Reprinted from

Mappings of Operator Algebras

Proceedings of the Japan-U.S. Joint Seminar,
University of Pennsylvania, 1988

Edited by
Huzihiro Araki
Richard V. Kadison

Dedicated to Shôichirô Sakai

1990

Birkhäuser
Boston · Basel · Berlin

PROPER ACTIONS OF GROUPS ON C^*-ALGEBRAS

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Dedicated to Shôichirô Sakai
on the occasion of his sixtieth birthday

Recently I have been attempting to formulate a suitable C^*-algebraic framework for the subject of deformation quantization of Poisson manifolds [1,13]. Some of the main examples which I have constructed within this framework [27] involve "proper" actions of groups on C^*-algebras, where "proper" actions are to be defined as a generalization of proper actions of groups on locally compact spaces. Much of the material on proper actions which I have developed for this purpose is of a general nature which may be useful in other situations, so it has seemed appropriate to give a separate exposition of it, in the present article.

The notion of "proper" action which we introduce in this article is closely related to various notions of "integrable" actions which are discussed in the literature [5,7,15]. The main difference is that our notion of "proper" action emphasizes a natural inner-product having values in the crossed product algebra for the action. It turns out that because of this, our notion of "proper" actions is more closely related to reduced crossed products

*Supported in part by NSF grant DMS-8601900.

than full crossed products.

Section 1 of this article is devoted to the definition and basic properties of "proper" actions, especially a strong Morita equivalence between a certain ideal of the crossed product algebra and the generalized fixed-point algebra which we associate to a proper action. Section 2 contains several general examples. The first of these, which provides some clarifying counter-examples, consists of the action of a group by conjugation by its left regular representation on the algebra of compact operators. The second, for Abelian groups, consists of the dual action of the dual group on a crossed product algebra. The Morita equivalence alluded to above provides in this case what can be considered to be another manifestation of Takesaki duality as it was extended to C^*-algebras by Takai (as in 7.9 of [15]). In Section 3 we use the Morita equivalence alluded to above to study when the field of generalized fixed-point algebras corresponding to a continuous field of proper actions will be continuous. The results so obtained play a key role in the construction of the examples of deformation quantization discussed in [27]. In particular, [27] contains further interesting examples of proper actions.

§1 PROPER ACTIONS ON C*-ALGEBRAS

We recall that an action, α, of a locally compact group G on a locally compact space M is said to be proper if the map from $G \times M$ to $M \times M$ defined by $(x,m) \longmapsto (x, \alpha_x(m))$ is proper, in the sense that preimages of compact sets are compact. (A recent paper concerning proper actions, containing references to earlier papers, is [17].) A basic fact about proper actions is that the space of orbits, X/α, with the quotient topology, is again locally compact and Hausdorff (see propositions 3 and 9 in 3.4 of [2]).

Let $A = C_\infty(M)$, the algebra of continuous complex-valued functions on M vanishing at infinity, and let α also denote the action of G on A defined by $(\alpha_x f)(m) = f(\alpha_x^{-1}(m))$. Let $A^\alpha = C_\infty(X/\alpha)$. How are A and A^α related via α? Well, the elements of A^α can be viewed as continuous bounded functions on X which are constant on α-orbits. Thus A^α is a subalgebra of the multiplier algebra, $M(A)$, of A. The action α on A defines a corresponding action on $M(A)$ (which need not be strong-operator continuous). Let $M(A)^\alpha$ denote the subalgebra of fixed points in $M(A)$ for this action. Then it is easily seen that $A^\alpha \subseteq M(A)^\alpha$. But how do we characterize A^α as a subalgebra of $M(A)$? Intuitively, one obtains elements of A^α by "averaging", that is by integrating, elements of A over G. But if G is not compact, only elements in the dense subalgebra $A_0 = C_c(M)$ of A consisting of functions of compact support can be so integrated, and even then, the integration is not with respect to the norm topology but rather with respect to the strict topology. That is, if $f, g \in A_0$, then the function $x \longmapsto \alpha_x(f)$ is

not norm-integrable, but rather the function $x \longmapsto g\alpha_x(f)$ is, since it has compact support.

Generalizations of the above situation to the case in which A is non-commutative have been considered by various authors, generally under some variation of the name "integrable" actions. (See 7.84 of [15] with the note at the end of [14], definition II.2.1 of [5], and [7].) However, there is another important aspect of the commutative case $A = C_\infty(M)$ which does not seem to have been considered in the generalizations of properness to non-commutative A, but which we need to stress here, and which, when it is present, will lead us to use the term "proper" action instead of "integrable" action. This aspect is that for α a proper action of G on M, there is an inner-product on $C_c(M)$ with values in the transformation group C^*-algebra $E = C^*(G, M)$. If for simplicity we assume for the moment that G is unimodular, this inner-product is defined by

$$\langle f, g\rangle_E(x, m) = f(m)\bar{g}(\alpha_x^{-1}(m)) \quad ,$$

where the properness of α ensures that $\langle f, g\rangle_E$ has compact support, and so is in $L^1(G, C_\infty(M))$. This kind of inner-product already plays a key role in various commutative situations, such as those in [24]; and variants of it have appeared in related contexts, such as equation 2.3 of [20] and theorem 6.3 of [17]. In the case where G is compact, so that every action of G should be proper, this kind of inner-product has also been used for actions on non-commutative C^*-algebras; see section 7.1 of [16] and the references given there.

Suppose now that G is not necessarily compact, and that α is an action of G on some C^*-algebra A. We will need to consider a dense subalgebra of A, but we do not want to insist on an analogue of compact support (e.g. using the Pedersen ideal [15]), because of interesting examples that do not permit this, such as that of theorem 2.18 of [25], and the second example of the next section. But then we must be careful about the treatment of modular functions in case G is not unimodular. Guidance for this can be obtained from §2 of [20], which treats a special case of the situation we will consider here. Anyway, our first crucial assumption is that there is a dense α-invariant subalgebra, A_0, of A such that for any $a, b \in A_0$ both the functions $x \longmapsto a\alpha_x(b^*)$ and

$$\langle a, b\rangle_E(x) = \Delta(x)^{-1/2}a\alpha_x(b^*)$$

are norm integrable on G as A-valued functions, where Δ denotes the modular function of G.

As our notation suggests, we wish to view $\langle\ ,\ \rangle_E$ as an inner-product on A_0 with values in a C^*-algebra completion of $L^1(G, A)$. For this inner-product to be useful, we need to know that $\langle a, a\rangle_E$ is appropriately positive for any $a \in A_0$. Now if (π, U) is any covariant representation of (G, A) on a Hilbert space Ξ, then for $\xi \in \Xi$ we have, using module notation for the integrated form of (π, U),

$$\langle\langle a, a\rangle_E \xi, \xi\rangle = \int\langle\pi(a\alpha_x(a^*))U_x\xi, \xi\rangle \Delta(x)^{-1/2}dx$$
$$= \int\langle U_x\pi(a^*)\xi, \pi(a^*)\xi\rangle \Delta(x)^{-1/2}dx .$$

If G is unimodular, the integrand is clearly a function of positive type on G which is integrable. But when G is not amenable, it is easy to construct functions of positive type and of compact support whose integral over G is strictly negative. To see this, suppose for simplicity that G is discrete, and recall [15] that if G is not amenable then the trivial representation, τ, is not contained in the left regular representation, λ. This means that we can find $a \in C^*(G)$ such that $\|a\| = 1$, $a = a^*$ and $\tau(a) > 0$ while $\lambda(a) = 0$. Then we can find $f \in C_c(G)$ such that $\|f\| = 1$, $f = f^*$ and $\|a - f\| < \tau(a)/4$. This means that $\|\lambda(f)\| < \tau(a)/4$ while $\tau(f) \geq 3\tau(a)/4$. Let $g = (\tau(a)/2)\delta_e - f$ where δ_e is the delta-function at the identity element. Then $g \in C_c(G)$, and $\lambda(g)$ is positive so that g is of positive type. But

$$\int_G g = \tau(g) \leq -\tau(a)/4 .$$

Thus the integral we are examining above does not appear to be automatically non-negative. Suppose, however, that the covariant representation (π, U) is induced from a representation ρ of A on a Hilbert space H, as described in 7.7.1 of [15], so that $\Xi = L^2(G, H)$. Consider any $\xi \in \Xi$ which is actually in $C_c(G, H)$. Then

$$(1.1) \qquad \int\langle U_x\pi(a^*)\xi, \pi(a^*)\xi\rangle \Delta(x)^{-1/2}dx$$

