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NON-COMMUTATIVE TORI — A CASE STUDY OF NON-COMMUTATIVE DIFFERENTIABLE MANIFOLDS

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ABSTRACT. The non-commutative tori are perhaps the most accessible and best studied interesting examples of non-commutative differentiable manifolds. We give a survey of many of the results which have been obtained about them.

Led by Alain Connes [Cn1, Cn3, Cn4], we have during this decade gotten a glimpse of a new kind of mathematical object, namely non-commutative differentiable manifolds. So far no one has given a satisfactory definition for these objects. But by now a number of naturally arising examples are known which will surely be included when a good definition is found. There is every indication that large parts of the mathematics which one does on ordinary manifolds will be extended to non-commutative manifolds. Substantial motivation for doing this comes from the contributions to issues in other areas of mathematics which such extensions will provide [Be1, Cn3, CM, RS1, Rs2]. In particular, one will study the index of elliptic operators on non-commutative differentiable manifolds. This is already explicitly indicated in Connes' original paper on non-commutative differential geometry [Cn1], and this is one of the ideas lying behind his work on index theorems for foliated manifolds [Cn2, CS].

Probably the most accessible interesting class of non-commutative differentiable manifolds are the non-commutative tori; they are surely the best understood, although many questions about them still remain open. In this report I will try to survey much of what is known, and indicate some of the open questions. It is my hope that readers of this report will come to feel that non-commutative tori are not exotic objects, but rather are attractive well-behaved objects closely associated with classical situations.

1. THE DEFINITION OF NON-COMMUTATIVE TORI.

Non-commutative tori arise naturally in a number of different situations. For example, they play a certain universal role in the representation theory of Lie groups [Pg], and they

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provide a convenient framework within which to study Schrödinger operators with quasi-periodic potentials [Bel]. But here we will choose to approach them from the direction which most strongly suggests their relationship to ordinary manifolds, namely from the direction of deformation quantization [Rf8, Rf9].

Let T^n be an ordinary n-torus. We will often use real coordinates for T^n, that is, view T as R/Z. Let $C^\infty(T^n)$ be the algebra of infinitely differentiable complex-valued functions on T^n. Pick a real skew-symmetric $n\times n$ matrix θ. Then we can use θ to define a Poisson bracket on $C^\infty(T^n)$ by

$$\{f, g\} = \Sigma\, \theta_{jk}(\partial f/\partial x_j)(\partial g/\partial x_k)$$

for $f, g \in C^\infty(T^n)$. The idea of deformation quantization is to seek to deform the pointwise product of $C^\infty(T^n)$ to a one-parameter family of associative products, $*_\hbar$, in such a way that

$$\lim_{\hbar\to 0} (f*_\hbar g - g*_\hbar f)/i\hbar = \{f, g\} .$$

To this end, use the Fourier transform to carry $C^\infty(T^n)$ to $S(Z^n)$, the space of complex-valued Schwartz functions on Z^n. Then the Fourier transform carries the pointwise multiplication on $C^\infty(T^n)$ to convolution on $S(Z^n)$. A simple computation shows that it also carries the Poisson bracket to

$$\{\phi, \psi\}(p) = -4\pi^2 \sum_q \phi(q)\psi(p - q)\gamma(q, p - q)$$

for $\phi, \psi \in S(Z^n)$ and $p, q \in Z^n$, where

$$\gamma(p, q) = \Sigma\, \theta_{jk}p_j q_k ,$$

and where the factor $4\pi^2$ comes from our convention that the Fourier transform, $\hat{f}$, for $f \in C^\infty(T^n)$ is defined by

$$\hat{f}(p) = \int_{T^n} \exp(-2\pi i x\cdot p)f(x)dx .$$

For every $\hbar \in R$ define a bicharacter, $\sigma_\hbar$, on Z^n by

$$\sigma_\hbar(p, q) = \exp(-\pi i\hbar\gamma(p, q)) ,$$

and then set

$$(\phi *_\hbar \psi)(p) = \sum_q \phi(q)\psi(p - q)\sigma_\hbar(q, p - q) .$$

Define the involution on $S(Z^n)$, independent of $\hbar$, to be that coming from complex conjugation on $C^\infty(T^n)$, so that

$$\phi^*(p) = \bar{\phi}(-p) .$$

For each $\hbar$ define the norm $\| \ \|_\hbar$ on $S(Z^n)$ to be the operator norm for the action of $S(Z^n)$ on $\ell^2(Z^n)$ given by the same formula as used above to define the product $*_\hbar$. Let $C_\hbar$ be $C^\infty(T^n)$ but with product, involution, and norm obtained by pulling back through the inverse Fourier transform the product $*_\hbar$, involution, and norm $\| \ \|_\hbar$. Then one can show [Rf8, Rf9] that the completions of the $C_\hbar$'s form a continuous field of C^*-algebras, and that for $f, g \in C^\infty(T^n)$ one has

$$\|(f*_\hbar g - g*_\hbar f)/i\hbar - \{f, g\}\|_\hbar \longrightarrow 0$$

as $\hbar \longrightarrow 0$. Since C_0 is just $C^\infty(T^n)$ with its usual pointwise multiplication and supremum norm, this all means that the $C_\hbar$'s form a strict deformation quantization of $C^\infty(T^n)$ in the direction of the Poisson bracket coming from θ, as defined in [Rf8].

We denote the algebra for $\hbar = 1$ by A_θ. Since A_θ is obtained by deforming $C^\infty(T^n)$, it is natural to call A_θ a non-commutative n-torus — the "C^∞" version. Let $\bar{A}_\theta$ denote the norm completion of A_θ, so that $\bar{A}_\theta$ is a C^*-algebra. This will be a deformation of $C(T^n)$, and so is considered the "topological" version of the non-commutative torus determined by θ.

