, 2]$, and let B be X viewed as a dense subalgebra of the C^*-algebra of continuous functions on the interval $[0, 1]$, with the obvious actions of E and B on X. Define E- and B-valued inner products on X by

$$\langle f, g\rangle_E = f\bar{g}, \qquad \langle f, g\rangle_B = \bar{f}g = g\bar{f}.$$

Then it is easily verified that conditions 1 and 2 of Definition 6.7 hold, but that condition 3 does not. If we exchange the intervals on which E and B are viewed as being algebras of functions, then conditions 1 and 3 will hold but condition 2 will not.

We remark that conditions 2 and 3 can be given an alternate formulation along the lines of Proposition 4.27.

The next series of results leads, among other things, to the fact that if X is an E–B-imprimitivity bimodule, then X is both a pre-Hermitian right B-rigged left E-module, and a pre-Hermitian left E-rigged right B-module.

LEMMA 6.12. *If X is an E–B-imprimitivity bimodule, then*

$$\langle ex, y\rangle_B = \langle x, e^*y\rangle_B, \qquad \langle xb, y\rangle_E = \langle x, yb^*\rangle_E$$

for $x, y \in X$, $e \in E$ and $b \in B$.

Proof. Let $u, v, x, y \in X$. Then a routine calculation using condition 1 of Definition 6.10 shows that

$$\langle\langle u, v\rangle_E\, x, y\rangle_B = \langle x, \langle u, v\rangle_E^*\, y\rangle_B,$$

so that the first equality of the lemma holds for any element of the range of $\langle\ ,\ \rangle_E$, and so for the algebra generated by this range. But the algebra generated by this range is by assumption dense in E, and both sides of the first equality are continuous functions of e, as can be seen by using the generalized Cauchy–Schwartz inequality of Proposition 2.9 in the same way that it was used in the first part of the proof of Proposition 4.27. Thus the first equality is true for all $e \in E$. The proof of the second equality is similar. Q.E.D.

LEMMA 6.13. *Let B be a pre-C^*-algebra, and let X be a B-rigged space. Let B_0 be the span of the range of $\langle\ ,\ \rangle_B$. Then XB_0 is dense in X with respect to the B seminorm on X.*

Proof. Let $x \in X$ and $b \in B_0$. Then

$$\langle x - xb, x - xb\rangle_B = \langle x, x\rangle_B - \langle x, x\rangle_B\, b - b^*\langle x, x\rangle_B + b^*\langle x, x\rangle b,$$

which can be made as small as desired by letting b be close to appropriate elements of an approximate identity for the completion of B. Q.E.D.

PROPOSITION 6.14. *Let X be an E–B-imprimitivity bimodule. Then X is a pre-Hermitian right B-rigged left E-module, and a pre-Hermitian left E-rigged right B-module.*

Proof. The fact that X is a pre-Hermitian B-rigged E-module is clear except for the nondegeneracy of X as an E-module. To show the nondegeneracy it suffices to note that condition 1 of Definition 6.10 implies that $E_0X = XB_0$, where E_0 denotes the linear span in E of the range of $\langle\ ,\ \rangle_E$. It then suffices to apply Lemma 6.13. The proof of the last part of the proposition is similar. Q.E.D.

COROLLARY 6.15. *Let X be an E–B-imprimitivity bimodule. Then* *for every Hermitian B-module V, we can form the induced module* ${}^{E}_{X}V$. *We obtain in this way a functor from the category of Hermitian B-modules to the category of Hermitian E-modules.*

Proof. Apply Theorems 5.1 and 5.3. Q.E.D.

The Morita theorems suggest that we might be able to obtain an inverse for this functor by using an appropriate form of the B-dual of X, which should be a left-B-right-E-bimodule (see the definition of Q on p. 67 of [1]). Accordingly, we might let $X^* = \mathrm{Hom}_B(X, B)$, forgetting all topologies. This is certainly a B–E-bimodule if actions are defined by

$$(bf)(x) = b(f(x)), \qquad (fe)(x) = f(ex) \tag{6.16}$$

for $f \in X^*$, $b \in B$, $e \in E$ and $x \in X$.

However, in the spirit of our subject, we would expect to have to restrict attention to elements of X^* which are continuous in some sense. In fact, in analogy with the situation for ordinary Hilbert spaces, it is natural to take as the dual of X the set X itself with an appropriate conjugate structure defined so that the assignment to each $y \in X$ of the map $x \mapsto \langle y, x\rangle_B$ is a B–E-linear map of this dual into $\mathrm{Hom}_B(X, B)$. This turns out to be the appropriate definition.

Definition 6.17. Let $\tilde{X}$ denote the additive group X with the conjugate operations of E, B and the complex numbers, and the corresponding preinner products. When an element x of X is viewed as an element of $\tilde{X}$, we write it as $\tilde{x}$. Then these conjugate operations and preinner products on $\tilde{X}$ are defined by

$$b\tilde{x} = (xb^*)^\sim, \qquad \tilde{x}e = (e^*x)^\sim,$$

$$\langle \tilde{x}, y\rangle_B = \langle x, y\rangle_B, \qquad \langle \tilde{x}, y\rangle_E = \langle x, y\rangle_E,$$

for $x, y \in X$, $b \in B$, $e \in E$. Whenever convenient, we will also let $\tilde{x}$ denote the corresponding element of $\mathrm{Hom}_B(X, B)$ defined by $\tilde{x}(y) = \langle x, y\rangle_B$. We will call $\tilde{X}$ the *dual* of the imprimitivity bimodule X.

Proposition 6.18. *With the operations defined in Definition* 6.17, $\tilde{X}$ *is a B–E-imprimitivity bimodule.*

This is verified by routine computations.