$$= \int\int\langle(U_x\pi(a^*)\xi(y), (\pi(a^*)\xi)(y)\rangle dy\, \Delta(x)^{-1/2}dx$$

$$= \int\langle\int\rho(\alpha_{y^{-1}}(\alpha_x(a^*)))\xi(x^{-1}y)\Delta(x)^{-1/2}dx, \rho(\alpha_{y^{-1}}(a^*))\xi(y)\rangle dy$$

$$= \langle\int\rho(\alpha_x(a^*))\xi(x^{-1})\Delta(x)^{-1/2}dx, \int\rho(\alpha_y(a^*))\xi(y^{-1})\Delta(y)^{-1/2}dy\rangle$$

$$\geq 0 .$$

For this particular calculation we did not need to assume that $a, b \in A_0$. But if we do assume this, then $x \longmapsto \Delta(x)^{-1/2}a\alpha_x(a^*)$ is integrable, so $\langle a, a\rangle_E$ defines a bounded operator on Ξ. Then from the above calculation we conclude that this bounded operator is positive on Ξ. Thus we see that the appropriate place to view the values of $\langle\ ,\ \rangle_E$ is in the reduced C^*-algebra $C_r^*(G, A)$, since this algebra is defined in terms of these induced representations [15]. (Our notation will not explicitly indicate the action α involved, as there will be no ambiguity about this in what follows.) A simple calculation shows that

$$\langle a, b\rangle_E^* = \langle b, a\rangle_E .$$

We want the linear space, E_0, of finite linear combinations of elements of the form $\langle a, b\rangle_E$ to be a subalgebra of $C_r^*(G, A)$, and we want this subalgebra to act on A_0 on the left. But none of this is evident without further hypotheses.

To see what these hypotheses should be, let us remark first that A-valued functions f on G act on the left on A, the appropriate formula being

$$fa = \int f(x)\alpha_x(a)\Delta^{-1/2}(x)dx \quad ,$$

when this makes sense. In particular, for $a, b, c \in A_0$ we have

$$\langle a, b\rangle_E c = \int a\alpha_x(b^*c)dx \ ,$$

which does make sense by our hypotheses on A_0. We need to assume that such integrals are again in A_0, so that A_0 has a chance of being a left E_0-module. Since on A_0 we already have defined an E_0-valued inner product, A_0 will then be a left E_0-rigged space in the terminology of 2.8 of [21]. We can then look for its imprimitivity algebra, D_0, in the sense of definition 6.4 of [21] except using left modules. This will be generated by the operators $\langle b, c\rangle_D$ acting on the right on A_0 and defined by the formula

$$a\langle b, c\rangle_D = \langle a, b\rangle_E c \quad .$$

We want these operators to be nicely related to the situation. By slight abuse of notation, let $M(A_0)$ denote the subalgebra of $M(A)$ consisting of the multipliers which carry A_0 into itself, and let $M(A_0)^\alpha$ denote its subalgebra of α-invariant elements. We will require that $\langle b, c\rangle_D$ come from an element of $M(A_0)^\alpha$. Thus:

1.2 DEFINITION. Let α be an action of a locally compact group G on a C^*-algebra A. We say that α is a *proper* action if there is a dense α-invariant $*$-subalgebra A_0 of A such that

1) for any $a, b \in A_0$ the function $\langle a, b\rangle_E(x) = \Delta(x)^{-1/2}a\alpha_x(b^*)$ is in $L^1(G, A)$, as is the function $x \longmapsto a\alpha_x(b^*)$.

2) For any $a, b \in A_0$ there is a (uniquely determined) element $\langle a, b\rangle_D$ of $M(A_0)^\alpha$ such that for every $c \in A_0$ we have
$$\int c\alpha_x(a^*b)dx = c\langle a, b\rangle_D \quad .$$

Under these hypotheses we can now show that E_0, as defined earlier, is an algebra. Indeed, for $a, b, c, d \in A_0$, we have

$$\begin{aligned}\langle a, b\rangle_E\langle c, d\rangle_E(x) &= \int\langle a, b\rangle_E(y)\alpha_y(\langle c, d\rangle_E(y^{-1}x))dy \\ &= \int\Delta(y)^{-1/2}a\alpha_y(b^*)\alpha_y(\Delta(y^{-1}x)^{-1/2}c\alpha_{y^{-1}x}(d^*))dy \\ &= \int\Delta(x)^{-1/2}\int a\alpha_y(b^*c)dy\ \alpha_x(d^*) \\ &= \langle a\langle b, c\rangle_D, d\rangle_E(x) \quad .\end{aligned}$$

Since $a\langle b, c\rangle_D$ is by hypotheses again in A_0, it follows that E is an algebra. But, from a slightly earlier calculation, the above calculation can be restated as giving

$$\langle\langle a, b\rangle_E c, d\rangle_E = \langle a, b\rangle_E \langle c, d\rangle_E \quad .$$

From this it follows that for any $e \in E_0$ we have

$$\langle ea, b\rangle_E = e\langle a, b\rangle_E \quad .$$

If we equip E_0 with the norm from $C^*_r(G, A)$, so that E_0 is a pre-C^*-algebra, then the above observations show that A_0 is a left E_0-rigged space in the terminology of definition 2.8 of [21].

As discussed in §2 of [21], we can define a norm on A_0 by

$$\|a\|_E = \|\langle a, a\rangle_E\|^{1/2} \quad ,$$

where the norm on the right-hand side is that of E_0, and so of $C^*_r(G, A)$. We let $\bar{A}_0$ denote the completion of A_0 with respect to this norm. Let E denote the closure of E_0 in $C^*_r(G, A)$. Then the action of E_0 on A_0 defined above extends by continuity to an action of E on $\bar{A}_0$. A simple argument, given in lemma 6.13 of [21], shows that this action is non-degenerate in the sense that $E\bar{A}_0$ is dense in $\bar{A}_0$.

We will show now that E is an ideal in $C^*_r(G, A)$. For any $a \in A$ let m_a denote a viewed as a multiplier of $C^*_r(G, A)$. Then for $a, b, c \in A_0$ a simple calculation shows that

$$m_a\langle b, c\rangle_E = \langle ab, c\rangle_E \quad .$$

It follows by continuity that $m_a E \subseteq E$ for any $a \in A$. For any $y \in G$ let δ_y denote y viewed as a multiplier of $C^*_r(G, A)$. Another simple calculation shows that

$$\delta_y\langle a, b\rangle_E = \langle \alpha_y(a), b\rangle_E$$

for any $a, b \in A_0$. It follows by continuity that $\delta_y E \subseteq E$ for every $y \in G$. But for any $f \in C_c(G, A)$ and any $\eta \in C^*_r(G, A)$ their product in $C^*_r(G, A)$ can be written as

$$f\eta = \int m_{f(y)}\delta_y \eta \, dy \quad ,$$

so that if $\eta \in E$ then $f\eta \in E$. It follows by continuity that E is a left ideal, and so a two-sided ideal since it is a $*$-subalgebra. Since $\bar{A}_0$ is a non-degenerate left E-module, the action of E on $\bar{A}_0$ extends uniquely to an action of $C^*_r(G, A)$ on $\bar{A}_0$.

We now consider further the imprimitivity algebra for the situation. Let $d \in M(A_0)^\alpha$. Then from calculation (1.1), with the notation used there, we find that for any $a \in A_0$,

$$\langle\langle ad, ad\rangle_E \xi, \xi\rangle =$$

$$= \langle \rho(d^*)\int \rho(\alpha_x(a^*))\xi(x^{-1})\Delta(x)^{-1/2}dx,$$

$$\rho(d^*)\int \rho(\alpha_y(a^*))\xi(y^{-1})\Delta(y)^{-1/2}dy\rangle$$

$$\le \|d\|^2 \langle\langle a, a\rangle_E \xi, \xi\rangle \quad .$$

Consequently, as elements of E,

$$\langle ad, ad\rangle_E \le \|d\|^2\langle a, a\rangle_E \quad ,$$

so that d is a bounded operator on A_0 in the sense of

definition 2.3 of [21], because d^* is easily seen to serve as an adjoint for d with respect to $\langle\ ,\ \rangle_E$. Consequently, d extends to a bounded operator on $\bar{A}_0$. We show next that the norm of this operator is the same as the norm of d in $M(A)$. To this end, choose ρ acting on H and a unit vector $v \in H$ such that $\|\rho(d)v\|$ is close to $\|d\|$. Then choose an $a \in A_0$ close to a suitable element of an approximate identity for A. Finally, let ξ be a unit vector in $\Xi = L^2(G, H)$ supported in a small neighborhood of the identity element of G and constant there with value a multiple of v. Then the calculation made above shows that $\langle\langle ad, ad\rangle_E\xi, \xi\rangle$ is close to $\|d\|^2\langle\langle a, a\rangle_E\xi, \xi\rangle$. Consequently the norm of d as a bounded operator on $\bar{A}_0$ is $\|d\|$. We have thus established:

1.3 LEMMA. *With notation as above, each element* $d \in M(A_0)^\alpha$ *determines an element of the algebra,* $L(\bar{A}_0)$, *of bounded operators on* A_0, *defined by* $a \mapsto ad$ *for* $a \in A_0$. *The corresponding anti-homomorphism of* $M(A_0)^\alpha$ *into* $L(\bar{A}_0)$ *is isometric.*

The imprimitivity algebra, D, of $\bar{A}_0$ is by definition the closure in $L(\bar{A}_0)$ of the linear span, D_0, of the operators $\langle a, b\rangle_D$ defined by

$$c\langle a, b\rangle_D = \langle c, a\rangle_E b$$

for $a,b,c \in A_0$. The norm on D_0 is that from $L(\bar{A}_0)$. Notice that D_0 is already a $*$-subalgebra of $L(\bar{A}_0)$, by simple calculations using the relation of $\langle\ ,\ \rangle_D$ to $\langle\ ,\ \rangle_E$, as indicated in proposition 6.3 of [21]. But by hypothesis, the operators $\langle a, b\rangle_D$ for $a, b \in A_0$ are all in $M(A_0)^\alpha$, which we have just seen is isometrically embedded in $L(A_0)$. Thus we can now simply view D as the closure in $M(A)$ of the linear span of the $\langle a, b\rangle_D$'s for $a, b \in A_0$.

In view of the fact that for $a,b,c \in A_0$ we have

$$c\langle a, b\rangle_D = \int c\alpha_x(a^*b)dx \quad ,$$

it is natural to write symbolically that

$$\langle a, b\rangle_D = \int \alpha_x(a^*b)dx \quad ,$$

even though the integral on the right will not converge unless G is compact. But this suggests that the elements of D should be viewed as the generalized fixed-points for α, much as happens for proper actions on locally compact spaces, as discussed at the beginning of this section. Thus we make:

1.4 DEFINITION. Let α be a proper action of G on A. Then by the *generalized fixed-point algebra* of α we will mean the closure, D, in $M(A)$ of the linear span, D_0, of the elements $\langle a, b\rangle_D$ for $a, b \in A_0$. When convenient we will, by slight abuse of notation, denote D by A^α.

The above discussion together with proposition 6.6 of [21] shows that A_0 is an E_0-D_0-imprimitivity bimodule, as defined in 6.10 of [21]. Taking completions, we obtain a strong Morita equivalence as defined in [23]:

1.5 THEOREM. *Let* α *be a proper action of a locally compact group* G *on a* C^**-algebra* A. *Then, with notation as above,* A^α *is strongly Morita equivalent to the ideal* E *of* $C_r^*(G, A)$ *defined above, with* $\bar{A}_0$ *serving as an equivalence (i.e. imprimitivity) bimodule.*

We remark that E can easily fail to be all of $C_r^*(G, A)$. For instance, if G is compact and α is the trivial action, then E will consist of exactly the A-valued functions on G which are constant. If G is a finite group acting on a compact space M, and thus on $A = C(M)$, then it is easily checked that $E = C^*(G, A)$ exactly if the action is free. Since there are other possible ways to try to generalize the notion of freeness, as discussed in great detail in [16] (see also §10.8 of [17]), we will not use the term "free" here, but will rather use the following terminology, which is consistent with that used for compact groups as discussed in §7.1 of [16]:

1.6 DEFINITION. Let α be a proper action of a locally compact group G on a C^*-algebra A. We will say that α is *saturated* if $E = C_r^*(G, A)$, in the notation used above.

1.7 COROLLARY. *Let* α *be a saturated proper action of a locally compact group* G *on a* C^**-algebra* A. *Then, with notation as above,* A^α *is strongly Morita equivalent to* $C_r^*(G, A)$.

In §2 we will give some interesting examples of saturated proper actions.

In concluding this section, let us remark that it is not very clear how the results of this section depend on the choice of the dense subalgebra A_0. It would be desirable to have a more intrinsic definition of proper actions, which produces the subalgebra A_0 by some canonical construction. It is also not clear how often the integrable actions implicit in 7.8.4 of [15] will be proper.

§2 EXAMPLES

We now give several general examples of proper actions. The first of these will clarify the following issue. We have been careful in §1 to respect the distinction between $C^*(G, A)$ and $C_r^*(G, A)$. Of course, if G is amenable then this distinction disappears (theorem 7.7.7 of [15]). But actually, Phillips has shown (theorem 6.1 of [17]) that if α is any proper action of an arbitrary G on any locally compact space M, then $C^*(G, A) = C_r^*(G, A)$ for $A = C_\infty(M)$. This suggests that this might also happen for A non-commutative. But we now give a class of examples which show that this is not always the case, and that our earlier emphasis on the reduced algebras was appropriate. (Let me record here my thanks to Chris Phillips for helpful discussions about this matter.)

2.1 EXAMPLE. Let G be any locally compact group, and let λ denote the left regular representation of G

on $L^2(G)$. Let $A = K(L^2(G))$, the algebra of compact operators on $L^2(G)$, and let α be the action of G on A consisting of conjugation by λ. We show now that α is a proper action. Let A_0 be the subalgebra of A consisting of compact operators defined by kernels $F \in C_c(G\times G)$, where

$$(F\xi)(x) = \int F(x, y)\xi(y)dy$$

for $\xi \in L^2(G)$. It is easily calculated that

$$(\alpha_z(F))(x, y) = F(z^{-1}x, z^{-1}y) ,$$

so α carries A_0 into itself. For any $F, F' \in A_0$ we have

$$\begin{aligned}\langle F, F'\rangle_E(z)(x, y) &= \Delta(z)^{-1/2}(F\alpha_z(F'^*))(x, y)\\ &= \Delta(z)^{-1/2} \int F(x, w)(\alpha_z)(F'^*))(w, y)dw\\ &= \Delta(z)^{-1/2} \int F(x, w)\bar{F}'(z^{-1}y, z^{-1}w)dw ,\end{aligned}$$

which is easily seen to have compact support in all variables, as will be true also when the modular function is omitted. Thus condition 1 of Definition 1.2 holds. Next, we can prove a property slightly stronger than condition 2, namely, that for any $F \in A_0$ there is a $\Phi \in M(A_0)^\alpha$ such that

$$\int F'\alpha_z(F)dz = F'\Phi$$

for all $F' \in A_0$ (where the juxtaposition means product of operators, not pointwise product). As above, the integrand has compact support, so the integral is well-defined since the integrand is easily seen to be norm continuous. The integral will be a compact operator, and it is reasonable

to hope that it is given by a kernel function. If we calculate pointwise at the level of functions, we find that

$$\begin{aligned}(\int F'\alpha_z(F)dz)(x, y) &= \int\int F'(x, w)F(z^{-1}w, z^{-1}y)dw\, dz\\ &= \int F'(x, w) \int F(z^{-1}, z^{-1}w^{-1}y)dz\, dw .\end{aligned}$$

Set

$$f(u) = \int F(z^{-1}, z^{-1}u)dz .$$

Then it is easily seen that $f \in C_c(G)$, while, of course,

$$(\int F'\alpha_z(F)dz)(x, y) = \int F'(x, w)f(w^{-1}y)dw .$$

Applying this to a $\xi \in L^2(G)$, still at the pointwise level, we obtain

$$\begin{aligned}((\int F'\alpha_z(F)dz)\xi)(x) &= \int(\int F'(x, w)f(w^{-1}y)dw)\xi(y)dy\\ &= \int F'(x, w)(\int f(y)\xi(wy)dy)dw\\ &= (F'(\rho_f\xi))(x) ,\end{aligned}$$

where ρ denotes the *right* regular representation of G on $L^2(G)$. This makes sense since $M(A) = B(L^2(G))$, the algebra of all bounded operators on $L^2(G)$, and we expect to obtain something in $M(A)^\alpha$, that is, in this case, an operator commuting with λ. But the operators commuting with λ are exactly those in the von Neumann algebra generated by the *right* regular representation, ρ, of G. Anyway, we have found above, symbolically, that

$$\int \alpha_x(F) = \rho_f .$$

Now that we see what the answer must be, it is straight-

forward to justify the above calculations at the level of operator norms, rather than just pointwise. (One technique for this is given in the next example.) In this way condition 2 is verified. Furthermore, it is not difficult to see that every f can be approximated arbitrarily closely in the L^1-norm by functions of the form

$$u \longmapsto \int F(z^{-1}, z^{-1}u)dz$$

for $F \in C_c(G\times G)$. It follows that A^α, as defined earlier, will be all of $C_r^*(G)$, acting by the right regular representation on $L^2(G)$.