It will be important for us at various points that, as is easily seen, the action of T^n by translation on $C^\infty(T^n)$ is also an action by continuous $*$-algebra automorphisms for the product $*_\hbar$ (that is, the deformation quantization constructed above is T^n-invariant), and so gives an action of T^n on $\bar{A}_\theta$, called the dual action. This dual action is easily seen to be ergodic in the sense that the only invariant elements of $\bar{A}_\theta$ are the scalar multiples of the identity element. In fact, it is shown in [OPT] that the $\bar{A}_\theta$'s are exactly all unitial C^*-algebras admitting ergodic actions of T^n (with full spectrum if the action is to be faithful).

Now for each $p \in Z^n$ the function $t \longmapsto \exp(2\pi i t \cdot p)$ will correspond to a unitary operator, U_p, in $\bar{A}_\theta$, and the mapping $p \longmapsto U_p$ will be a projective unitary representation of Z^n. Thus $\bar{A}_\theta$ can be viewed as the C^*-algebra generated by this representation. Alternatively, if we let $U_1, \cdots, U_n$ denote the unitary operators corresponding to the standard basis for Z^n, then they already generate $\bar{A}_\theta$ and satisfy the relation

$$U_k U_j = \exp(2\pi i \theta_{jk}) U_j U_k \quad ,$$

and it is not difficult to show that $\bar{A}_\theta$ is (isomorphic to) the universal C^*-algebra generated by n unitary operators which satisfy the above relations.

When $n = 2$, the skew-symmetric matrix θ is just determined by a real number, again denoted θ, and A_θ will be isomorphic to the crossed product C^*-algebra for the action of Z on the circle T coming from rotation by angle $2\pi\theta$. For this reason these algebras are often called "rotation algebras" [AP], or "irrational rotation algebras" when θ is irrational [Rf2].

Going back to the general case, let us define a functional, τ, on $S(Z^n)$ by

$$\tau(\phi) = \phi(0) \quad .$$

Note that τ is just the Fourier transform of Lebesgue measure on T^n. Let α denote the dual action of T^n on $\bar{A}_\theta$ defined earlier. Then it is easily seen that τ is invariant for α, and in fact that

$$\tau(\phi) = \int_{T^n} \alpha_t(\phi)dt \quad ,$$

where the left-hand side should initially be viewed as $\tau(\phi)I$, where $I = U_0$ is the identity element of $\bar{A}_\theta$. From this, one sees that τ extends to a trace τ on $\bar{A}_\theta$, which is faithful. And that the only α-invariant traces on $\bar{A}_\theta$ are the scalar multiples of τ [Sl, Gr, OPT]. The Hilbert space obtained by applying the GNS construction [KR] to τ is easily identified with $\ell^2(Z^n)$, with the U_p's as an orthonormal basis.

For every $a \in \bar{A}_\theta$ one can define its "Fourier coefficients" $\{a_p\}$ by

$$a_p = \tau(aU_p^*) \quad .$$

Because τ is faithful, an element of $\bar{A}_\theta$ will be determined by its "Fourier coefficients", though, just as with $C(T^n)$, it can be difficult to tell when a given function on Z^n is the set of "Fourier coefficients" for an element of $\bar{A}_\theta$. But at any rate, one can do for non-commutative tori many of the same maneuvers which one does for ordinary Fourier series in n variables. This is additional justification for using the term "non-commutative tori".

Because the non-commutative tori are so closely related to ordinary tori, it is natural to expect that many of their properties will be similar to those for ordinary tori. In the next sections we will first discuss this question at the "topological" level, and then at the "C^∞" level.

2. "TOPOLOGICAL" PROPERTIES OF NON-COMMUTATIVE TORI. If θ has sufficient irrationality so that whenever $\theta(p, Z^n) \subset Z$ for some $p \in Z^n$ it follows that $p = 0$, then $\bar{A}_\theta$ is simple (as is A_θ), i.e. has no proper two-sided ideal [Sl, Gr, OTP]. In this sense A_θ is then as far away as possible from being an algebra of functions on a topological space. One can show in this case that the only traces on $\bar{A}_\theta$ are the scalar multiples of τ [Sl, Gr, OTP].

If θ is rational, then $\bar{A}_\theta$ is a bundle of full matrix algebras over a T^n, the matrix size depending on the denominators of θ [Gr, HS, OTP, Rf4]. In particular, $\bar{A}_\theta$ is strongly Morita equivalent to $C(T^n)$ [Rf3]. But these bundles are almost never trivial (i.e. product) bundles [HK, Rf4, DB].

Even though $\bar{A}_\theta$ is far from being a function algebra when it is simple, we will see in section 4 that it is still important to extend to it the notion of the classical dimension of a compact space [Pr]. This can be done in the following way, described in more detail in [Rf4]. Let M be a compact space. Then a standard theorem from classical dimension theory (proposition 3.3.2 of [Pr]) says that the classical dimension of M is the least integer n such that every continuous function f from M into R^{n+1} can be approximated arbitrarily closely by functions which do not contain the origin in their ranges. Now such a function f is an (n+1)-tuple, f_1, ---, f_{n+1}, of real-valued functions, and the condition that f miss the origin is just the condition that the f_i's nowhere vanish simultaneously. Let $C_R(M)$ denote the Banach algebra of real-valued continuous functions on M. Then this last condition just says that the ideal of $C_R(M)$ generated by the f_i's all together is $C_R(M)$. To generalize this to non-commutative algebras, one must choose to use either left or right ideals, though for algebras with involution this choice will make no difference. For A an algebra with identity element, we let $Lg_n(A)$ denote the set of n-tuples of A which generate A as a left ideal. When A is a Banach algebra, the theorem from classical dimension theory stated above indicates that we are interested in when $Lg_n(A)$ is dense in A^n. But we are interested in algebras over the complex numbers, and a complex-valued function on a compact space M will correspond to two real-valued functions. Because of this, it will not be appropriate to use the term "dimension", and so instead we will use the term "topological stable rank".

