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NON-STABLE K-THEORY AND NON-COMMUTATIVE TORI

Dedicated to

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in Celebration of their Completing Sixty Circumnavigations of the Sun

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During the last five years there has been much progress in understanding and calculating the K-groups of various C*-algebras. But once this has been accomplished in any given situation, there remain many interesting questions concerning finer structure, or what I call non-stable K-theory. My purpose here is to list some of these questions, and then to discuss the progress which has been made in answering them for non-commutative tori and for a few other examples.

1. THE QUESTIONS

Most of the questions which we will consider are only of interest for C*-algebras with identity element, and so we will assume the presence of an identity element throughout. Actually, as far as the K_0 group is concerned, we can usually work with any algebra with identity element. We recall [1, 19] that there are two equivalent definitions of the $K_0(A)$, of an algebra A. In the most natural of these two definitions, one considers the set, S(A), of

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isomorphism classes of finitely generated projective (say right) A-modules. Under formation of direct sums of modules, S(A) becomes a commutative semigroup, with the (class of the) zero module serving as identity element. Then $K_0(A)$ is defined to be the enveloping (or Grothendieck) group of the semigroup S(A). The image of S(A) in $K_0(A)$ provides $K_0(A)$ with a "positive cone", which can be badly behaved if A is not finite in some sense [3]. The study and calculation of this positive cone can be very interesting, but we consider it to be still part of the stable K-theory of A. Rather it is the study of S(A) itself, and of the passage from S(A) to $K_0(A)$, which we consider to constitute the non-stable part of K_0-theory. Thus for a given algebra A the fundamental question of non-stable K_0-theory is:

QUESTION 1. *What is the structure of the semigroup* S(A)?

This question is usually too hard to answer, and so one first considers special aspects of it. For example:

QUESTION 2. *Does* S(A) *satisfy cancellation, that is, if* U, V *and* W *are finitely generated projective* A*-modules such that* $U \oplus W \cong V \oplus W$, *does it follow that* $U \cong V$?

If cancellation holds, then the map from S(A) to $K_0(A)$ is injective. Thus if one knows what is the positive cone of $K_0(A)$, then one knows S(A). However cancellation usually fails, and so one instead asks weaker questions. Recall, for example, that modules U and V represent the same element of $K_0(A)$ exactly if there is some integer n such that $U \oplus A^n \cong V \oplus A^n$, where A^n denotes the free A-module on n generators. One can then ask, for a given algebra, whether there is an upper bound on the needed n's. That is:

QUESTION 3. *Is there a positive integer* N *such that whenever* U *and* V *represent the same element of* $K_0(A)$, *then* $U \oplus A^N \cong V \oplus A^N$?

This question can be viewed as asking whether cancellation holds as soon as the modules involved are "large enough". Here "large enough" means that the modules contain A^N as a summand. For C*-algebras there is a closely related way of defining the size of a module, namely by means of a trace on the algebra. We recall that the second equivalent definition [19] of the K_0-group is in terms of projections in matrix algebras, $M_n(A)$, over A. If p is a projection in $M_n(A)$, then pA^n will be a finitely generated projective right A-module. If τ be a (finite, positive) trace on A, then τ extends in an evident way to a trace on each $M_n(A)$, and thus if p is a projection in any $M_n(A)$ then the positive number $\tau(p)$ is defined. One can show easily that this number depends only on the isomorphism class of the module pA^n. Thus τ defines a homomorphism (again denoted by τ) of S(A) into the group of real numbers, which then factors through $K_0(A)$. In analogy with Question 3 one can ask:

QUESTION 4. *For a given trace* τ *on* A, *is there a number* N *such that, if* $U \oplus W \cong V \oplus W$ *and if* $\tau([U]) \geq N$ (*so also* $\tau([W]) \geq N$), *then* $U \cong V$?

In a slightly different direction, one can ask about cancellation for special classes of modules. The most commonly discussed class consists of the stably free modules. Specifically:

QUESTION 5. *Are stably free modules free? That is, if the module* U *is such that*

$$U \oplus A^n \cong A^{m+n}$$

for some m *and* n, *does it always follow that*

$$U \cong A^m?$$

If not, then one can ask, as before, whether there is a bound on the n's that are needed, that is:

QUESTION 6. *Is there an integer* N *such that whenever* U *is a stably free module then* $U \oplus A^N$ *is free?*

The next question is best phrased in terms of projections, and is most appropriate to ask for C*-algebras where cancellation holds. Here and later we let $U_n(A)$ denote the group of unitary elements of $M_n(A)$, and we let $U_n^\circ(A)$ denote the connected component of the identity element in $U_n(A)$.