It is easily checked that E_0 contains an approximate identity for $C_r^*(G, A)$, so that $E = C_r^*(G, A)$, and α is saturated. Since α is an inner action, that is, comes from a representation of G into the group of unitary elements of $M(A)$, and since inner actions are not usually viewed as being analogues of free actions, this indicates some of the limitations of viewing saturation as an extension of freeness (even when G is compact). However, the above α certainly acts in some respects more like a free action than does, say, the trivial action, even though these actions are closely related, as we next discuss.

It is well known that the crossed product algebra for an inner action is isomorphic to the corresponding crossed product for the trivial action, the isomorphism being given at the level of functions by the map

$$f \longmapsto (x \longmapsto f(x)\lambda_x)$$

for $f \in C_c(G, A)$, where λ is the unitary representation

defining the action. This also works for reduced crossed products, as is seen by applying the mapping

$$\xi \longmapsto (x \longmapsto \lambda_x\xi(x))$$

to vectors $\xi \in L^2(G, H)$ where H is the Hilbert space of a faithful representation of A, and $L^2(G, H)$ is the Hilbert space of the corresponding induced representation. Thus in our specific situation where $A = K$, we find that

$$C^*(G, A) \cong C^*(G) \otimes K$$

$$C_r^*(G, A) \cong C_r^*(G) \otimes K \quad .$$

From this it is clear that if G is not amenable, then the full and reduced crossed product algebras do not coincide for the action α of conjugation by the left regular representation. For example, if G is the free group on two generators, Powers has shown [18] that $C_r^*(G)$ is simple, whereas $C^*(G)$ certainly is not simple since G has finite dimensional unitary representations. Consequently, theorem 6.1 of [17] does not generalize to non-commutative A. Since we saw above that the generalized fixed point algebra is $C_r^*(G)$, this example also supports the emphasis which we put on reduced crossed products in the definition of proper actions, since if matters had worked out using full crossed products, we would have found that $C^*(G)$ is strongly Morita equivalent to $C_r^*(G)$, which fails here since strong Morita equivalence preserves simplicity (by theorem 3.1 of [22]). These considerations show that, in part, it is our insistence that the generalized fixed-point algebra be a

closed subalgebra of M(A) which is forcing us to use reduced crossed products.

Our next example involves G which are Abelian, and can be considered to be another manifestation of Takesaki-Takai duality [15]. For G Abelian, $C^*_r(G, A) = C^*(G, A)$ for any action α of G on a C^*-algebra A, and so for simplicity of notation we will here denote this crossed product algebra by $A\times_\alpha G$. Let $\hat{G}$ denote the dual group of G, and let $\hat{\alpha}$ denote the dual action [15] of $\hat{G}$ on $A\times_\alpha G$. For $f \in L^2(G, A)$ this is defined by

$$\hat{\alpha}_t(f)(x) = \langle x, t\rangle f(x)$$

for $x \in G$ and $t \in \hat{G}$, where $\langle\ ,\ \rangle$ denotes here the duality between G and $\hat{G}$. For simplictiy of exposition we will actually assume that G is elementary in the sense of no. 11 of [28], that is, that G is a compactly generated Abelian Lie group, so of the form $R^p\times Z^q\times T^m\times F$ where R, Z, T and F denote respectively the reals, integers, circle group, and a finite Abelian group. There seems to be little doubt that the results obtained here can be extended to all locally compact Abelian groups by using their Schwartz space as defined by Bruhat [3] and employed in [28] and [25], but I have not checked this carefully. We state this example as:

2.2 THEOREM. *Let α be an action of an elementary locally compact Abelian group G on a C^*-algebra A. Then the dual action, $\hat{\alpha}$, is proper. Furthermore, $\hat{\alpha}$ is saturated, and the generalized fixed-point algebra of $\hat{\alpha}$ is naturally identified with A. Thus $(A\times_\alpha G)\times_{\hat{\alpha}}\hat{G}$ is strongly Morita equivalent to A.*

Proof. Notation is now a bit confusing because here the roles of G, A and α in Definition 1.2 are played by $\hat{G}$, $A\times_\alpha G$ and $\hat{\alpha}$. To verify the conditions of Definition 1.2 we must first pick a suitable dense subalgebra of $A\times_\alpha G$. In preparation for this and for later needs, we consider the following slightly more general situation. Let H be another elementary group (which may also be G), and let $C_\infty(H, A)$ denote the Banach space of continuous A-valued functions on H vanishing at infinity, with the supremum norm. Define a strongly continuous action β of $G\times H$ on $C_\infty(H, A)$ by

$$\beta_{(x,\ y)}(f)(z) = \alpha_x(f(z-y)) .$$

We let $S_\alpha(H, A)$ denote the space of elements of $C_\infty(H, A)$ which are infinitely differentiable for the action β, and which, with all of their higher partial derivatives for either G or H, vanish more rapidly at infinity than any polynomial on H grows. Here derivatives are taken in the R and T directions of G and H, whereas polynomials are taken with respect to the R and Z directions of H. Of course $S_\alpha(H, A) \subseteq L^1(H, A)$, and the values of functions in $S_\alpha(H, A)$ are in A^∞, the space of C^∞-vectors for the action α. Furthermore, there are plenty of elements in $S_\alpha(H, A)$, since any function of form $a\phi$ for $a \in A^\infty$ and $\phi \in S(H)$ will be in $S_\alpha(H, A)$.

The algebra which will play the role of the A_0 of Definition 1.2 is $S_\alpha(G, A)$. Some straightforward calculations show that $S_\alpha(G, A)$ is, in fact, a $*$-subalgebra of $A\times_\alpha G$, and that $S_\alpha(G, A)$ is carried into itself by the dual action $\hat{\alpha}$.

We must now examine the functions on $\hat{G}$ of the form

$$\langle f, g\rangle_E(t) = f\hat{\alpha}_t(g)$$

for $f, g \in S_\alpha(G, A)$. As functions of both t and x we have

$$(f\hat{\alpha}_t(g))(x) = \int f(y)\alpha_y(\hat{\alpha}_t(g(x-y)))dy$$
$$= \int f(x-y)\alpha_{x-y}(g(y))\langle y, t\rangle dy ,$$

which we recognize as just the partial Fourier transform (for appropriate conventions) with respect to the second variable, of the function

$$\Phi(x, y) = f(x-y)\alpha_{x-y}(g(y)) \quad .$$

A simple linear change of coordinates converts this function to the function

$$(x, y) \longmapsto f(x)\alpha_x(g(y)) \quad ,$$

which is easily seen to be in $S_\alpha(G\times G, A)$, and from this it follows easily that Φ itself is in $S_\alpha(G\times G, A)$. Standard arguments show, much as indicated at the top of page 159 of [28], that the partial Fourier transform of Φ in its second variable will then be in $S_\alpha(G\times\hat{G}, A)$, that is, the function

$$(x, t) \longmapsto (f\hat{\alpha}_t(g))(x)$$

is in $S_\alpha(G\times\hat{G}, A)$. In particular, it is in $L^1(G\times\hat{G}, A)$, and so $\langle f, g\rangle_E(t)$ is in $S_\alpha(G, A) \subseteq L^1(G, A)$ for each fixed t. It follows from the Fubini theorem that

$$\langle f, g\rangle_E \in L^1(\hat{G}, A\times_\alpha G).$$

We have thus verified condition 1 of Definition 1.2.