COROLLARY 6.19. *$\tilde{X}$ is a pre-Hermitian right E-rigged left B-module.*

COROLLARY 6.20. *For any Hermitian E-module W, we can form the corresponding induced module ${}^B_X W$. We obtain in this way a functor from the category of Hermitian E-modules to the category of Hermitian B-modules.*

We wish to show that this functor is an inverse for the functor defined in Corollary 6.15. The next series of results will show that the functor $V \mapsto {}^B_X({}^E_X V)$ is naturally equivalent to the identity functor on the category of Hermitian E-modules. The proof that the composition of the functors in the opposite order is naturally equivalent to the identity functor on the category of Hermitian E-modules is then obtained just by interchanging the roles of X and $\tilde{X}$. (Note that the dual of $\tilde{X}$ is X.)

Now it is easily verified that the pre-Hermitian B-module $\tilde{X} \otimes_E (X \otimes_B V)$ is preequivalent to ${}^B_X({}^E_X V)$ under the obvious map (where they are viewed as $\mathbb{C}$-rigged spaces for the purposes of Definition 5.6). We need next the fact that the preinner products on tensor products (as defined in Theorem 5.9) of Hermitian rigged modules behave properly with respect to the associativity property of tensor products.

PROPOSITION 6.21. *Let A, B, C, D be pre-C^*-algebras, let X be a pre-Hermitian B-rigged A-module, let Y be a pre-Hermitian C-rigged B-module, and let Z be a pre-Hermitian D-rigged C-module. Then the pre-Hermitian D-rigged A-modules $X \otimes_B (Y \otimes_C Z)$ and $(X \otimes_B Y) \otimes_C Z$ are naturally isomorphic (so preequivalent) under the obvious map.*

This is verified by routine calculations.

It follows that $\tilde{X} \otimes_E (X \otimes_B V)$ is preequivalent to $(\tilde{X} \otimes_E X) \otimes_B V$. We are thus led to examine the pre-Hermitian B-rigged B-module $\tilde{X} \otimes_E X$. Now as in Example 2.14 we can view B itself as a pre-Hermitian B-rigged B-module.

LEMMA 6.22. *The pre-Hermitian B-rigged B-module $\tilde{X} \otimes_E X$ is preequivalent to B viewed as a pre-Hermitian B-rigged B-module, under the map R whose value on elementary tensors is given by*

$$R(\tilde{y} \otimes x) = \langle y, x \rangle_B$$

for all $x, y \in X$.

This also is verified by routine calculations.

It follows that $\tilde{X} \otimes_E (X \otimes_B V)$ is preequivalent to the pre-Hermitian B-module $B \otimes_B V$, which itself is easily seen to be preequivalent to V under the map $b \otimes v \mapsto bv$. From all of this together with Lemma 5.7, we conclude that ${}^B_{\tilde{X}}({}^E_X V)$ is unitarily equivalent to V. That this equivalence is natural is easily seen. Thus the functor $V \mapsto {}^B_{\tilde{X}}({}^E_X V)$ is naturally equivalent to the identity functor as desired. As mentioned earlier, we can also interchange the roles of X and $\tilde{X}$ to show that the functor $W \mapsto {}^E_X({}^B_{\tilde{X}} W)$ is naturally equivalent to the identity functor. We thus obtain the proof of one of our main theorems.

THE EQUIVALENCE THEOREM 6.23. *Let E and B be pre-C^*-algebras, and let X be an E–B-imprimitivity bimodule. Then the functor $V \mapsto {}^E_X V$ from Hermitian B-modules to Hermitian E-modules and the functor $W \mapsto {}^B_{\tilde{X}} W$ from Hermitian E-modules to Hermitian B-modules establish an equivalence between the category of Hermitian E-modules and the category of Hermitian B-modules.*

We remark that a slightly different way of developing the proof of the above theorem is to show that X is the analog of what Bass calls an invertible bimodule (p. 60 of [1]), with $\tilde{X}$ as its inverse. Specifically, one shows that $\tilde{X} \otimes_E X$ viewed now as a B–B-imprimitivity bimodule in the obvious way is preequivalent (in a sense analogous to Definition 5.6) to B viewed as a B–B-imprimitivity bimodule (as in Example 6.7 with $B = A = L$). Similarly, one shows that the E–E-imprimitivity bimodule $X \otimes_B \tilde{X}$ is preequivalent to E viewed as an E–E-imprimitivity bimodule. In this connection, we remark that the preinner products on $\tilde{X} \otimes_E X$ and $X \otimes_B \tilde{X}$ need not be definite.

We give now an example to show that the map R of Lemma 6.22 need not be injective.

EXAMPLE 6.24. Let A be the dense subalgebra of the C^*-algebra of continuous functions on $[-2, -1] \cup [1, 2]$ which vanish at both -1 and 1 consisting of functions which on $[-2, -1]$ agree with some polynomial vanishing at -1 and which on $[1, 2]$ agree with some (possibly different) polynomial vanishing at 1. Let $E = X = B = A$ with E- and B-valued inner products on X defined as in Example 6.7 with $L = A$. Let f be the function which agrees with 0 on $[-2, -1]$ and with $x - 1$ on $[1, 2]$, and let g be the function which agrees with $1 + x$ on $[-2, -1]$ and with 0 on $[1, 2]$. Then $\tilde{f} \otimes g$ is carried to zero by R, but $\tilde{f} \otimes g$ is not zero in $\tilde{X} \otimes_E X$ because the B-balanced bilinear map $(\tilde{h}, k) \mapsto \tilde{h}'(1)\, k'(-1)$ for $h, k \in A$ does not vanish on $\tilde{f} \otimes g$ (when the primes denote the first derivative). (The fact that E does not have an identity element is crucial for this example to work.)