QUESTION 7. *If* A *satisfies cancellation, and if* p *and* q *are (self-adjoint) projections in* $M_n(A)$ *which represent the same class in* $K_0(A)$, *then are* p *and* q *in the same connected component of the set of projections in* $M_n(A)$? *Equivalently, is there a unitary* u *in* $U_n^\circ(A)$ *such that* $upu^* = q$?

A question about $K_0(A)$ which goes in a rather different direction is:

QUESTION 8. *What is the smallest* n *such that the projections in* $M_n(A)$ *generate* $K_0(A)$?

We remark that the answers to Questions 1 and 2 are invariant under Morita equivalence of algebras [12], whereas most of the other questions above involve, in some sense, the position of the free module of rank one as an order unit in S(A).

We recall [19] that $K_1(A)$ is defined as the limit of the groups

$$U_n(A)/U_n^\circ(A) \to U_{n+1}(A)/U_{n+1}^\circ(A).$$

There are two quite evident questions to ask about the non-stable behavior for K_1, namely:

QUESTION 9. *What is the smallest* n *such that the homomorphism from* $U_k(A)/U_k^\circ(A)$ *to* $K_1(A)$ *is injective for all* $k \geq n$?

QUESTION 10. *What is the smallest* n *such that the homomorphism from* $U_n(A)/U_n^\circ(A)$ *to* $K_1(A)$ *is surjective*?

There is a substantial literature in algebraic K-theory concerning these last two questions. See [20] and the references therein.

2. TECHNIQUES AND INTERRELATIONS

For a commutative C*-algebra, of form $A \cong C(X)$ for X a compact space, the finitely generated projective modules correspond exactly to the complex vector bundles, by a theorem of Swan [18, 12]. Thus in this case the vast apparatus of algebraic topology, especially topological K-theory, can be brought to bear on answering the above questions. But it is known that the answers are usually complicated. There are a small number of general results. For example, from Theorem 1.5 of Chapter 8 of [8] one obtains immediately the following answer to Question 4 (where one may take as the trace, evaluation of functions at some fixed point):

THEOREM 1. *Let* X *be a compact connected* CW *complex of dimension* d. *If* V *and* W *are complex vector bundles over* X *which represent the same element of* $K^0(X)$, *and if their dimension is* $\geqslant d/2$, *then* $V \cong W$.

But usually, even when one can compute for specific examples, it is difficult to find general patterns. One must then expect that this will be all the more the situation for non-commutative C*-algebras.

In the case of C*-algebras which are postliminal (i.e. GCR), and so are fairly closely related to commutative C*-algebras, one can hope that results in topological K-theory will provide some guidance as to what to expect, as well as results upon which one can build by inductive arguments. For example, Albert J. Sheu [17] has studied the unitized C*-algebras $\tilde{C}^*(G)$ where G is a simply connected nilpotent Lie group of form $R^n \rtimes R$. These are GCR, and can be considered to be "non-commutative spheres". He obtains a good answer to Question 4, where as trace he uses evaluation at the adjoined "point of infinity". He also shows that cancellation holds for certain of the G of arbitrarily high dimension. But he has an example of a four-dimensional G for which cancellation fails, though he can nevertheless describe the structure of its semigroup of projective modules.

Theorem 1 above suggests that some notion of dimension in the non-commutative context might play a role in non-stable K-theory. It is far from clear whether there should be a unique notion of dimension in this context, but one notion has already played an important role in algebraic K-theory, namely the notion of Bass stable rank. We omit the definition, since for topological algebras it is more convenient to use the notion of topological stable rank (tsr) which was introduced in [13] and shown there to dominate the Bass stable rank. Subsequently, it was shown by Herman and Vaserstein [7] that for C*-algebras the topological stable rank coincides with the Bass stable rank. To define the topological stable rank, we let $\mathrm{Gen}_k(V)$, for any module V and positive integer k, denote the collection of k-tuples of elements, $\{v_j\}$, in V^k which collectively generate V algebraically, that is, such that

$$\Sigma\, v_j A = V.$$

Note next that if A is a topological algebra, then any finitely generated projective A-module, being realizable as a summand of some A^n, is a topological module (independently of the realization).