To verify condition 2 we wish to evaluate

$$\int_{\hat{G}} f\hat{\alpha}_t(g)dt$$

for any $f, g \in S_\alpha(G, A)$. For this purpose, let $LC(G, A)$ be the Banach space $L^1(G, A) \cap C_\infty(G, A)$ with the norm $\|\cdot\|_1 + \|\cdot\|_\infty$. Note that $S_\alpha(G, A) \subseteq LC(G, A)$, and that the injection of $LC(G, A)$ into $L^1(G, A)$, and so into $A\times_\alpha G$, is continuous. The utility of the above norm is that evaluations at points of G are clearly continuous for it. To take advantage of this we must also notice that if $f \in LC(G, A)$, then $t \longmapsto \hat{\alpha}_t(f)$ is continuous not only for $\|\cdot\|_1$, but also for $\|\cdot\|_\infty$. But this is true essentially by the definition of the topology on $\hat{G}$. Note also that convolution is clearly at least separately continuous for the above norm, so that

$$t \longmapsto f\hat{\alpha}_t(g)$$

is continuous for this norm. Thus, to calculate the integral of this function with values in $A\times_\alpha G$, under the assumption that $f, g \in S_\alpha(G, A)$, it suffices to view its values as being in $LC(G, A)$, and calculate pointwise. But, calculating as we did somewhat earlier, we see that for $x \in G$ we have

$$\int(\int f(y)\alpha_y(\hat{\alpha}_t(g)(x-y))dy)dt$$
$$= \int(\int f(x-y)\alpha_{x-y}(g(y))\langle y, t\rangle dy)dt \quad .$$

For our fixed x, define a function h by

$$h(y) = f(x-y)\alpha_{x-y}(g(y)) \quad .$$

It is easily seen that $h \in S_\alpha(G, A)$. Furthermore, the above calculation shows that

$$\left(\int_{\hat{G}} f\hat{\alpha}_t(g)dt\right)(x) = (\hat{h})^{\check{}}(0) \quad ,$$

where $\check{}$ denotes the inverse Fourier transform. Thus we need to know that the Fourier inversion formula can be applied to h. But by composing h with continuous linear functionals on A, we obtain complex-valued functions in S(G), and it is well known [28] that the Fourier inversion theorem applies to these. It follows that it applies to h, and so we find that

$$\left(\int_{\hat{G}} f\hat{\alpha}_t(g)dt\right)(x) = h(0) = f(x)\alpha_x(g(0)) \quad .$$

Now for $a \in A$ let m_a denote a viewed as an element of $M(A \times_\alpha G)$. Then we recognize the right-hand side above to be just $(fm_{g(0)})(x)$. Thus we have found that

$$\int_{\hat{G}} f\hat{\alpha}_t(g)dt = fm_{g(0)} \quad ,$$

or, symbolically,

$$\int_{\hat{G}} \hat{\alpha}_t(g)dt = m_{g(0)} \quad ,$$

(this latter integral not converging in norm unless $\hat{G}$ is compact). From this it is easily seen that condition 2 of Definition 1.2 is satisfied, and that for $f, g \in S_\alpha(G, A)$ we have

$$\langle f, g\rangle_D = m_{(f*g)(0)} \quad .$$

By taking f and g of the form $a\phi$ for $a \in A^\infty$ and $\phi \in S(G)$, and letting the corresponding ϕ's range over an approximate identity for $L^1(G)$, one sees easily that the elements of A of the form $\langle f, g\rangle_D$ are dense in A. Here we identify $a \in A$ with m_a, which is appropriate, as it is easily seen that the norms coming from A and $M(A\times_\alpha G)$ coincide. It follows that the generalized fixed-point algebra of $\hat{\alpha}$ is A.

The one fact which remains to be verified is the density of E_0, that is, the fact that $\hat{\alpha}$ is saturated. To this end we show first that E_0 is invariant under the action $\hat{\hat{\alpha}}$ dual to $\hat{\alpha}$. It suffices to verify this for elements of E_0 of the form $\langle f, g\rangle_E$ for $f, g \in S_\alpha(G, A)$. But for $z \in G$,

$$\begin{aligned}(\hat{\hat{\alpha}}_z(\langle f, g\rangle_E))(t) &= \langle z, t\rangle\langle f, g\rangle_E(t)\\ &= \langle z, t\rangle f\hat{\alpha}_t(g^*)\\ &= \langle z, t\rangle(f\delta_z^*)\delta_z\hat{\alpha}_t(g^*) \quad .\end{aligned}$$

But $\hat{\alpha}_t(\delta_z) = \langle z, t\rangle\delta_z$, so the above

$$= (f\delta_z^*)\hat{\alpha}_t((g\delta_z^*)^*) = \langle f\delta_z^*, g\delta_z^*\rangle_E(t) \ .$$

Since $f\delta_z^*$ and $g\delta_z^*$ are both again in $S_\alpha(G, A)$, it follows that E_0 is $\hat{\hat{\alpha}}$-invariant. Of course, then also E is $\hat{\hat{\alpha}}$-invariant.

The proof of Theorem 2.2 will thus be complete once we have given a proof of the following proposition, which returns to the notation of Definition 1.2 rather than that

used just above.

2.3 PROPOSITION. *Let* α *be a proper action of a locally compact Abelian group* G *on a* C^**-algebra* A. *Let* E *be the ideal of* $A\times_\alpha G$ *defined as above. Then the following conditions are equivalent:*

1) α *is saturated.*
2) E *contains an approximate identity for* $A\times_\alpha G$.
3) E *is carried into itself by the dual action* $\hat{\alpha}$.

Proof. Since saturation means just that $E = A\times_\alpha G$, and since E is an ideal, it is clear that conditions 1 and 2 are equivalent. We have included condition 2 because it has been a frequently used method for showing this kind of thing — not only earlier in this paper, but also in, for example, the proof of situation 10 of [24], and lemma 2.4 of [20]. It is also clear that conditions 1 and 2 imply condition 3.

To show that condition 3 implies condition 1, we use proposition 6.3.9 of [16] (which for the separable case also appears as corollary 2.2 of [10]). This proposition 6.3.9 tells us that because E is $\hat{\alpha}$-invariant, it must be of the form $I\times_\alpha G$ for some α-invariant ideal I in A. Then the image of E in $(A/I)\times_\alpha G$ must be (0). But if $I \neq A$, we can find $a, b \in A_0\backslash I$, and then the image of $\langle a, b\rangle_E$ in $(A/I)\times_\alpha G$ will clearly not be 0. Thus $I = A$, so $E = A\times_\alpha G$. Q.E.D.

For the case of actions for which G is not Abelian, one has instead of a dual action of the dual group, a dual

coaction of G itself, as discussed in [11] and the references therein. It seems to be an interesting challenge to formulate an appropriate definition of what it means for a coaction to be proper, and then to show that the dual coaction to an action will be proper.* Also, the dual action to a coaction should be proper. All this should also be possible for duality of twisted crossed products, as discussed in [19].

Let us also remark that for G Abelian the strong Morita equivalence of Theorem 2.2 fits into the framework of [4] and [6]. To be specific, with notation as earlier, define an action $\bar{\alpha}$ of G on $A_0 = S_\alpha(G, A)$ by

$$\bar{\alpha}_z f = f\delta_z^* \quad .$$

Then the calculation used above to show that E_0 is $\hat{\alpha}$ invariant shows that

$$\langle\bar{\alpha}_z f, \bar{\alpha}_z g\rangle_E = \hat{\hat{\alpha}}_z(\langle f, g\rangle_E) \quad .$$

Furthermore, we have (identifying D with A)

$$\langle\bar{\alpha}_z f, \bar{\alpha}_z g\rangle_D = (\bar{\alpha}_z f)^*(\bar{\alpha}_z g)(0) = (f\delta_z^*)^*(g\delta_z^*)(0)$$

$$= \delta_z(f^*g)\delta_z^*(0) = \alpha_z((f^*g)(0)) = \alpha_z(\langle f, g\rangle_D) \quad .$$

Thus the actions $\hat{\hat{\alpha}}$, $\bar{\alpha}$ and α on $(A\times_\alpha G)\times_{\hat{\alpha}}\hat{G}$, $\bar{A}_0$, and A respectively satisfy exactly the relations of theorem 1 of

*This has now been verified by Kevin Mansfield in "Induced representations of crossed products by coactions," preprint.

[6], that is, $\bar{\alpha}$ provides a Morita equivalence between $\hat{\hat{\alpha}}$ and α as defined in §3 of [4]. Thus we can conclude from these articles the unsurprising fact that $((A\times_\alpha G)\times_{\hat{\alpha}}\hat{G})\times_{\hat{\hat{\alpha}}}G$ is strongly Morita equivalent to $A\times_\alpha G$.

Because the ideas involved will be useful in [27], let us see how condition 3) of Proposition 2.3 can be used to obtain the following well-known result (see [24], where it is seen to be true for any locally compact G).