The Equivalence Theorem applied to the imprimitivity bimodule obtained from a B-rigged space can be viewed as the generalization to the present context of Mackey's Theorem 6.4 in [38], which was generalized by Blattner to the nonseparable case in [3]. It is also a generalization of the well-known fact that the one-dimensional C^*-algebra $\mathbb{C}$ and the C^*-algebra of all compact operators on a Hilbert space have equivalent categories of Hermitian modules.

Any properties of Hermitian modules which can be defined in terms of morphisms must, of course, be preserved under the equivalence of categories given above. For example:

COROLLARY 6.25. *A Hermitian B-module V is irreducible if and only if* ${}^{E}V$ *is irreducible.*

But the equivalence of categories given in Theorem 6.23 also preserves certain properties which, as far as we can determine, are not definable just in terms of morphisms. Weak containment (see 3.4.5 of [13]) is such a property.

PROPOSITION 6.26. *Let X be an E–B-imprimitivity bimodule, let V be a Hermitian B-module, and let* $\{V_k\}$ *be a family of Hermitian B-modules. Then the family* $\{V_k\}$ *weakly contains V if and only if the family* $\{{}^{E}V_k\}$ *weakly contains* ${}^{E}V$.

Proof. Suppose that the family $\{V_k\}$ weakly contains V. Fix some $v \in V$ of unit length, and let p_v denote the corresponding vector state of B. Then by assumption there is a net, p_j, of states of B, each element of which is a convex combination of vector states from the V_k, which converges to p_v in the weak-$*$ topology. Let $p_j = \sum_{i=1}^{n_j} r_{ij}\, p_{v_{ij}}$ with $r_{ij} > 0$, $\sum_i r_{ij} = 1$, and v_{ij} of unit length in $V_{k_{ij}}$.

Now for any $x \in X$ for which the elementary tensor $w = x \otimes v$ in ${}^{E}V$ is of unit length, the corresponding state, q_w, of E is easily seen to have value on $e \in E$ given by

$$q_w(e) = p_v(\langle x, ex\rangle_B),$$

and, of course,

$$p_v(\langle x, ex\rangle_B) = \lim_j p_j(\langle x, ex\rangle_B).$$

Now

$$\lim_j p_j(\langle x, x\rangle_B) = p_v(\langle x, x\rangle_B) = \langle x \otimes v, x \otimes v\rangle = 1,$$

so that eventually $p_j(\langle x, x\rangle_B)$ is always nonzero. For the rest of the proof we assume that $p_j(\langle x, x\rangle_B) \neq 0$ for all j.

If $x \otimes v_{ij}$ has nonzero length, let

$$w_{ij} = (x \otimes v_{ij})/\| x \otimes v_{ij} \|,$$

and let

$$s_{ij} = r_{ij} \| x \otimes v_{ij} \|^2 / p_j(\langle x, x\rangle_B).$$

Let $q_j = \sum_i s_{ij} q_{w_{ij}}$, where the sum is taken only over nonzero terms. Then q_j is a convex combination of vector states from the ${}^E V_k$. But it is easily verified that q_w is the limit in the weak-$*$ topology of the q_j.

Thus every vector state of ${}^E V$ corresponding to an elementary tensor of unit length is a limit of convex combinations of vector states from the ${}^E V_k$. Now if V is cyclic, then the elementary tensors of unit length in ${}^E V$ are dense in the set of all vectors of unit length in ${}^E V$ by the arguments used in the proof of Lemma 1.7. It is easily seen that because of this, every vector state of ${}^E V$ is a weak-$*$ limit of convex combinations of vector states from the ${}^E V_j$, so that ${}^E V$ is weakly contained in $\{{}^E V_j\}$. In general, V is the direct sum of cyclic modules, and to show the weak containment of a direct sum of modules it is easily seen to be sufficient to show the weak containment of each summand.

The converse follows by applying the above result to the functor from E-modules to B-modules. Q.E.D.

COROLLARY 6.27. *Let E and B be pre-C^*-algebras. If there exists an E–B-imprimitivity bimodule then the spectrums of E and B are homeomorphic.*

One property of modules which is not preserved by the equivalence established by an imprimitivity bimodule is that of being cyclic, as is shown by the following example.

EXAMPLE 6.28. Let B be the one-dimensional C^*-algebra $\mathbb{C}$, let $X = \mathbb{C}^2$ with the usual action of $\mathbb{C}$ (viewed as a right action) and the usual B-inner product. Then X is a right B-rigged space whose imprimitivity algebra is the algebra of 2×2 complex matrices acting on X in the usual way, and with E-valued inner product defined in the usual way by forming operators of rank one. Now let V be the Hermitian B-module $\mathbb{C}^2$, which is not cyclic. Then ${}^E V$ is the four dimensional representation of E, which is cyclic. If we interchange the roles of E and B and take $\tilde{X}$ instead of X, and if we let V now be the four-dimensional cyclic E-module, then it follows that ${}^B V$ is not cyclic.

We remark however, that we do not know whether or not if H is a closed subgroup of a locally compact group G, and if V is a cyclic unitary H-module, then ${}^G V$ will always be cyclic as an E-module. (Of course, it need not be cyclic as a G-module—consider the regular representation of any compact group which is not second countable.) See also [27, 70].

The equivalence theorem (Theorem 6.23) is the heart of the proof of the imprimitivity theorem.

THE IMPRIMITIVITY THEOREM 6.29. *Let A and B be pre-C^*-algebras, and let X be a Hermitian B-rigged A-module. Let E be the imprimitivity algebra of the B-rigged space X. Then a Hermitian A-module W is unitarily equivalent to a Hermitian A-module induced from some Hermitian B-module via X if and only if W can be made into a Hermitian E-module in such a way that*

$$a(ex) = (ae)x \tag{6.30}$$

for all $a \in A$, $e \in E$, $x \in X$, where by ae we mean the product of a and e as elements of $L(X)$. (This product will be an element of E.)