2.1 Definition [Rf4]. Let A be a Banach algebra with identity element. By the ***left topological stable rank*** of A, denoted ltsr(A), we mean the least integer n such that $Lg_n(A)$ is dense in A^n (= ∞ if no such integer exists).

Then for a compact space M we will have $tsr(C(M)) = [dim(M)/2] + 1$ where [ ] denotes "integer part of". (We use *tsr* instead of *ltsr* for algebras with involution, where the choice of left or right ideals gives the same result.)

An upper bound on the tsr of non-commutative tori can be obtained as follows. Any non-commutative torus can be constructed as a succession of crossed products by actions of the group Z, starting from a trivial action on the one-dimensional algebra $\mathbb{C}$ [El2, Rf6]. But a basis result [Rf4] about tsr is that if α is an action of Z on a C^*-algebra A, then

$$tsr(A \times_\alpha Z) \leq tsr(A) + 1 \quad .$$

Thus we see that if $\bar{A}_\theta$ is a non-commutative torus based on Z^n, then $tsr(\bar{A}_\theta) \leq n + 1$. Actually, when θ is not rational one can show that $tsr(\bar{A}_\theta)$ is no larger than 2, and must be equal to 2 if $\bar{A}_\theta$ is not simple.

But Riedel [Rdl, AP] has shown that, surprisingly, for some simple $\bar{A}_\theta$ one actually has $\mathrm{tsr}(\bar{A}_\theta) = 1$. How often this happens is unclear. That is, a very open question is:

2.2 QUESTION: Is $\mathrm{tsr}(\bar{A}_\theta) = 1$ whenever $\bar{A}_\theta$ is simple?

Let us mention that having $\mathrm{tsr}(A) = 1$ is equivalent to the invertible elements being dense in A [Rf4]. While I was making final corrections to this manuscript, Ian Putnam told me by telephone that he believes he has a proof that the answer to Question 2.2 is always affirmative for irrational rotation C^*-algebras.

Related to the above is the very recent result of Choi and Elliott [CE] that for a dense set of θ's, any self-adjoint element of the irrational rotation C^*-algebra $\bar{A}_\theta$ can be approximated in norm by self-adjoint elements which have finite spectrum.

3. VECTOR BUNDLES AND K-THEORY. Let M be a compact space and let E be a complex vector-bundle over M. Then the space, $\Gamma(E)$, of continuous cross-sections of E is a finitely generated projective module over M, and all finitely generated projective $C(M)$-modules arise in this way, according to a theorem of Swan [Sw, Rf3]. Since we think of C^*-algebras with identity element as being "non-commutative compact spaces", it is then natural to view projective modules over them as being the generalization of vector bundles. (For brevity we will say "projective" when we mean "finitely generated projective".)

Given any ring A with identity element, it is of great interest to classify the isomorphism classes of projective A-modules. With the operation of forming direct sums, these classes form an Abelian semigroup, which we will denote by $S(A)$. So one would like to describe $S(A)$. But many examples show that $S(A)$ need not satisfy the cancellation law, including the case of $A = C(T^n)$ for $n \geq 5$. This tends to make it difficult to describe $S(A)$. A potentially easier problem is first to describe the cancellative semigroup, $C(A)$, formed as the quotient of $S(A)$ by forcing cancellation, that is, by decreeing that two elements, r and s, of $S(A)$ are equivalent if there is a $t \in S(A)$ such that $r + t = s + t$. But I believe that this problem also is unsolved for $C(T^n)$ for large n — it is certainly known that the situation becomes complicated [Rf6, Bl]. Finally, $C(A)$ embeds in its enveloping Abelian group ("Grothendieck groups"), denoted $K_0(A)$. This group is part of a homology theory periodic of period 2. The other group, $K_1(A)$, is defined to be the inductive limit of the groups $GL_k(A)/GL_k^0(A)$, where $GL_k(A)$ is the group of invertible $k \times k$ matrices over A while $GL_k^0(A)$ is its connected component, and $GL_k(A)$ is embedded in $GL_{k+1}(A)$ by $T \longmapsto T\oplus 1$. See [Bl]. When X is a compact space and $A = C(X)$, these groups are just the groups of topological K-theory.

The first major break-through in finding techniques for calculating the K-groups of non-commutative C^*-algebras was made by Pimsner and Voiculescu [PV1, Bl]. They obtained a periodic exact sequence for the K-groups of a crossed product of form $A\times_\alpha Z$, in terms of the K-groups of A and of the effect of α on the K-groups of A. Because, as mentioned in the last section, a non-commutative torus $\bar{A}_\theta$ can be constructed as a succession of crossed products for actions of Z, it is easy to deduce from the Pimsner-Voiculescu exact sequence that

$$K_0(\bar{A}_\theta) \cong Z^{2^{n-1}} \cong K_1(\bar{A}_\theta) \quad ,$$

just as happens for the topological K-groups of an ordinary n-torus T^n. Note in particular that the K-groups do not distinguish between the algebras $\bar{A}_\theta$ for different θ.

Another notable result of Pimsner and Voiculescu [PV2] of the same vintage as their exact sequence, is that an irrational rotation algebra $\bar{A}_\theta$ can be embedded in a specific AF C^*-algebra, B_θ, in such a way that the corresponding map on the K_0-groups is an isomorphism. We recall that an AF C^*-algebra is just an inductive limit of finite dimensional C^*-algebras, and that the AF C^*-algebras are considered to be the non-commutative analogues of Cantor sets. (Note that $\bar{A}_\theta$ itself is not an AF C^*-algebra since the K_1 group of any AF C^*-algebra is trivial.) That this embedding is remarkable is seen by observing that the corresponding statement for spaces would say that one has a compact space which is the quotient of a Cantor set in such a way that the quotient map gives an isomorphism of their K_0 groups. This seldom happens, and in particular does not happen for ordinary tori.