2.4 COROLLARY. *Let G be a discrete countable Abelian group, and let α be a proper action of G on a locally compact space M. Let α also denote the corresponding proper action of G on the C^*-algebra $C_\infty(M)$. If the action α on M is free, then the proper action α on $C_\infty(M)$ is saturated.*

Proof. We will show that the ideal E is carried into itself by $\hat{\alpha}$. We assume the dense subalgebra is $C_c(M)$, but the same proof would work for various other choices, such as $C_c^\infty(M)$ if M is a manifold. Anyway, let $\phi, \psi \in C_c(M)$, and let $t \in \hat{G}$. We need to show that $\hat{\alpha}_t(\langle\phi, \psi\rangle_E)$ is again in E. Since α is free and proper, and G is discrete, we can assume that all the translates of the closure of the support of ψ are disjoint, for if this is not the case, then we can express ψ as a finite sum of elements of $C_c(M)$ which do have this property. Likewise we can then assume that the support of ϕ meets the support of at most one translate of ψ. If it meets no translate, then $\langle\phi,\psi\rangle_E = 0$ and so $\hat{\alpha}$ applied to it is in E.

Otherwise, there is a unique $k_0 \in G$ such that $\langle\phi, \psi\rangle_G(k_0) \neq 0$. Let $\theta = \langle k_0, t\rangle \bar{\psi}$. Then for $x \in M$ and $k \in G$,

$$(\hat{\alpha}_t(\langle\phi, \psi\rangle_E))(x, k) = \langle k, t\rangle\langle\phi, \psi\rangle_E(x, k)$$
$$= \langle k, t\rangle\phi(x)\bar{\psi}(\alpha_k^{-1}(x)) \quad ,$$

which is 0 if $k \neq k_0$, and so can be rewritten as

$$= \phi(x)\langle k_0, t\rangle\bar{\psi}(\alpha_k^{-1}(x)) = \phi(x)\bar{\theta}(\alpha_k^{-1}(x)) = \langle\phi, \theta\rangle_E(x, k)$$

as desired. Q.E.D.

Our next two examples are somewhat less interesting since they explicitly involve proper actions on spaces. Hence we present them in somewhat sketchy fashion.

2.5 EXAMPLE. This one comes from theorem 2 of [6]. The set-up used there, and in situation 10 of [24], consists of a locally compact space P and locally compact groups G and H having commuting proper and free actions on P, which we now denote by juxtaposition. One can then define an action α of G on the (non-commutative) transformation group C^*-algebra $A = C^*(H, P)$ by

$$\alpha_x(f)(t, p) = f(x, x^{-1}p)$$

for $f \in C_c(H, P) = A_0$. Because of the properness of the action of G on P, it is easily seen that $x \longmapsto f\alpha_x(g)$ has compact support for $f, g \in A_0$, so that condition 1 of

Definition 1.2 is satisfied. Furthermore, it is not difficult to check that for $f, g \in A_0$,

$$\int_G f\alpha_x(g)dx = f\tilde{g}$$

where $\tilde{g} \in C_c(H, P/G)$ is defined by

$$\tilde{g}(t, p) = \int_G g(t, x^{-1}p)dx \quad ,$$

and elements of $C_c(H, P/G)$ act as multipliers on $C^*(H, P)$ in the evident way. Thus α is a proper action. But what norm does $C_c(H, P/G)$ obtain when it acts as multipliers on $C^*(H, P)$? Now $C^*(H, P) = C_r^*(H, P)$ according to theorem 6.1 of [17]. Thus we can choose a faithful representation of $C_\infty(P)$ on some Hilbert space Ξ, and then the corresponding induced representation on $L^2(H, \Xi)$ will give a faithful representation of $C^*(H, P)$, and so of $M(C^*(H, P))$. So $C_c(H, P/G)$ obtains the norm from its corresponding action on $L^2(H, \Xi)$. But the representation of $C_\infty(P)$ on Ξ extends to a faithful representation of $M(C_\infty(P))$, and so of $C_\infty(P/G) \subseteq M(C_\infty(P))$. And it is easily seen that if this faithful representation of $C_\infty(P/G)$ is induced up to H, one obtains just the representation of $C_c(H, P/G)$ considered above. Since it is reduced crossed products which are defined in terms of such induced representations [15], it is now clear that the norm on $C_c(H, P/G)$ is that of $C_r^*(H, P/G)$. It is not difficult to then show that the generalized fixed-point algebra for α is all of $C_r^*(H, P/G)$. But it takes some serious work,

along the lines given in [24], to show that α is saturated. Once this is done, one obtains the fact that $C_r^*(G\times H, P)$ is strongly Morita equivalent to $C_r^*(H, P/G)$, since, using theorem 6.1 of [17],

$$C_r^*(G, C^*(H, P)) = C_r^*(G, C_r^*(H, P)) = C_r^*(G\times H, P) \ .$$

This should be contrasted with theorem 2 of [6], which gives the corresponding conclusion for the full crossed products.

We remark that if we consider the special case in which $P = G = H$, so that G acts on the left and right on itself, then we obtain essentially the first example of this section. And one sees, as done there, that it must be the reduced crossed products which are involved here.

2.6 EXAMPLE. This one comes from theorem 2.2 of [20]. The set-up used there consists of a locally compact space P with a free and proper action of a locally compact group G, as well as of an action β of G on a C^*-algebra B. Let $A = C_\infty(P, B)$, and let α be the diagonal action of G on A defined by

$$\alpha_x(f)(p) = \beta_x(f(x^{-1}p))$$

for $f \in C_c(P, A) = A_0$. Then it is not difficult to verify that α is proper, with generalized fixed-point algebra $GC(P, A)^\alpha$ as defined in [20]. Again, it requires some serious work, along the lines found in [20], to show that the action is saturated, so that one obtains a strong

Morita equivalence of $C_r^*(G, C_\infty(P, A))$ with $GC(P, A)^\alpha$. But in theorem 2.2 of [20] the assertion is that it is the full algebra of $C^*(G, C_\infty(P, A))$ which is strongly Morita equivalent to $GC(P, A)^\alpha$. Thus one can expect that $C_r^*(G, C_\infty(P, A)) = C^*(G, C_\infty(P, A))$, in mild generalization of theorem 6.1 of [17].

§3 CONTINUOUS FIELDS OF PROPER ACTIONS

In this section we consider continuous fields of proper actions, and how they lead to continuous fields of the corresponding generalized fixed-point algebras. We work in the setting of [26], especially theorems 3.1 and 3.4. Thus $\{A_\omega\}$ will be an upper semi-continuous field [8] of C^*-algebras over a locally compact Hausdorff space Ω, and A will be the corresponding maximal C^*-algebra of sections. For each $\omega \in \Omega$ the evaluation map from A to A_ω will be denoted by π_ω, and its kernel by K_ω. By an upper semi-continuous field of actions on $\{A_\omega\}$ we mean an action α of a locally compact group G on A which carries each K_ω into itself, and so defines an action α_ω on A_ω for each $\omega \in \Omega$.

3.1 DEFINITION. Let α be an upper semi-continuous field of actions of G on the field $\{A_\omega\}$. We will say that α is an *upper semi-continuous field of proper actions* if there is given a dense $*$-subalgebra A_0 of A with respect to which α is a proper action, and if, in addition, the evident action of $C_\infty(\Omega)$ on A carries A_0 into itself.

A glance at Definition 1.2 shows that the action α_ω on each A_ω is then proper, where as dense subalgebra we take the image A_ω^0 of A_0 under π. The only point which needs a few moment's thought is that each π_ω extends to a homomorphism, $\hat{\pi}_\omega$, from $M(A)$ into $M(A_\omega)$, which carries $M(A_0)^\alpha$ into $M(A_\omega^0)^\alpha$, so that for any $a, b \in A_0$ the element $\langle a, b\rangle_D$ in $M(A_0)^\alpha$ determines corresponding elements in each $M(A_\omega^0)^\alpha$, which depend only on the images of a and b in A_ω^0. Accordingly, for each ω we can define the corresponding generalized fixed-point algebra, D_ω, which from the above comments will just be the image of D under $\hat{\pi}_\omega$. Then $\{D_\omega\}$ is a field of C^*-algebras over Ω, with D identified as an algebra of sections of this field, but without any obvious continuity properties.

We remark that since upper semi-continuity is associated with full crossed products, as seen in theorem 3.1 of [26], while proper actions involve reduced C^*-algebras as seen in §1, it can be expected that in order to obtain upper semi-continuity for a field of generalized fixed-point algebras, we will need to assume that the full and reduced crossed products agree. The next theorem will play a key role in our discussion of deformation quantization in [27], and so is the main result of the present paper. The term "Hilbert-continuous" used here is defined in definition 3.2 of [26].