DEFINITION. *Let* A *be a topological algebra, and let* V *be a finitely generated projective* A-*module. Then* tsr(V) *is defined to be the least integer* k, *if it exists, such that* $\mathrm{Gen}_k(V)$ *is dense in* V^k. *In particular,* tsr(A) *is defined to be the* tsr *of* A *as a right* A-*module.*

Motivation for the above definition can be found in [13]. There is a substantial literature in algebraic K-theory (see [1, 20]) relating the Bass stable rank to the non-stable behavior of K_1, especially Questions 9 and 10. Applied to C*-algebras, these results yield, for example:

THEOREM 2 (Theorem 10.12 of [13]). *If* $n \geqslant \mathrm{tsr}(A) + 2$, *then the map from* $U_n(A)/U_n^{\circ}(A)$ *to* $K_1(A)$ *is an isomorphism.*

Warfield [21] seems to have been the first to notice a direct general relationship between the Bass stable rank and the cancellation property for projective modules. A distinctive feature of his results is that it is not the Bass stable rank of the algebra which is important, but rather that of the endomorphism algebra of the module being cancelled. His results are in the spirit of Questions 3 and 4 to the effect that cancellation holds for "sufficiently large" modules, but now size is measured in terms of the size of the module being cancelled. For example, from his results one obtains:

THEOREM 3. *Let* W *be a projective* A-*module and let* n *be the Bass stable rank of* $\mathrm{End}_A(W)$. *If* U *and* V *are projective modules such that*

$$(U \oplus W^n) \oplus W \cong V \oplus W,$$

then $U \oplus W^n \cong V$.

To apply such stable rank techniques, one needs to be able to estimate stable ranks, and this is often very difficult. But Sheu's work mentioned above depends heavily on successful estimates of stable ranks, and the same is true for the results about non-commutative tori to be discussed in the next section. A crucial tool is provided by an estimate in the case of crossed products by the integers, which can be considered the most important result of [13] (see Theorem 7.1). Specifically:

THEOREM 4. *Let* $A \times_\alpha Z$ *denote the crossed product of a* C^**-algebra* A *by an action* α *of the integers. Then*

$$\mathrm{tsr}(A \times_\alpha Z) \leq \mathrm{tsr}(A) + 1.$$

Let us mention here that Blackadar [2] has used several of the techniques indicated above to show that cancellation holds for tensor products with algebras having ample small projections. For example, his Theorem A2 is:

THEOREM 5. *Let* A *be a simple unital* C^**-algebra, and let* B *be a* UHF C^**-algebra with* $B \otimes B \cong B$. *If* $\mathrm{tsr}(A \otimes B) < \infty$, *then* $A \otimes B$ *has cancellation.*

A somewhat analogous result concerning tensor products has been obtained by Sheu [17] using rather different methods. To state his result, let $B_n = C_\infty(R^n) \otimes K$ where K denotes the algebra of compact operators and R denotes the real line. For any C^*-algebra C without identity element we let $C^\sim$ denote the algebra obtained by adjoining an identity to C.

THEOREM 6 (Theorem 4.11 of [17]). *For any* C^**-algebra* A *and any* $n \geq 1$, *cancellation holds for* $(A \otimes B_n)^\sim$.

Notably missing are techniques for obtaining a lower bound for tsr in the absence of either a directly relevant compact space or of proper isometries. In particular, no finite simple C^*-algebras are known for which one can prove that $\mathrm{tsr}(A) \geq 2$, although there are many possible candidates. This is related to the lack of any example of a finite simple C^*-algebra for which cancellation fails, since tsr(A) = 1 is equivalent to the invertible elements being dense, and one has (see III.2.4 of [4] and 4.5.2 of [3]):

THEOREM 7. *If invertible elements are dense in* A, *then* A *satisfies cancellation.*

Notice, for example, that this implies that AF C*-algebras have cancellation. There is correspondingly a lack of any example of a finite simple C*-algebra for which one can show that the invertible elements are not dense.

Let us discuss next the fact that the various questions stated in §1 are somewhat interrelated. We give two examples, whose proofs will appear in [16]. The first involves Questions 8 and 10.

THEOREM 8. *Let* α *be an automorphism of the unital* C*-*algebra* A *which is in the connected component of the identity automorphism of* A, *and let* α *also denote the corresponding action of* Z *on* A. *Suppose that*

1. *Every element of* $K_1(A)$ *is represented by an invertible element in* A *itself.*
2. *The projections in* A *generate* $K_0(A)$.

Then every element in $K_1(A \times_\alpha Z)$ *is represented by an invertible element in* $A \times_\alpha Z$.

For the next result we let TA denote the C*-algebra of continuous functions from the circle, T, to A. We remark that then $TA \cong A \times_\alpha Z$ for α the trivial action, and it is an interesting question as to whether the next theorem can be generalized to the case of non-trivial α. This theorem involves Questions 2 and 9.