To carry the story further, Kumjian has shown [Km1] that B_θ can be embedded (unitally) in $\bar{A}_\theta$. By iterating this embedding and that of Pimsner and Voiculescu, he concludes in [Km2] that the AF algebra B_θ is the increasing limit of a sequence of subalgebras each of which is isomorphic to $\bar{A}_\theta$, with the inclusion maps all giving isomorphisms of the K_0 groups.

4. NON-STABLE K-THEORY. The K-groups can be viewed as defined by suitable stabilizations. The study of what happens before stabilizing, that is, of the semigroup $S(A)$ itself, and of the groups $GL_k(A)/GL_k^0(A)$ before taking the limit, is often called non-stable K-theory. Although for ordinary n-tori T^n the semigroup $S(T^n)$ is badly behaved and not yet fully understood, for large n, it turns out that as soon as θ has any irrational entries, $S(\bar{A}_\theta)$ is well-

behaved, and can be fully described [Rf6]. To explain this, we first mention that the canonical trace τ on $\bar{A}_\theta$ (the "Lebesgue measure") determines a group homomorphism, denoted again by τ, of $K_0(\bar{A}_\theta)$ into R, obtained by extending τ to a trace on matrices over $\bar{A}_\theta$ in the evident way, and then by evaluating this extended trace on the projection matrices which define projective modules. The main result of [Rf6] says that as soon as θ has at least one irrational entry, then cancellation holds in $S(\bar{A}_\theta)$ (so $S(\bar{A}_\theta) = C(\bar{A}_\theta)$), and that the embedding of $S(\bar{A}_\theta)$ into $K_0(\bar{A}_\theta)$ identifies $S(\bar{A}_\theta)$ with exactly the set of elements of $K_0(\bar{A}_\theta)$ on which τ is positive. Furthermore, it is shown in this case how to construct all projective modules over $\bar{A}_\theta$, up to isomorphism (theorems 6.1 and 7.1 and corollary 7.2 of [Rf6]). The proof depends in a crucial way on the information about $tsr(\bar{A}_\theta)$ described in the previous section, as well as the differential geometric techniques which we will discuss shortly.

The following is a prototypical example of a (non-free) projective module for a non-commutative two-torus [Cn1, Cn3, Rf2, Rf5]. In this case θ is specified by just one real number, which we again denote by θ. Let $\lambda = \exp(2\pi i\theta)$, and let U and V be two unitary generators for $\bar{A}_\theta$ satisfying the commutation relation $VU = \lambda UV$. Let S(R) denote the space of Schwartz functions on R. We let U act on S(R) by translation by θ, and V act by multiplication by $t \longmapsto \exp(2\pi it)$, and extend these actions to finite sums of products of powers of U and V. In the next section we will indicate how to define an inner-product on S(R) with values in $\bar{A}_\theta$, and a corresponding norm. When S(R) is completed for this norm, it becomes a projective $\bar{A}_\theta$-module which is not free [Cn1, Cn3, Rf2, Rf5]. This is certainly not an exotic object. The projective modules over non-commutative n-tori for higher n, for θ not rational, can all be decomposed as finite direct sums of projective modules which are suitable higher dimensional generalizations of the above module (corollary 7.2 of [Rf6]).

Already for 2-tori the contrast in non-stable K-theory between the rational and irrational cases is quite interesting. For T^2 the cancellation property does hold (by theorem 1.5 of chapter 8 of [Hs]), and $S(T^2)$ defines a positive cone in $K^0(T^2)$, as does $S(\bar{A}_\theta)$ in $K_0(\bar{A}_\theta)$. But for T^2, if one identifies $K^0(T^2)$ ($\cong Z^2$) with the integer lattice points in the plane, this positive cone can be identified with the integer lattice points in the upper half plane. In particular, $K^0(T^2)$ is not totally ordered by $S(T^2)$. But for θ irrational it can be shown [PV1, Rf2] that τ identifies $K_0(\bar{A}_\theta) \cong Z^2$ with the dense subgroup $Z + Z\theta$ of the real line, and that then $S(\bar{A}_\theta)$ is identified with the elements of $Z + Z\theta$ which are positive real numbers. In particular, $K_0(\bar{A}_\theta)$

is totally ordered by $S(\bar{A}_\theta)$.

One important consequence of this information about the positive cone is an answer to the question of when, given two different irrational numbers θ and θ', the corresponding algebras are isomorphic (a question which had remained open for some years). The answer is that they are isomorphic if and only if $\theta' = \pm(\theta + k)$ for some integer k [PV1, PV2, Rf2]. This answer was one of the first striking applications of K-theoretic methods to C^*-algebras.

For higher dimensional non-commutative tori these techniques are not powerful enough to settle the isomorphism question, and the situation remains tantalizingly unclear in spite of considerable effort [CEGJ, DEKR, Th], that is:

4.1 QUESTION: Given skew n×n matrices, θ and θ', when are $\bar{A}_\theta$ and $\bar{A}_{\theta'}$ isomorphic?

The strongest partial results I am aware of are given in [BCEN], which also contains partial results concerning when the corresponding smooth algebras A_θ are isomorphic. For information on the case when θ is rational see [DEKR, Br3, Db, Rf5, Ym]. See also [Rh1, Rh2].

While the range of the trace on $K_0(\bar{A}_\theta)$ does not give enough information to answer the isomorphism question for $n > 2$, it is still of much interest. Elliott [El2] has given the following elegant description of the range of the trace. View θ as a (nilpotent) element of the even exterior algebra $\Lambda^e R^n$, so that we can form the element $\exp(\theta)$ of $\Lambda^e R^n$. Let D denote the integral lattice in L^*, so that we can view $\Lambda^e D$ as the integral lattice in $\Lambda^e L^*$. Then the range of the trace is obtained by the pairing $\langle \exp(\theta), \Lambda^e D\rangle$. A proof of this within the context of Connes theory of n-traces is contained in Pimsner's paper [Pm].