3.2 THEOREM. *Let α be an upper semi-continuous field of proper actions on a field $\{A_\omega\}$, and let D be the generalized fixed-point algebra for the action α on the maximal algebra, A, of sections. Assume that* $C_r^*(G, A_\omega) =$

$C^*(G, A_\omega)$ for all ω. Then the field $\{D_\omega\}$ of generalized fixed-point algebras is upper semi-continuous, when D is used to define its continuity structure. Furthermore, D can be identified with the maximal algebra of sections of $\{D_\omega\}$. If, in addition, the field $\{A_\omega\}$ is actually Hilbert-continuous, then the field $\{D_\omega\}$ is continuous.

Proof. According to Theorem 3.1 of [26], $\{C^*(G, A_\omega)\}$ is an upper semi-continuous field, with $C^*(G, A)$ as its maximal C^*-algebra of sections. By exactly the same argument as in the second paragraph of the proof of Theorem 3.4 of [26], our assumption that $C_r^*(G, A_\omega) = C^*(G, A_\omega)$ implies that $C_r^*(G, A) = C^*(G, A)$. Consequently, in those places where $C_r^*(G, A)$ is needed in the treatment of proper actions, we will here write $C^*(G, A)$ instead. Accordingly, let E denote the ideal in $C^*(G, A)$ spanned by the elements $\langle a, b\rangle_E$ for $a, b \in A_0$, just as in §1. For each ω let E_ω be the image of E in $C^*(G, A_\omega)$. It is clear that, equivalently, E_ω is the closed ideal of $C^*(G, A_\omega)$ spanned by the functions $\langle a, b\rangle_{E_\omega}$ for $a, b \in A_\omega^0$, in the way described in §1.

We need the following simple result, in part because it is crucial for us to know that $E_\omega \cong E/EI_\omega$, where, as in [26], I_ω is the ideal in $C_\infty(\Omega)$ of functions which vanish at ω.

3.3 PROPOSITION. *Let $\{B_\omega\}$ be an upper semi-continuous field of C^*-algebras over Ω, and let B be its maximal C^*-algebra of sections. Let E be an ideal of B with the property that if $c \in C_\infty(\Omega)$ and if $cE = \{0\}$, then $c = 0$.*

For each ω let E_ω be the image of E in B_ω. Then $\{E_\omega\}$ is an upper semi-continuous field of C^*-algebras over Ω, which is continuous if $\{B_\omega\}$ is. In particular, $E_\omega = E/EI_\omega$ for each ω, and E is the maximal C^*-algebra of sections of $\{E_\omega\}$.

Proof. Multipliers of B act as multipliers of E. Because of the special hypotheses made on E, the corresponding homomorphism of $C_\infty(\Omega)$ into $M(E)$ is isometric, so that $C_\infty(\Omega)$ can be viewed as a subalgebra of $M(E)$, and the theory of §1 of [26] applies. Now the kernel of the evaluation map π_ω restricted to E is, of course, $E \cap BI_\omega$. But the latter is clearly an essential I_ω-module, and so $E \cap BI_\omega = EI_\omega$. Thus $E_\omega = E/EI_\omega$. The rest of the assertions now follow immediately from the results of §1 of [26]. Q.E.D.

We now return to the proof of Theorem 3.2. It is clear that, in this setting, $\{E_\omega\} \neq \{0\}$, because we tacitly assume that $A_\omega \neq \{0\}$. Thus Proposition 3.3 applies, and we conclude that $\{E_\omega\}$ is an upper semi-continuous field, with E as its maximal C^*-algebra of sections. Now E is strongly Morita equivalent to D according to Theorem 2.4, with $\bar{A}_0$ serving as equivalence bimodule. Using this strong Morita equivalence, we should be able to transfer the field structure of E to D. Recall from theorem 3.1 of [22] that if A and B are C^*-algebras, and if X is an A-B-equivalence bimodule (i.e. imprimitivity bimodule), then X establishes a canonical inclusion-preserving bijection between the (two-sided) ideals of A and those

of B, which is given, for example, by sending the ideal J of A to $\langle X, JX\rangle_B$.

3.4 THEOREM. *Let $\{A_\omega\}$ be an upper semi-continuous field of C^*-algebras over a locally compact space Ω, with A its maximal C^*-algebra of sections. Let B be a C^*-algebra which is strongly Morita equivalent to A via an A-B-equivalence bimodule X, and let h denote the corresponding bijection from ideals of A to ideals of B. For each ω let K_ω be the kernel of the evaluation map from A to A_ω, and let $B_\omega = B/h(K_\omega)$. Then $\{B_\omega\}$ is an upper semi-continuous field of C^*-algebras over Ω, with B as its maximal C^*-algebra of sections. If $\{A_\omega\}$ is actually continuous, then so is $\{B_\omega\}$.*

<u>Proof</u>. From corollary 3.3 of [22], the restriction of h to primitive ideals gives a homeomorphism, also denoted by h, from $\mathrm{Prim}(A)$ to $\mathrm{Prim}(B)$. Now $C_\infty(\Omega)$ can be viewed as a subalgebra of bounded continuous functions on $\mathrm{Prim}(A)$, and so, by this homeomorphism, as a subalgebra of bounded continuous functions on $\mathrm{Prim}(B)$. The points of Ω then correspond to closed subsets of $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ which correspond under h. Then the commonly called Dauns-Hofmann theorem [9] (which, as discussed in [8], is just a special result on the way to the main theorem of Dauns and Hofmann) says that $C_\infty(\Omega)$ can be viewed as a central subalgebra of $M(B)$. Let I_ω denote, as before, the ideal in $C_\infty(\Omega)$ of functions vanishing at ω. Then $\{B/BI_\omega\}$ will be an upper semi-continuous field with maximal algebra B, by Proposition 1.2 of [26]. But it is easily seen from the way h is involved here that $BI_\omega = h(AI_\omega) = h(K_\omega)$, thus demonstrating the first assertion. Suppose now that $\{A_\omega\}$ is lower semi-continuous, hence continuous. According to theorem 4 of [12] this is equivalent to the assertion that the evident continuous map p from $\mathrm{Prim}(A)$ to Ω coming from $C_\infty(\Omega) \subseteq A$ is open. But the corresponding map from $\mathrm{Prim}(B)$ to Ω is easily seen to be ph^{-1}, and h is a homeomorphism, so this map also is open. Then, again by theorem 4 of [12], $\{B_\omega\}$ will be continuous. Q.E.D.

We return now to the proof of Theorem 3.2. From the strong Morita equivalence of D with E, we obtain, by Theorem 3.4, a structure for D as an upper semi-continuous field over Ω. For any ω the ideal in D corresponding to EI_ω will be $\langle \bar{A}_0, EI_\omega\bar{A}_0\rangle_D = \langle \bar{A}_0, \bar{A}_0\rangle_D I_\omega = DI_\omega$, so the fiber algebra is D/DI_ω.

It might seem that we have now almost completed the proof, but in fact we have only just begun, for we must show that this field structure coincides with that given by the field of generalized fixed point algebras as in the statement of Theorem 3.2. The proof of this is somewhat slippery, because of the many identifications which one is tempted to assume, but which need to be verified. Thus we proceed with some care. Also, perhaps we should comment here that the reason for the approach we are taking is that it appears difficult to prove directly the upper semi-continuity of the field of generalized fixed-point algebras, whereas, as we have just seen, the upper semi-continuity of the field coming from the strong Morita equivalence follows from quite straightforward considera-

tions.

The main identification which we must carefully check is the following

3.5 LEMMA. *Let* $\bar{A}^0_\omega$ *and* $\bar{A}_0$ *denote the completions of* A^0_ω *and* A_0 *for* $\langle\ ,\ \rangle_{E_\omega}$ *and* $\langle\ ,\ \rangle_E$ *respectively. Let* $\bar{\pi}_\omega$ *denote the restriction of* π_ω *to* A_0, *mapping onto* A^0_ω. *Then* $\bar{\pi}_\omega$ *extends to an* E_ω*-module homomorphism*

$$\bar{\pi}_\omega \colon \bar{A}_0/\bar{A}_0 I_\omega \longrightarrow \bar{A}^0_\omega \quad ,$$

which is an isometric isomorphism preserving the E_ω*-valued inner products.*

Let us remark that if we were to make precise the idea of a field of modules with C^*-algebra valued inner products, along lines similar to those used in §3 of [8], then this lemma would signify that the field $\{\bar{A}^0_\omega\}$ is upper semi-continuous.