Proof. If W is induced from a Hermitian B-module, then it follows from Theorem 5.1 that W will also be a Hermitian E-module satisfying 6.30.

Suppose, conversely, that W can be made into a Hermitian E-module in a way satisfying 6.30. Then by Theorem 6.23 there is a Hermitian B-module V such that as E-modules ${}^E V$ is unitarily equivalent to W. Let S be a unitary E-isomorphism of ${}^E V$ onto W. Now by Theorem 5.1 ${}^E V$ is an A-module in such a way that

$$a(eu) = (ae)u$$

for $a \in A$, $e \in E$, $u \in {}^E V$. Since the action of A on W is assumed to satisfy the same relation, we have

$$S(a(eu)) = S((ae)u) = (ae)\, S(u) = aS(eu)$$

for all $a \in A$, $e \in E$, and $u \in {}^E V$. But the linear span of the elements of the form eu in ${}^E V$ is dense in ${}^E V$, and so S is a unitary A-isomorphism as well. Q.E.D.

7. The Imprimitivity Theorem for Groups

In this section we will show how Mackey's imprimitivity theorem for induced representations of locally compact groups can be derived from the imprimitivity theorem for induced representations of C^*-algebras (Theorem 6.29). Many of the maneuvers which we will carry out in showing this are counterparts of maneuvers occuring in [33, 4, 16, 19, 44, 71].

Our first task will be to obtain an explicit description of the imprimitivity algebra E of Definition 6.4 in the case of the rigged space obtained from a group and a closed subgroup. As before, we will let H be a closed subgroup of the locally compact group G, and we will let A and B denote the pre-C^*-algebras $C_c(G)$ and $C_c(H)$ respectively, with A viewed as a pre-Hermitian B-rigged A-module in the way described in Section 4.

Let $C(G/H)$ denote the C^*-algebra of bounded continuous complex-valued functions on G/H with pointwise multiplication and supremum norm $\| \ \|_\infty$, and let $C_\infty(G/H)$ denote its C^*-subalgebra of functions vanishing at infinity. Whenever convenient we will tacitly identify elements of $C(G/H)$ with the corresponding bounded continuous functions on G which are constant on the cosets of H. To facilitate this, our notation will not distinguish between points of G and points of G/H. We recall that according to Blattner's formulation [4] of Mackey's imprimitivity theorem, a system of imprimitivity based on G/H for a unitary G-module W is a representation of $C_\infty(G/H)$ on W such that

$$x(Fw) = (xF)(xw) \tag{7.1}$$

for all $x \in G$, $F \in C_\infty(G/H)$, and $w \in W$, where by definition $(xF)(y) = F(x^{-1}y)$ for all $y \in G$.

Now the elements of $C(G/H)$ act on $C_c(G)$ by pointwise multiplication. We show that under this action, the B-rigged space $C_c(G)$ becomes a pre-Hermitian B-rigged $C_\infty(G/H)$-module, and that the action of $C_\infty(G/H)$ on $C_c(G)$ satisfies relation 7.1.

PROPOSITION 7.2. *Let $F \in C(G/H)$ and let M_F denote the operator on $C_c(G)$ consisting of pointwise multiplication by F. Then $M_F \in L(A)$ (where $A = C_c(G)$ is viewed as a $C_c(H)$-rigged space), and in fact $\| M_F \| = \| F \|_\infty$. Thus $C_c(G)$ becomes a pre-Hermitian $C_c(H)$-rigged $C(G/H)$-module. Furthermore, as a $C_\infty(G/H)$-module $C_c(G)$ is non-degenerate, and so it is also a pre-Hermitian $C_\infty(G/H)$-module.*

Proof. Let $F \in C(G/H)$. It is easily verified that, since F is constant on cosets of H, the operator M_F commutes with the right action of $C_c(H)$ on $C_c(G)$. Let $f \in C_c(G)$. We must show[3] that

$$\langle M_F f, M_F f \rangle_B \leqslant \| F \|_\infty^2 \langle f, f \rangle_B .$$

Let V be a unitary H-module, and let $v \in V$. From the fact that F is constant on cosets of H it is easily calculated that

$$\langle P((M_F f)_x)v, P((M_F f)_x)v \rangle = | F(x)|^2 \langle P(f_x)v, P(f_x)v \rangle,$$

where we use the notation of Theorem 4.4. Then the desired inequality is easily seen to follow from the second statement of Theorem 4.4. Furthermore, it is easily verified that the adjoint of M_F is $M_{\bar{F}}$.

It is clear that the $*$-homomorphism of $C(G/H)$ into $L(A)$ so obtained is injective, and so according to 1.8.3 of [13] it is isometric. Finally, for any $f \in C_c(G)$ it is clear that we can find $F \in C_\infty(G/H)$ such that $M_F f = f$. It follows that as a $C_\infty(G/H)$-module $C_c(G)$ is still non-degenerate. Q.E.D.

From Proposition 4.10 and routine calculations we obtain:

PROPOSITION 7.3. *Let $x \in G$, and let L_x denote the operator on $C_c(G)$ consisting of convolving on the left by the unit measure concentrated at x (which is the same as left translation by x, so that $L_x f = xf$ for $f \in C_c(G)$). Then $L_x \in L(A)$, and L_x is an isometric operator with respect to the B-valued inner-product on A. Furthermore, if $F \in C(G/H)$, then*

$$L_x M_F = M_{xF} L_x ,$$

when by definition $(xF)(y) = F(x^{-1}y)$ for all $y \in G$.

[3] Here is a simpler proof: First show that $\bar{F}$ acts as the adjoint of F. Next, let $F_0 = (\| F \|^2 - \bar{F}F)^{1/2}$. Then

$$\| F \|_\infty^2 \langle f, f \rangle_B - \langle Ff, Ff \rangle_B = \langle (\| F \|_\infty^2 - \bar{F}F)f, f \rangle_B = \langle F_0 f, F_0 f \rangle_B \geqslant 0.$$

We remark that the above relation between L_x and M_F is just the analog of relation 7.1. Thus we could view the representation M of $C(G/H)$ on $C_c(G)$ as being a "system of imprimitivity" for the "B-unitary" representation $x \mapsto L_x$.