The non-stable K-theory of non-commutative tori for θ not rational has further attractive properties. For example, the projective submodules of $\bar{A}_\theta$ as right module over itself already generate $K_0(\bar{A}_\theta)$ (corollary 7.10 of [Rf5]), and any two projections in a matrix algebra $M_k(\bar{A}_0)$ which determine isomorphic projective modules will be in the same path component of the set of projections in $M_k(\bar{A}_\theta)$ (theorem 8.13 of [Rf5]). For K_1 one finds that the natural map from $GL_k(\bar{A}_\theta)/GL_k^0(\bar{A}_\theta)$ to $K_1(\bar{A}_\theta)$ is an isomorphism for all $k \geq 1$ (theorem 8.3 of [Rf5]).

For any C^*-algebra A the group $K_1(A)$ is closely related to the homotopy groups of the groups $GL_k(A)$ [Bl]. By using the results on the non-stable K-theory of A_θ described above, one can show [Rf7] that for θ not rational one has

$$\pi_m(GL_k(\bar{A}_\theta)) \cong Z^{2^{n-1}}$$

for all integers $m \geq 0$ and $k \geq 1$.

5. HERMITIAN METRICS. Just as it is useful to equip a vector space with an inner-product, it is useful to equip a vector bundle, E, with inner-products on each fiber chosen in a continuous way, that is, with a Hermitian metric. We need a similar structure for projective modules over a C^*-algebra [Cn1]. To see what this should be, we note that it is natural to consider, for $\xi, \eta \in \Gamma(E)$, the function $\langle \xi, \eta \rangle_A$ on M defined by

$$\langle \xi, \eta \rangle_A(m) = \langle \xi(m), \eta(m) \rangle \quad .$$

It will be continuous, so in $A = C(M)$. Then $\langle \ , \ \rangle_A$ can be considered to be an A-valued inner product on $\Xi = \Gamma(E)$. If the inner products on the fibers are chosen to be linear in the second variable, as will be convenient, then $\langle \ , \ \rangle_A$ satisfies

1) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$
2) $\langle \xi, \eta \rangle_A^* = \langle \eta, \xi \rangle_A$
3) $\langle \xi, \xi \rangle_A \geq 0$

for $\xi, \eta \in \Xi$ and $a \in A$. One will also have definiteness, that is, if $\langle \xi, \xi \rangle_A = 0$ then $\xi = 0$. But this would still be satisfied if the fiber inner-product at one non-isolated point of M were zero while all other fiber inner-products were definite. Thus we need to consider the stronger property of self-duality, which holds exactly if all the fiber inner-products are definite, namely:

4) For any linear map, ϕ, from Ξ to A such that $\phi(\xi a) = \phi(\xi)a$ for all $\xi \in \Xi$ and $a \in A$, there is an $\eta \in \Xi$ such that $\phi(\xi) = \langle \eta, \xi \rangle_A$ for all $\xi \in \Xi$.

5.1 **Definition.** Let A be a unital C^*-algebra, and let Ξ be a projective right A-module. By a Hermitian metric on Ξ we mean a bi-additive A-valued function $\langle \ , \ \rangle_A$ on $\Xi \times \Xi$ which satisfies properties 1 to 4 above.

A projective module can always be equipped with a Hermitian metric (in many ways) by viewing it as a summand of a free module and restricting to it the standard Hermitian metric on the free module. Given a Hermitian metric on a projective right A-module Ξ, it is natural to define a norm on Ξ by

$$\|\xi\| = \|\langle \xi, \xi \rangle_A\|^{1/2} \quad .$$

As an example, consider the projective module defined in the previous section. With the notation used there, we want to define a Hermitian metric on S(R) (and then complete for the corresponding norm). For $\xi, \eta \in S(R)$ we can hope to write $\langle \xi, \eta \rangle_A$ as a finite sum

$$\langle \xi, \eta \rangle_A = \Sigma \langle \xi, \eta \rangle_A(m, n) U^m V^n$$

for suitable coefficients $\langle \xi, \eta \rangle_A(m, n)$. It turns out that the appropriate formula is

$$\langle \xi, \eta \rangle_A(m, n) = \int \xi(r)\eta(r - m\theta)\exp(-2\pi i n r)dr \quad .$$

Motivation for this formula can be found in [Rf1]. I hope that the reader will consider this formula to be nothing especially exotic, but rather quite similar to formulas found in traditional harmonic analysis.

6. SMOOTH STRUCTURE. As mentioned earlier, the action of T^n by translation on $C^\infty(T^n)$ gives an action, α, on $\bar{A}_\theta$, which leaves the canonical trace τ invariant. On the Fourier space $S(Z^n)$ this action is given by

$$(\alpha_t(\phi))(p) = \exp(2\pi i t\cdot p)\phi(p)$$

for $t \in T^n$, $p \in Z^n$ and $\phi \in S(Z^n)$. From this it is not difficult to see that the space of C^∞-vectors for this action on $\bar{A}_\theta$ is just $C^\infty(T^n) \sim S(Z^n)$ itself, that is, our original A_θ. Let δ_k denote differentiation on A_θ in the k^{th} direction of T^n, that is, the infinitesimal generator for the action of translation on A_θ in the k^{th} direction of T^n. Then δ_k is given by

$$(\delta_k(\phi))(p) = 2\pi i p_k \phi(p) \quad .$$

Each δ_k is a $*$-derivation of A_θ, that is, satisfies

1) $(\delta_k(a))^* = \delta_k(a^*)$

2) $\delta_k(ab) = \delta_k(a)b + a\delta_k(b)$.