<u>Proof of Lemma 3.5</u>. For $a, b \in A_0$ we have

$$\begin{aligned}\langle\bar{\pi}_\omega(a), \bar{\pi}_\omega(b)\rangle_{E_\omega}(x) &= \Delta(x)^{-1/2}\,\pi_\omega(a)\alpha_x(\pi_\omega(b^*)) \\ &= \Delta(x)^{-1/2}\,\pi_\omega(a\alpha_x(b^*)) \quad .\end{aligned}$$

If we let $\tilde{\pi}_\omega$ denote the quotient map from E to E_ω coming from π_ω, then the above says that

$$\langle\bar{\pi}_\omega(a), \bar{\pi}_\omega(b)\rangle_{E_\omega} = \tilde{\pi}_\omega(\langle a, b\rangle_E) \quad ,$$

that is, $\bar{\pi}_\omega$ respects inner products via $\tilde{\pi}_\omega$. In particular,

$$\|\bar{\pi}_\omega(a)\|_{\bar{A}^0_\omega} \le \|a\|_{\bar{A}_0} \quad ,$$

so that $\bar{\pi}_\omega$ extends to a map, again $\bar{\pi}_\omega$, between the completions, which will again respect the inner-products. Now $\pi_\omega(A_0 I_\omega) = 0$, and so by continuity $\bar{\pi}_\omega(\bar{A}_0 I_\omega) = 0$. Thus $\bar{\pi}_\omega$ drops to a map from $\bar{A}_0/\bar{A}_0 I_\omega$. Now $E_\omega = E/EI_\omega$ by Proposition 3.3, so that E_ω acts on $\bar{A}_0/\bar{A}_0 I_\omega$, and the inner-product there can be viewed as having values in E_ω. From this and the earlier calculations, it is easily checked that $\bar{\pi}_\omega$ is an E_ω-module homomorphism which preserves the E_ω-valued inner-products, and hence is isometric. Since $\bar{\pi}_\omega$ clearly has dense range, it follows that $\bar{\pi}_\omega$ is a module isomorphism. Q.E.D.

We are now prepared to deal with D. As earlier, let $\hat{\pi}_\omega$ denote the homomorphism from D onto D_ω obtained by extending π_ω to a homomorphism $\hat{\pi}_\omega$ from $M(A)$ to $M(A_\omega)$. Since $\hat{\pi}_\omega$ clearly has I_ω in its kernel, $\hat{\pi}_\omega$ drops to a homomorphism, $\check{\pi}_\omega$, of D/DI_ω onto D_ω. Our objective is to show that $\check{\pi}_\omega$ is an isomorphism. Now D/DI_ω clearly acts on the right on $\bar{A}_0/\bar{A}_0 I_\omega$, as does D_ω on $\bar{A}^0_\omega$, while $\bar{\pi}_\omega$ and $\check{\pi}_\omega$ are easily seen to be compatible for these actions in the sense that

$$\bar{\pi}_\omega((x + \bar{A}_0 I_\omega)(d + DI_\omega)) = \bar{\pi}_\omega(x + \bar{A}_0 I_\omega)\check{\pi}_\omega(d + DI_\omega)$$

for $x \in \bar{A}_0$ and $d \in D$. But $\bar{\pi}_\omega$ is an isomorphism by Lemma 3.5. Thus the kernel of $\check{\pi}_\omega$ must act as zero operators on $\bar{A}_0/\bar{A}_0 I_\omega$. But D/DI_ω is exactly the imprimitivity algebra for $\bar{A}_0/\bar{A}_0 I_\omega$ as left E_ω-rigged space by corollary 3.2 of [22], and so D/DI_ω is faithfully represented on $\bar{A}_0/\bar{A}_0 I_\omega$. Thus $\check{\pi}_\omega$ must be an isomorphism, as desired. Consequently, $\{D_\omega\}$ is upper semi-continuous.

Suppose now that $\{A_\omega\}$ is Hilbert-continuous, with faithful representations σ_ω of A_ω on the fixed Hilbert space H. Then $\{C^*(G, A_\omega)\}$ is Hilbert-continuous by theorem 3.4 of [26] with faithful representations ρ_ω on $L^2(G, H)$ as in the proof of that theorem. From Proposition 3.3 it follows that $\{E_\omega\}$ is continuous. But then from Theorem 3.4 it follows that $\{D_\omega\}$ is continuous, because of the fact that we have shown earlier in this proof that $D_\omega = D/DI_\omega$. (It is not clear to me whether $\{D_\omega\}$ will always, in fact, be Hilbert-continuous.) Q.E.D.

No examples are given here of the application of Theorem 3.2, because such examples will be given in [27].

REFERENCES

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, "Deformation theory and quantization, I, II," *Ann. Physics* 110 (1978), 61-110, 111-151.

[2] N. Bourbaki, *Topologie Générale*, ch. III and IV (Hermann, Paris, 1963).

[3] F. Bruhat, "Distributions sur un group localement compact et applications à l'étude des représentations des groupes p-adiques," *Bull. Soc. Math. France* 89 (1961), 43-75.

[4] F. Combes, "Crossed products and Morita equivalence," *Proc. London Math. Soc.* 49 (1984), 289-306.

[5] A. Connes and M. Takesaki, "The flow of weights on factors of type III," *Tohoku Math. J.* 29 (1977), 473-575.

[6] R. E. Curto, P. S. Muhly and D. P. Williams, "Cross products of strongly Morita equivalent C*-algebras," *Proc. AMS* 90 (1984), 528-530.

[7] D. DeSchreye, "Integrable-ergodic C*-dynamical systems on Abelian groups," *Math. Scand.* 57 (1985), 189-205.

[8] M. J. Dupré and R. M. Gillette, "Banach bundles, Banach modules and automorphisms of C*-algebras," *Research Notes in Math.* 92 (Pitman Publishing, London, 1983).

[9] G. A. Elliott and D. Olesen, "A simple proof of the Dauns-Hofmann theorem," *Math. Scand.* 34 (1974), 231-234.

[10] E. C. Gootman and A. J. Lazar, "Crossed products of type I AF algebras by Abelian groups," *Israel J. Math.* 56 (1986), 267-279.

[11] ———, "Applications of non-commutative duality to crossed product algebras determined by an action or coaction," *Proc. London Math. Soc.*, to appear.

[12] R.-Y. Lee, "On the C*-algebras of operator fields," *Indiana U. Math. J.* 25 (1976), 303-314.

[13] A. Lichnerowicz, "Deformations and quantization," in *Geometry and Physics* (ed. M. Modugno), pp.103-116 (Pitagora Editrice, Bologna, 1983).

[14] D. Olesen and G. K. Pedersen, "Application of the Connes spectrum to C*-dynamical systems, II," *J. Funct. Anal.* 36 (1980), 18-32.

[15] G. K. Pedersen, "C*-algebras and their automorphism groups," *London Math Soc. Monographs* 14 (Academic Press, London, 1979).

[16] N. C. Phillips, "Equivariant K-theory and freeness of group actions on C*-algebras," *Lecture Notes in Math.* 1274 (Springer-Verlag, 1987).

[17] ———, "Equivariant K-theory for proper actions," *Pitman Research Notes Math.* 178 (1988).

[18] R. T. Powers, "Simplicity of the C*-algebra associated with the free group on two generators," *Duke Math. J.* 42 (1975), 151-156.

[19] J. C. Quigg, "Duality for reduced twisted crossed products," *Indiana U. Math. J.* 35 (1986), 549-571.

[20] I. Raeburn and D. P. Williams, "Pull-backs of C*-algebras and crossed products by certain diagonal actions," *Trans. A.M.S.* 287 (1985), 755-777.

[21] M. A. Rieffel, "Induced representations of C*-algebras," *Adv. Math.* 13 (1974), 176-257.

[22] ———, "Unitary representations of group extensions; an algebraic approach to the theory of Mackey and Blattner," *Studies in Analysis, Adv. Math. Suppl. Series* 4 (1979), 43-82.

[23] ———, "Morita equivalence for operator algebras," in *Operator Algebras and Applications* (R.V.Kadison, ed.), pp.285-298; *Proc. Symp. Pure Math.* 38 (Amer. Math. Society, Providence, 1982).

[24] ———, "Applications of strong Morita equivalence to transformation group C*-algebras," in *Operator Algebras and Applications* (R.V.Kadison, ed.), pp. 299-310; *Proc. Symp. Pure Math.* 38 (American Math. Society, Providence, 1982).

[25] ———, "Projective modules over higher dimensional non-commutative tori," *Canadian J. Math.* 40 (1988), 257-338.

[26] ———, "Continuous fields of C*-algebras coming from group cocycles and actions," *Math. Ann.* 283 (1989), 631-643.

[27] ———, "Deformation quantization for Heisenberg manifolds," *Comm. Math. Phys.* 122 (1989), 531-562.

[28] A. Weil, "Sur certains groupes d'opérateurs unitaire," *Acta Math.* 111 (1964), 143-211.

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