THEOREM 9. *For a unital* C*-*algebra* A *the following are equivalent*:

1. TA *satisfies cancellation.*
2. *Both* a) A *satisfies cancellation, and*
   b) *For every projective* A-*module* V *the natural map from* $Aut_A(V)/Aut_A^\circ(V)$ *to* $K_1(A)$ *is injective.*

The proof of this last theorem comes from examining the familiar "clutching" construction which to any automorphism of an A-module associates a TA-module.

3. NON-COMMUTATIVE TORI

By definition a non-commutative torus, A_θ, is a C*-algebra defined as follows. Let θ be a skew bilinear form on R^n, and define a skew cocycle σ on Z^n by

$$\sigma(x,y) = \exp(\pi i \theta(x, y))$$

for $x, y \in Z^n$. Let A_θ be the group C*-algebra of Z^n twisted by σ, i.e., $C^*(Z^n,\sigma)$. Thus to each $x \in Z^n$ there is a unitary, u_x, in A_θ, and these unitaries satisfy the relation

$$u_y u_x = \sigma(x, y) u_{x+y} .$$

For $n = 2$ one obtains the more familiar irrational (and rational) rotation C*-algebras [14].

By the work of Pimsner and Voiculescu [11] concerning the computation of the K-groups of crossed products with the integers, one finds that the K-groups of an A_θ are the same as those for an ordinary n-torus T^n (which is the A_θ for which $\theta \equiv 0$). In particular,

$$K_0(A_\theta) \cong Z^{2^{n-1}} .$$

This still leaves quite open the problem of determining what is the positive cone of $K_0(A_\theta)$. By using techniques from topological K-theory, one can show that the answer for T^n becomes complicated for $n = 4$ and 5 (see [16]), and I do not know if the answer is known for dimensions much above that. (Also, cancellation already fails for T^5.)

It turns out however that there is a nice answer when θ is not entirely rational, in the sense that the range of θ on the integer lattice $Z^n \subset R^n$ is not entirely contained in the rational numbers. Notice that there is a canonical trace, τ, on A_θ, corresponding to evaluating at the identity element of Z^n, with its associated homomorphism, τ, from $K_0(A_\theta)$ into the group of real numbers.

The latter is positive on the positive cone of $K_0(A_\theta)$. In [16] it is shown that:

THEOREM A. *If* θ *is not rational, then the positive cone of* $K_0(A_\theta)$ *consists of exactly the elements on which* τ *is positive.*

We should mention that the range of τ on $K_0(A_\theta)$ has been elucidated by Elliott [6], whose work is an important ingredient of the proofs of most of the theorems stated in this section. Much of the proof of the above theorem involves a specific construction, sketched in [15], of finitely generated projective modules over A_θ, together with a classification of the modules so constructed, by means of Connes' Chern character introduced in [5]. In fact, one finds that every element of $K_0(A_\theta)$ with positive trace is represented by a module obtained by the construction. If one examines the construction further so as to obtain, among other things, information about the topological stable rank of the endomorphism algebras of the constructed modules, one finds that one can apply Warfield's theorem (Theorem 2 above) to answer Question 2:

THEOREM B. *If* θ *is not rational, then* $S(A_\theta)$ *satisfies cancellation.*

Thus for such θ one can answer Question 1, that is, one can describe $S(A_\theta)$. Even more, one has an explicit construction of all finitely generated projective A_θ-modules up to isomorphism. For the special case $n = 2$ these results were obtained earlier in [14]. We also obtain in [16] an answer to Question 8:

THEOREM C. *If* θ *is not rational, then the projections in* A_θ *generate* $K_0(A_\theta)$.

By using Theorem 8 of the previous section with Theorem C above in an induction argument, we then obtain the following answer to Question 10:

THEOREM D. *If* θ *is not rational, then every element of* $K_1(A_\theta)$ *is represented by an invertible element of* A_θ.

By using Theorem 9 of the previous section with Theorem B we also obtain the following answer to Question 9:

THEOREM E. *If* θ *is not rational, then the natural map from* $U_1(A_\theta)/U_1^0(A_\theta)$ *to* $K_1(A_\theta)$ *is an isomorphism.*

From this theorem together with some additional argument, one obtains the following answer to Question 7:

THEOREM F. *If* θ *is not rational, then any two projections in* $M_m(A_\theta)$ *which represent the same element of* $K_0(A_\theta)$ *are in the same connected component of the set of projections in* $M_m(A_\theta)$.

In closing, let me mention that J. A. Packer [9, 10] has studied the algebras C*(G, σ) where G is the discrete Heisenberg group and σ is a cocycle on G. Among many other results, she has shown that for many σ's, these algebras satisfy cancellation. For this she uses, in part, the techniques of [13, 14].

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