These directional derivatives can be used to form "partial differential operators" on A_θ. For example, the Laplace operator is

$$\Delta = \sum_k \delta_k^2 \quad .$$

Note that Δ coincides in its action on the linear space A_θ, with the usual Laplace operator on $C^\infty(T^n)$. Hence, as Connes has indicated [C1], $(1 - \Delta)^{-1}$ is a compact operator on $L^2(\bar{A}_\theta, \tau)$. One can also define linear partial differential operators with "non-constant coefficients". For example a first order operator would be of the form

$$\Sigma\, a_k \delta_k$$

for $a_k \in A_\theta$. Connes indicates [C1] that one can also develop an appropriate calculus of pseudodifferential operators, index theorems for elliptic operators, etc. In a different direction, Ji has studied [Ji] a generalization to non-commutative tori of Toeplitz operators.

A very important property of A_θ as a subalgebra of $\bar{A}_\theta$ is that it is closed under the holomorphic functional calculus of $\bar{A}_\theta$, in the sense that if $a \in A_\theta$ and if f is a function analytic in a neighborhood of the spectrum of a viewed as an element of $\bar{A}_\theta$, then $f(a) \in A_\theta$ (and similarly for the $k \times k$ matrix algebras). This is an immediate consequence of results in the appendix of [Cn2]. Much of its importance lies in the fact that it implies that the embedding of A_θ in $\bar{A}_\theta$ gives an isomorphism of their K-theory [Cn2].

Another attractive property of A_θ as a "smooth" algebra is [BEJ] that any derivation, δ, of A_θ into itself has a decomposition $\delta = \delta_0 + \tilde{\delta}$ where δ_0 is a linear combination of the generators $\delta_1, \cdots, \delta_n$ with coefficients in the center of A_θ, while $\tilde{\delta}$ is an approximately inner derivation in an appropriate sense. Even more, if θ satisfies a suitable diophantine approximation condition, then $\tilde{\delta}$ must, in fact, be inner [BEJ]. These results are used in [BEGJ] to classify the possible smooth actions of Lie groups on non-commutative two-tori. See also [Jr].

Just as diffeomorphisms of ordinary manifolds are of interest, so should be those of non-commutative tori. Now the (generalization of) homeomorphisms consist just of the automorphisms of $\bar{A}_\theta$, while the (generalization of) diffeomorphisms consist of the automorphisms which carry A_θ onto itself. Elliott [El3] has shown that for non-commutative two-tori for which θ satisfies a suitable diophantine approximation condition, any diffeomorphism of A_θ is the product of an inner automorphism coming from a unitary in A_θ, a diffeomorphism coming from the action of T^2 on A_θ, and the diffeomorphism coming from an element of SL(2, Z) acting on the generators in the natural way [Br1, Br2]. But Kodaka [Kd] has shown that this very nice description can fail when θ does not satisfy Elliott's diophantine approximation condition. The "entropy" of the diffeomorphisms coming from elements of SL(2,Z) has been discussed by Watatani in [Wt].

In section 1 we used the fact that ordinary tori carry natural Poisson structures. In [Xu], a definition is given of a Poisson structure on a non-commutative algebra, and then it is shown how non-commutative two-tori carry such Poisson structures, and their Poisson cohomology is calculated.

7. DeRHAM HOMOLOGY. Connes has shown [Cn4] that the way to generalize to non-commutative algebras the deRham homology (defined in terms of currents) of a differentiable manifold, is by means of his cyclic cohomology. A k-cochain for the cyclic cohomology of an algebra A is a $(k + 1)$-multilinear functional ϕ on A which satisfies the cyclic condition

$$\phi(a_0, \cdots, a_k) = (-1)^k \phi(a_1, \cdots, a_k, a_0) \quad .$$

Since Hochschild cohomology is defined in terms of general multi-linear functionals, one can apply the Hochschild coboundary operator to cyclic cochains. One finds then that the cyclic cochains form a subcomplex of the Hochschild complex. The cohomology groups of this subcomplex, denoted $HC^k(A)$, are by definition the cyclic cohomology groups of A. There are natural maps from $HC^k(A)$ to $HC^{k+2}(A)$ for each k, and the limit groups for these maps of the even groups and of the odd groups are the even and odd parts of the non-commutative deRham cohomology of A, and their direct sum is the non-commutative deRham cohomology of A, which we denote by $H_{dR}(A)$. Although the definition of cyclic cohomology makes sense for C^*-algebras, it is not for C^*-algebras that it has been of primary interest so far, but rather it is the cyclic cohomology of suitable dense subalgebras ("smooth structures") on which interest has focused.

Connes calculated the cyclic cohomology and deRham cohomology of (the smooth version of) non-commutative 2-tori in [Cn4], and more recently Nest [Ns] has calculated the cyclic cohomology of general non-commutative tori A_θ. One obtains

$$H_{dR}(A_\theta) \cong \wedge L \cong C^{2^{n-1}}$$

where L is the complexified Lie algebra of T^n, so $L \cong C^n$. Just as happens for an ordinary n-torus, every cohomology class can be represented by cocycles which are invariant under the action of T^n. To describe these invariant cocycles, define δ_X for any $X \in L$ by

$$\delta_X = \Sigma\, c_k \delta_k$$

where the c_k's are the coefficients of X in the standard basis for L. Then for any $\mu = X_1 \wedge \cdots \wedge X_m$ with the X_j's in L, define ϕ_μ on A_θ^{m+1} by

$$\phi_\mu(a_0, \cdots, a_m) = \Sigma\, \mathrm{sgn}(\sigma)\tau(a_0 \delta_{X_1}(a_{\sigma(1)}) \cdots \delta_{X_m}(a_{\sigma(m)}))$$

where σ runs over the permutations of m elements. Then the ϕ_μ's are invariant, and finite sums of them represent all cohomology classes.