Now by the action $F \mapsto xF$ used above (which comes from the natural action of G on G/H) G acts as a group of isometric $*$-automorphisms of the C^*-algebra $C(G/H)$. Furthermore, this action is strongly continuous on the subalgebra $C_\infty(G/H)$ and on its dense $*$-subalgebra $C_c(G/H)$. Let us for the moment denote the left regular representation of $C_c(G)$ on itself also by L, since it is the integrated form of the representation $x \mapsto L_x$. Then the algebraic tensor product $C_c(G/H) \otimes C_c(G)$ over the complex numbers can be viewed as a vector space of operators on $C_c(G)$ by $F \otimes f \mapsto M_F L_f$. This collection of operators will in general not quite be an algebra of operators. However, the elements of $C_c(G/H) \otimes C_c(G)$ can be identified in an obvious way with certain elements of $C_c(G, C_c(G/H))$, the space of continuous $C_c(G/H)$-valued functions on G of compact support, and it is easily seen that the above action of $C_c(G/H) \otimes C_c(G)$ on $C_c(G)$ extends to an action of $C_c(G, C_c(G/H))$ given by

$$\Phi f = \int_G M_{\Phi(x)} L_x f \, dx \tag{7.4}$$

for $\Phi \in C_c(G, C_c(G/H))$ and $f \in C_c(G)$. Furthermore, this collection of operators on $C_c(G)$ is an algebra, for if $C_c(G, C_c(G/H))$ is identified with $C_c(G \times G/H)$ in the obvious way, then the composition of the operators corresponding to the elements Φ and Ψ of $C_c(G \times G/H)$ is easily seen (by using the analog of 7.1) to be the operator corresponding to the element $\Phi * \Psi$ of $C_c(G \times G/H)$ defined by

$$(\Phi * \Psi)(x, z) = \int_G \Phi(y, z)\, \Psi(y^{-1}x, y^{-1}z)\, dy \tag{7.5}$$

for $x \in G$, $z \in G/H$. But this product is just the usual product in the transformation group algebra of the transformation group $(G, G/H)$ (see for example Eq. 3.3 of [16]). In fact, if $C_\infty(G/H)$ is viewed as acting on itself rather than on $C_c(G)$ then analogous considerations can be used to motivate the definition of the product in an arbitrary transformation group algebra.

PROPOSITION 7.6. *Let $\Phi \in C_c(G \times G/H)$, and let S_Φ denote the* *operator on the B-rigged space $A = C_c(G)$ defined by 7.4. Then $S_\Phi \in L(A)$. In fact, viewing Φ as an element of $C_c(G, C_\infty(G/H))$, we have*

$$\| S_\Phi \| \leqslant \int_G \| \Phi(y) \|_\infty \, dy,$$

and the adjoint of S_Φ is the operator corresponding to Φ^*, where

$$\Phi^*(y, x) = \bar{\Phi}(y^{-1}, y^{-1}x) \, \Delta(y^{-1}).$$

With this operation as the involution and with the product defined in 7.5, $C_c(G \times G/H)$ becomes a $*$-algebra, and the map $\Phi \mapsto S_\Phi$ is an injective $*$-homomorphism of $C_c(G \times G/H)$ into $L(A)$.

Proof. View Φ as an element of $C_c(G, C_\infty(G/H))$, and for each $y \in G$, let $N_y = M_{\Phi(y)} L_y$, which is an element of $L(A)$ of norm $\| \Phi(y) \|_\infty$ by Propositions 7.2 and 7.3. The function $y \mapsto N_y$ is not in general norm continuous since L_y is not. However, it is "strongly continuous" in the sense that for any $f \in C_c(G)$, the function $y \mapsto N_y f$ is clearly continuous for the inductive limit topology on $C_c(G)$. Thus for any state p of B (that is, normalized positive-type function on H), it is continuous for the topology on A corresponding to the inner product $p(\langle \, , \, \rangle_B)$. Furthermore, if $\| \, \|_p$ denotes the norm from that inner product, we have

$$\begin{aligned} \| N_y f \|_p^2 &= p(\langle M_{\Phi(y)} L_y f, M_{\Phi(y)} L_y f \rangle_B) \\ &\leqslant \| \Phi(y) \|_\infty^2 \, \| f \|_p^2 \end{aligned}$$

by Proposition 7.2. It follows that

$$\left\| \int N_y f \, dy \right\|_p \leqslant \left(\int \| \Phi(y) \|_\infty \, dy \right) \| f \|_p .$$

Since this is true for all states p of B, it follows easily that $\| S_\Phi \| \leqslant \int \| \Phi(y) \|_\infty \, dy$ as desired. The remaining statements of the proposition are verified by routine calculations. Q.E.D.

We remark that the involution defined above is the usual one for transformation group algebras (see Eq. 3.5 of [16]), and that $\int \| \Phi(y) \|_\infty \, dy$ defines a norm on $C_c(G \times G/H)$ under which this algebra becomes a $*$-normed algebra (see 3.11 of [16] or Theorem 2.2 of [9]).

It is our impression that it has not been noticed before in the literature that $C_c(G \times G/H)$ can act as an algebra of operators on $C_c(G)$.