Let us mention that Connes shows in [Cn4] that the Hochschild cohomology groups of a non-commutative two-torus depend on the diophantine approximation properties of θ. We mentioned earlier that the diophantine approximation properties are also involved in whether the invertible elements are dense, whether approximately inner derivations are inner, and in the structure of diffeomorphisms. It would be interesting to find closer relations between these four phenomena.

The cyclic homology of an algebra A (corresponding to the deRham cohomology of a manifold defined in terms of differential forms rather than currents), is the homology of a quotient of the complex for Hochschild homology, which in turn has as k-chains elements of the (k + 1)-fold tensor product of A with itself. Thus elements of cyclic homology will have such tensors as representatives. The deRham homology group $H^{dR}(A)$ is again defined as a limit of the cyclic homology groups.

As far as I know, no paper has yet appeared which gives an explicit calculation of the cyclic or deRham homology of non-commutative tori. But there is little doubt that the deRham homology group will again be $C^{2^{n-1}}$. In section 8 we will describe many elements of this group, in which it appears in the guise of $\wedge L^*$, where L^* is the vector space dual of L.

8. CONNECTIONS AND CURVATURE. Given a projective module, we would like to attach to it a Chern character, having values in deRham homology. If the module, say $\tilde{\Xi}$, is over $\bar{A}_\theta$, we need to find a smooth version of it, that is, an A_θ-module Ξ such that $\tilde{\Xi} = \Xi \otimes_{A_\theta} \bar{A}_\theta$. This can always be done [Cn1, Rf6] because of the fact that A_θ is closed under the holomorphic functional calculus of $\bar{A}_\theta$.

Connes showed [Cn1, Cn4] that the Chern character can be constructed by the Chern-Weil approach in terms of connections and curvature. (See also [Ka]). If Ξ is a projective A_θ-module, then a connection for Ξ is a linear map, ∇, of Ξ into $L^* \otimes \Xi$ satisfying the Leibnitz rule

$$\nabla_X(\xi a) = (\nabla_X \xi)a + \xi \delta_X(a)$$

for all $X \in L$, $\xi \in \Xi$ and $a \in A_\theta$, where δ_X is as defined in the previous section. Connections always exist, for one can use δ component-wise on free modules, and then compress this to direct summands of free modules. If ∇ and ∇' are connections on Ξ, then a simple calculation shows that $\nabla_X - \nabla'_X$ is in $\mathrm{End}_{A_\theta}(\Xi)$ for all $X \in L$, so that the connections form an affine space over the linear maps from L to $\mathrm{End}_{A_\theta}(\Xi)$.

The curvature of a connection measures, in the context of [Cn1], the extent to which the connection fails to be a Lie algebra homomorphism. Since here L is Abelian, this means that the curvature, R^∇, of a connection ∇ is defined simply by

$$R^\nabla(X, Y) = [\nabla_X, \nabla_Y]$$

for $X, Y \in L$. It is easily seen that the values of R^∇ are in $\mathrm{End}_{A_\theta}(\Xi)$, so that it is a skew bilinear map on L with values there.

For the projective module S(R) over a non-commutative two-torus described in section 4, a connection can be defined [Cn1] by defining its values, ∇_1 and ∇_2, on the standard basis for L, to be

$$(\nabla_1\xi)(t) = (d\xi/dt)(t)$$

$$(\nabla_2\xi)(t) = (2\pi it/\theta)\xi(t) \quad .$$

Its curvature is then determined by its value on the wedge of the standard basis vectors, that is, by

$$[\nabla_1, \nabla_2] = (2\pi i/\theta)I \quad ,$$

where I is the identity operator on $S(R)$. Analogous connections on many projective modules over higher-dimensional non-commutative tori are explicitly constructed in [Rf6], and their curvatures calculated.

9. CHERN CHARACTER. Let Ξ be a projective A_θ-module, and let ∇ be a connection on Ξ with curvature R^∇. We wish to associate with this data a (non-homogeneous) even cycle on A_θ. This cycle will pair with the deRham cohomology, which we saw was ΛL, and so determines an element, ch^∇, in $\Lambda^e L^*$. We will specify ch^∇ by describing how its components, ch_k^∇, pair with the components, $\Lambda^{2k}L$, of the even deRham cohomology of A_θ. To this end we note that when Ξ is equipped with a Hermitian metric, then the canonical trace, τ, on A_θ, determines [Rf2] a (non-normalized) trace, τ', on $E = \mathrm{End}_{A_\theta}(\Xi)$ such that

$$\tau'(\langle\xi, \eta\rangle_E) = \tau(\langle\eta, \xi\rangle_A) \quad ,$$

where $\langle\xi, \eta\rangle_E$ is the element of E defined by $\langle\xi, \eta\rangle_E\zeta = \xi\langle\eta, \zeta\rangle_A$. Next, let $(R^\nabla)^{\wedge k}$ be the exterior (wedge) k^{th} power of R^∇, so that it is an alternating $2k$-form with values in E. Then ch_k^∇ is defined by

$$ch_k^\nabla = \tau'((R^\nabla/2\pi i)^{\wedge k})/k! \quad .$$

Thus ch_k^∇ is in $\Lambda^{2k}L^*$. Connes shows [Cn1] that ch_k^∇ is independent of ∇, so depends only on Ξ, and can thus be denoted by $ch_k(\Xi)$. Then he defines the total Chern character, $ch(\Xi)$, of Ξ by

$$ch(\Xi) = \oplus ch_k(\Xi) \quad ,$$

an element of $\Lambda^e L^*$.

For the projective module $\Xi = S(R)$ over a non-commutative two-torus described earlier, straightforward computation shows [Cn1] that

$$ch_1(\Xi) = \pm\bar{Z}_1\wedge\bar{Z}_2$$

where $\{\bar{Z}_j\}$ is the dual basis to the standard basis of L, and where the sign depends on the orientation chosen for the basis of L. As Connes showed [Cn1, Cn4] the Chern character of a projective module depends only on the class of the module in $K_0(A)$, and taking Chern characters gives a homomorphism from all of $K_0(A)$ into the even cyclic homology group. Elliott [El2] showed that for non-commutative tori this homomorphism is injective, and he gave an elegant description of its range inside $\Lambda^e L^*$, as follows. As earlier, view θ as a (nilpotent) element of the even exterior algebra $\Lambda^e L$, so that we can form $\exp(\theta)$ in this algebra. Then contraction of elements of $\Lambda^e L^*$ by $\exp(\theta)$ defines a vector space automorphism, $\exp(\theta)\lrcorner$, of the exterior algebra $\Lambda^e L^*$. With D the integral lattice in L^*, Elliott shows that the range of the Chern character on $K_0(A_\theta)$ is exactly

$$\exp(\theta)\lrcorner\Lambda^e D$$

inside $\Lambda^e L^*$. This fact is crucial for the proof of the results described in section 4 concerning the non-stable K-theory of non-commutative tori.

10. YANG-MILLS. The Yang-Mills problem can be posed and studied in the context of non-commutative tori [CR, Rf10, Sp]. Let Ξ be a projective A_θ-module, equipped with a Hermitian metric. A connection ∇ on Ξ is said to be compatible with the Hermitian metric if it satisfies the Leibnitz rule

$$\delta_X(\langle\xi, \eta\rangle_A) = \langle\nabla_X\xi, \eta\rangle_A + \langle\xi, \nabla_X\eta\rangle_A \quad .$$

We will let $CC(\Xi)$ denote the space of compatible connections on Ξ. It is easily seen to be an affine space over the linear maps from L into E_s, where E_s denotes the elements of $\mathrm{End}_{A_\theta}(\Xi)$ which are skew-adjoint for the Hermitian metric.

We wish to define a functional on $CC(\Xi)$ which measures the "strength" of a connection. For this purpose we need a "Riemannian metric" on A_θ. Since L is playing the role of the tangent space of A_θ, this means that we must choose an inner-product on L. This inner-product then determines an E-valued inner-product, $\{\ ,\ \}$, on the space of alternating E-valued 2-forms on L. We can then define a non-negative real-valued non-linear functional, YM, on $CC(\Xi)$ by

$$YM(\nabla) = -\tau'(\{R^\nabla, R^\nabla\}) \ .$$

The Yang-Mills problem is then to determine the minima and critical points for YM. The Yang-Mills equations are the Euler-Lagrange equations for the critical points.

Let UE denote the group of elements of E which are unitary (for the Hermitian metric). This is the gauge group for our context. It acts on $CC(\Xi)$ by conjugation, and simple calculations show that YM is invariant under this action. The set $MC(\Xi)$ of minima for YM is thus invariant under the action of UE. By definition the orbit space $MC(\Xi)/UE$ is the moduli space for the minima of YM. One has other moduli spaces for various families of critical points of YM.

For non-commutative two-tori all the above can be calculated. In this case every projective module is of the form Ξ^d where Ξ is not a multiple of any other projective module. Then it is shown in [CR] that $MC(\nabla)/UE$ is homeomorphic to $(\hat{T}^2)^d/S_d$ where S_d is the permutation group on d elements. (A somewhat different approach has recently been given in [Sp].) The moduli spaces for critical points have a similar but slightly more complicated description [Rf10].

For higher dimensional non-commutative tori there are certain projective modules to which the results for two-tori readily extend (the modules of form $V \otimes \Xi$ in theorem 5.6 of [Rf6] where V is an "elementary" module as in definition 4.2 of [Rf6]). However, for more general modules the situation is unclear at present, though it should be very interesting to investigate. I have obtained some very partial results, which indicate the usefulness of considering Einstein-Hermitian vector bundles, generalizing those defined, for example, in [Kb].

11. RELATIONS WITH MATHEMATICAL PHYSICS. The simplest discrete Schrodinger operators with almost-periodic potential, the almost Mathieu operators, are closely related to non-commutative two-tori [Be1, Be2, Be3, BLT], and so the latter have provided a convenient setting for their study. As earlier, let U and V be generators for a non-commutative two-torus satisfying $VU = \exp(2\pi i\theta)UV$. Let β be a real coupling constant. Then the corresponding almost Mathieu operator is

$$H = U + U^* + \beta(V + V^*) .$$

Clearly H is contained in A_θ. The main questions about H have to do with its spectrum, and in particular with how often its spectrum is a Cantor set. The K-theory of A_θ is relevant to this question because the spectral projections of H for intervals whose endpoints are in gaps in the spectrum of H will be elements of A_θ, and so will contribute to the K-theory of A_θ, and can be labeled by their Chern characters.

The strongest results to date which use A_θ were obtained very recently by Choi, Elliott and Yui [CEY], and state that for $\beta = 1$ and for θ a Liouville number the spectrum of H is indeed a Cantor set, and provide information on the labeling of the gaps in the spectrum of H. Other work in the framework of A_θ has been done by Riedel [Rd2, Rd3, Rd4], who has developed techniques which perhaps will eventually be able to be used to obtain examples of almost Mathieu operators whose spectrum is not a Cantor set.

Let α be the automorphism of A_θ which carries U to U^* and V to V^*. It is clear that H is invariant under α, and so is contained in the fixed-point algebra, A_θ^α, of α. It is thus very desirable to understand A_θ^α, but this has so far proved to be surprisingly elusive. In particular, it is not known at present how to calculate the K-theory of A_θ^α. Some very partial results about A_θ^α are contained in [BEEK].

Another situation in which non-commutative tori have been related to mathematical physics is in quantum diffusions. See [Ap1, Ap2, Ap3, HR1, HR2] and the references therein.

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