COROLLARY 7.7. *The inductive limit topology on $C_c(G \times G/H)$ is finer than the C^*-norm topology obtained from viewing $C_c(G \times G/H)$ as a subalgebra of $L(A)$.*

We now obtain an explicit form for the "rank one" operators which generate E. Let $f, g \in C_c(G)$, and let $T_{(f,g)}$ be the corresponding operator defined as in 6.1. If $h \in C_c(G)$, then it is easily calculated that

$$\begin{aligned}(T_{(f,g)}h)(x) &= (f \cdot \langle g, h \rangle_B)(x)\\ &= \int_G \left(\int_H f(xt)\, g^*(t^{-1}x^{-1}y)\, dt \right) h(y^{-1}x)\, dy.\end{aligned}$$

Let us define $K_{(f,g)}$ by

$$K_{(f,g)}(y, x) = \int_H f(xt)\, g^*(t^{-1}x^{-1}y)\, dt \tag{7.8}$$

From the left invariance of Haar measure on H, it follows that in its second variable, $K_{(f,g)}$ is constant on cosets of H, and so $K_{(f,g)}$ can be viewed as a continuous function on $G \times G/H$. It is easily verified that in fact $K_{(f,g)}$ is in $C_c(G \times G/H)$. Finally, we have carefully defined $K_{(f,g)}$ in such a way that its action on $C_c(G)$ corresponding to the action of $T_{(f,g)}$ is defined by 7.4. It follows that E is contained in the image in $L(A)$ of $C_c(G \times G/H)$ under the representation $\Phi \mapsto S_\Phi$. In other words:

PROPOSITION 7.9. *If $f, g \in C_c(G)$, then $\langle f, g \rangle_E$ is the operator on $C_c(G)$ whose value on $h \in C_c(G)$ is given by*

$$(\langle f, g \rangle_E h)(x) = \int_G K_{(f,g)}(y, x)\, h(y^{-1}x)\, dy,$$

where $K_{(f,g)}$ is defined as in 7.8. The map $T_{(f,g)} \mapsto K_{(f,g)}$ extends to an injective $*$-homomorphism of E onto a two sided ideal of $C_c(G \times G/H)$.

From now on we will tacitly identify E with its image in $C_c(G \times G/H)$ whenever this is convenient.

We would like to show that E is dense in $C_c(G \times G/H)$ in the C^*-norm from $L(A)$. In view of Corollary 7.7 it suffices to show that E is dense in the inductive limit topology. To do this, it suffices to show that E contains an approximate identity for $C_c(G \times G/H)$ in the inductive limit topology, because of the fact that E is a two-sided ideal in

$C_c(G \times G/H)$. The verification of the following description of a convenient type of approximate identity for $C_c(G \times G/H)$ is almost the same as the proof of Lemma 3.27 of [16].

LEMMA 7.10. *Let* $\Theta_{(N,C)}$ *be a net of non-negative elements of* $C_c(G \times G/H)$ *which is indexed by decreasing neighborhoods, N, of the identity element of G and by increasing compact subsets, C, of G/H, such that*

(1) $\Theta_{(N,C)}(x, y) = 0$ *if* $x \notin N$,

(2) $\int_G \Theta_{(N,C)}(x, y)\, dx = 1$ *if* $y \in C$.

Then $\Theta_{(N,C)}$ is an approximate identity for $C_c(G \times G/H)$ in the sense that $\Theta_{(N,C)} * \Phi$ converges in the inductive limit topology to Φ for any $\Phi \in C_c(G \times G/H)$.

PROPOSITION 7.11. *The two-sided ideal E is dense in* $C_c(G \times G/H)$ *in the inductive limit topology.*

Proof. Let N be a neighborhood of the identity element of G, and let C be a compact subset of G/H. Let b be a Bruhat approximate cross section (as described before Theorem 4.4) which has been truncated so that $b \in C_c(G)$ and $\int_H b(xs)\, ds = 1$ for all $x \in C$. Choose a neighborhood M of the identity element of G such that $M^2 \subseteq N$, and choose a partition of unity [5] for G which is finer than right translates of M, that is, so that each element of the partition is supported in a right translate of M. Multiply this partition pointwise with b, so that we obtain a finite number of not identically zero functions, $f_1, \ldots, f_n$, in $C_c(G)$ such that for each i, the support of f_i is contained in Mz_i for some $z_i \in G$, and

$$\sum_{i=1}^{n} \int_H f_i(ys)\, ds = 1 \qquad \text{for} \quad y \in C. \tag{7.12}$$

For each i, choose a non-negative function $g_i \in C_c(G)$ such that g_i^* is supported in $z_i^{-1}M$ and $\int_G g_i^*(x)\, dx = 1$. Let

$$\Theta_{(N,C)} = \sum_{i=1}^{n} K_{(f_i, g_i)}\,.$$

Then routine calculations show that $\Theta_{(N,C)}$ satisfies conditions 1 and 2 of Lemma 7.10. Thus E contains an approximate identity for $C_c(G \times G/H)$ and so is dense in $C_c(G \times G/H)$. Q.E.D.

COROLLARY 7.13. *The two-sided ideal E is dense in $C_c(G \times G/H)$ in the C^*-norm from $L(A)$.*

Thus, while $C_c(G \times G/H)$ will not in general be complete with respect to the C^*-norm from $L(A)$, or even closed in $L(A)$, we can think of it as a fuller version than E of the analog of the algebra of all compact operators on A. For example, the closure of $C_c(G \times G/H)$ in $L(A)$ will be a two-sided ideal, and in particular will be acted on on the left by the subalgebras $M_c(G)$ (see Proposition 4.10) and $C(G/H)$ of $L(A)$. These subalgebras actually carry $C_c(G \times G/H)$ into itself, and their actions are easily verified to be the following:

$$(m * \Phi)(z, x) = \int_G \Phi(y^{-1}z, y^{-1}x)\, dm(y), \tag{7.14}$$

$$(F\Phi)(z, x) = F(x)\, \Phi(z, x), \tag{7.15}$$

for $m \in M_c(G)$, $F \in C(G/H)$, and $\Phi \in C_c(G \times G/H)$. Except for differences in conventions, these are the same actions as considered by Blattner in the last paragraph of p. 425 of [4], or as defined in Eqs. 3.22 and 3.33 of [16] (see [12] and [24] also). Of course, Hermitian E-modules will coincide with Hermitian $C_c(G \times G/H)$-modules. Thus we could now replace E by $C_c(G \times G/H)$. Instead, to simplify our notation, we will simply let E denote $C_c(G \times G/H)$ from now on.

On the basis of Theorem 6.29, we could now state a preliminary version of the imprimitivity theorem for this setting, namely, that a Hermitian $C_c(G)$-module W is induced from a Hermitian $C_c(H)$-module if and only if the action of $C_c(G)$ on W can be extended to an action of E on W in such a way that W becomes a Hermitian E-module and that

$$f(\Phi w) = (f * \Phi)w$$

for all $f \in C_c(G)$, $\Phi \in E$, and $w \in W$ (where $f * \Phi$ is defined as in 7.14). The reason that this version is not very useful is that it requires that the representation of E on W be continuous for the $L(A)$-norm, and this norm is defined in a somewhat complicated way in terms of the representation theory of H. To obtain a more useful theorem we need to show that, instead, it is sufficient for the representation of E on W to be continuous with respect to the inductive limit topology.

THEOREM 7.16. *Let W be a Hilbert space which is an E-module ($E = C_c(G \times G/H)$) giving a nondegenerate $*$-representation of E by bounded operators. If this representation is continuous for the inductive limit topology, in the sense that $\Phi \mapsto \langle \Phi w, w' \rangle_W$ is continuous in the inductive limit topology for all $w, w' \in W$, then this representation is continuous for the $L(A)$-norm on E, so that W is a Hermitian E-module.*

Proof. The means by which we relate W to the representation theory of H is to "induce" W down to H. Specifically:

LEMMA 7.17. *Let W be an E-module satisfying the conditions of the above theorem. Let X denote the E–B-imprimitivity bimodule $C_c(G)$, and then let $\tilde{X}$ be defined as in Definition 6.17. Define a sesquilinear form on $\tilde{X} \otimes_E W$ whose value on elementary tensors is given by*

$$\langle \tilde{f} \otimes w, \tilde{g} \otimes w' \rangle = \langle \langle \tilde{g}, \tilde{f} \rangle_E w, w' \rangle.$$

Then this form defines a preinner product on $\tilde{X} \otimes_E W$, and with this preinner product $\tilde{X} \otimes_E W$ becomes a pre-Hermitian B-module.

Proof. Let an element, $\sum \tilde{f}_i \otimes w_i$, of $\tilde{X} \otimes_E W$ be given. We must show that the sesquilinear form evaluated on this element is non-negative. Now it is easily seen that we can find a net, g_k, of elements of $C_c(G)$ such that the net $\langle g_k, g_k \rangle_B$ of elements of $C_c(H)$ is an approximate identity for $C_c(G)$ in the sense that $h \cdot \langle g_k, g_k \rangle_B$ converges to h in the inductive limit topology for all $h \in C_c(G)$. It follows that $\langle f_j \cdot \langle g_k, g_k \rangle_B, f_i \rangle_E$ converges to $\langle f_j, f_i \rangle_E$ in the inductive limit topology on E for each i and j. Now a routine calculation using condition 1 of Definition 6.10 shows that for each k

$$\sum_{i,j} \langle \langle f_j \cdot \langle g_k, g_k \rangle_B, f_i \rangle_E w_i, w_j \rangle = \left\langle \sum_i \langle g_k, f_i \rangle_E w_i, \sum_j \langle g_k, f_j \rangle_E w_j \right\rangle,$$

which is clearly non-negative. But then the limit of this expression, which is

$$\sum_{i,j} \langle \langle f_j, f_i \rangle_E w_i, w_j \rangle = \left\langle \sum_i \tilde{f}_i \otimes w_i, \sum_j \tilde{f}_j \otimes w_j \right\rangle$$

since the representation on W is assumed continuous for the inductive limit topology, must be non-negative, as desired.

We show next that $\tilde{X} \otimes_E W$ is a pre-Hermitian $C_c(H)$-module.

Now $C_c(G)$ is not only a right $C_c(H)$-module, but also a right $M_c(H)$-module in the same way. In particular, if δ_s denotes the unit mass at the point s of H, then $f \cdot \delta_s$ is defined for $f \in C_c(G)$ by $(f \cdot \delta_s)(x) = \gamma(s) f(xs^{-1}) \Delta(s^{-1})$, and in this way $C_c(G)$ becomes a right H-module. It is easily verified that this action of H is "isometric" with respect to the E-valued inner-product on $C_c(G)$. Furthermore, the action of H is "strongly continuous" in the sense that $\langle f - f \cdot \delta_s, f - f \cdot \delta_s \rangle_E$ converges to zero in the inductive limit topology for any $f \in C_c(G)$ as s converges to the identity element of H (because $f - f \cdot \delta_s$ converges to zero in the inductive limit topology). From all this, it follows that $\tilde{X}$ becomes a left H-module on which the action of H is "isometric" and "strongly continuous."

From the above considerations, it follows that $\tilde{X} \otimes_E W$ is a left H-module, and now the action of H is easily verified to be isometric with respect to the ordinary preinner product defined above. Furthermore, from the "strong continuity" of the action of H on $C_c(G)$ together with the assumption that the action of E on W is continuous for the inductive limit topology, it is easily seen that the action of H on $\tilde{X} \otimes_E W$ is strongly continuous in the usual sense. Thus $\tilde{X} \otimes_E W$ is a "preunitary" H-module.

Now what we wish to show is that as a $C_c(H)$-module, $\tilde{X} \otimes_E W$ is pre-Hermitian. To do this, it clearly suffices to verify that the representation of $C_c(H)$ on $\tilde{X} \otimes_E W$ is just the integrated form of the "preunitary" representation of H defined just above. By linearity and polarization, it suffices to show that

$$\langle \langle g \cdot \phi, f \rangle_E w, w' \rangle = \int_H \phi(s) \langle \langle g \cdot \delta_S, f \rangle_E w, w' \rangle \, ds$$

for all $f, g \in C_c(G)$, $\phi \in C_c(H)$, and $w, w' \in W$. But this is easily verified by noting that

$$g \cdot \phi = \int_H \phi(s)(g \cdot \delta_S) \, ds,$$

and by using the fact that $g \mapsto \langle g, f \rangle_E$ is continuous and that the representation of E on W is assumed to be continuous in the inductive limit topology. Q.E.D.

We now return to the proof of Theorem 7.16. Since, as we have just seen, $\tilde{X} \otimes_E W$ is a pre-Hermitian B-module, it follows from the first part of Theorem 5.9 that $X \otimes_B (\tilde{X} \otimes_E W)$ is a pre-Hermitian

E-module. But a routine calculation shows that the map from $X \otimes_B (\tilde{X} \otimes_E W)$ into W which on elementary tensors is defined by

$$f \otimes (\tilde{g} \otimes w) \mapsto \langle f, g \rangle_E w,$$

is an isometric E-module homomorphism. (This calculation can be used to give an alternative proof of Theorem 6.28.) Furthermore, this map has dense range because of Proposition 7.11. It follows that W is a Hermitian E-module. Q.E.D.

We remark that the above theorem can be generalized to the setting of C^*-algebras if appropriate (somewhat cumbersome) hypotheses are made concerning approximate identities.

We could now state a useful imprimitivity theorem of a form very analogous to Theorem 6.29. Instead, we will go directly to Mackey's form of the imprimitivity theorem [34, 38] as generalized to not necessarily separable groups by Loomis and Blattner [33, 4].

THE IMPRIMITIVITY THEOREM FOR GROUPS 7.18. *Let G be a locally compact group, let H be a closed subgroup of G, and let W be a unitary G-module. Then W is unitarily equivalent to a unitary G-module induced from some unitary H-module if and only if W can be made into a Hermitian $C_\infty(G/H)$-module in such a way that*

$$x(Fw) = (xF)(xw) \tag{7.19}$$

for all $x \in G$, $F \in C_\infty(G/H)$, and $w \in W$ (see 7.1). (A representation of $C_\infty(G/H)$ on W satisfying 7.19 is called a system of imprimitivity for W based on G/H.)

Proof. The necessity of the conditions follows from our earlier considerations, particularly Proposition 7.3. Suppose now that W is a unitary G-module which satisfies the conditions of the theorem. We would like to make W into a Hermitian E-module. Now for any $\Phi \in E$ and $w \in W$, view Φ as an element of $C_c(G, C_\infty(G/H))$ and define Φw, in analogy with 7.4, by

$$\Phi w = \int_G \Phi(y)(yw)\, dy. \tag{7.20}$$

This definition can be justified by arguments analogous to those used in the proof of Proposition 7.6. Furthermore, as in the proof of that proposition, it is easily seen that W becomes an E-module giving a $*$-representation of E by bounded operators, and in fact that

$$\| \Phi w \| \leqslant \| w \| \int_G \| \Phi(y) \|_\infty \, dy.$$

In particular, this last fact shows that the representation is continuous for the inductive limit topology in the sense of Theorem 7.16. Finally, a standard argument shows that the representation is nondegenerate. (Proofs of most of the above facts can also be found in [24, 4, 16].) Applying Theorem 7.16, we conclude that W is a Hermitian E-module.

Now if $f \in A$, $\Phi \in E$, and $w \in W$ then, by using 7.14, 7.19, and 7.20, it is easily calculated that

$$f(\Phi w) = (f * \Phi) w.$$

An application of Theorem 6.29 concludes the proof. Q.E.D.

We conclude this section by indicating how the above results can be applied to projective representations. We will use the approach to projective representations which is described in Section 17 of [19] because it works smoothly in the nonseparable case. Let T denote the group of complex numbers of modulus one, and let G be a locally compact group. We will let ξ denote a central group extension of G by T, that is, a locally compact group, G_ξ, together with maps

$$T \xrightarrow[i]{} G_\xi \xrightarrow[\sigma]{} G$$

such that i is a (homeomorphic) isomorphism of T onto a closed central subgroup of G_ξ, and σ is an open continuous homomorphism of G_ξ onto G whose kernel is $i(T)$. (The connection with cocycles in the separable case can be made using [17] and [49].) Following the definition on p. 149 of [19] we will mean by a ξ-projective representation of G an ordinary representation of G_ξ which when restricted to $i(T)$, gives the standard representation of T (that is, the representation $c \mapsto cI$ for $c \in T$, where I is the identity operator on the space of the representation of G_ξ). If H is a closed subgroup of G, we will let H_ξ denote the preimage of H under σ, so that H_ξ is the group of a central group extension of H by T which we will also denote by ξ. If R is a ξ-projective representation of H (so an ordinary representation of H_ξ), then by ${}^G R$ we mean the ordinary representation of G_ξ induced from R. It is easily verified that ${}^G R$ is a ξ-projective representation of G.

We note that G_ξ/H_ξ is naturally homeomorphic to G/H. By a system of imprimitivity based on G/H for the ξ-projective representation S of G, we mean an ordinary system of imprimitivity based on G_ξ/H_ξ (as in Theorem 7.18) for the ordinary representation S of G_ξ. Then it is not difficult to show that the ordinary representation H_ξ from which S is induced according to Theorem 7.18, whose space comes from $\tilde{X} \otimes_E W$ where W is the space of S, is a ξ-projective representation of H. This is the imprimitivity theorem for ξ-projective representations.

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