# Unitary Representations of Group Extensions; an Algebraic Approach to the Theory of Mackey and Blattner†

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In [32] Mackey used his theory of induced representations for separable locally compact groups [30] to attack the problem of describing the unitary representation theory of a separable locally compact group $G$ in terms of the representation theories of a normal subgroup $N$ and of the quotient group $G/N$. In the case in which $N$ is nicely embedded in $G$, he obtained far-reaching generalizations of results obtained earlier for finite groups by Clifford [12]. Mackey's main theorems were generalized to nonseparable groups by Blattner [5, 6] with weakened hypotheses concerning how $N$ is embedded in $G$. This necessitated introducing methods of proof which were considerably less measure-theoretic than those of Mackey. In [36] a theory of induced representations for $C^*$-algebras was introduced which has Mackey's definition for groups as a special case. This theory for $C^*$-algebras can involve no measure theory, and so required techniques that are different from those of either Mackey or Blattner. In this chapter we prove a version of the central theorem of Mackey's analysis for group extensions by using the approach introduced in [36]. In particular, our proof is fairly algebraic, and involves no measure theory beyond the elementary facts about Haar measure needed, for example, to convolve continuous functions of compact support and to construct the integrated forms of unitary representations. Our version of Mackey's theorem is not quite as strong as Blattner's, in large part because some of Blattner's hypotheses are explicitly and essentially measure-theoretic. But our version is applicable in many of the situations that ordinarily arise, such as the case in which the normal subgroup is "regularly embedded." The rather algebraic nature of the proof given here makes some of the mechanisms involved in the situation

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somewhat more transparent. Also, some of the techniques used and intermediate results obtained are of independent interest. For example, all of Section 3 develops general information relating ideals and quotients of $C^*$-algebras to the imprimitivity bimodules introduced in [36], while in other places results on weak containment (Proposition 4.1) and on the ideal centers of $C^*$-algebras (Proposition 5.3) are obtained. (See *Note Added in Proof*, p. 80.)

The chapter is organized in the following way. In Section 1 basic definitions are given and our version of Mackey's theorem is stated. Section 2 is devoted to constructing the $C^*$-algebras involved in the proof of the main theorem and to establishing a crucial relation between them. Section 3, which can be read independently, was just described. In Section 4 the material of Section 3 is applied to the situation involved in the main theorem, while in Section 5 systems of imprimitivity are tied into the situation by means of ideal centers of $C^*$-algebras. The proof of the main theorem is then concluded in Section 6. Various alternative hypotheses (related to work of Effros [17] and Glimm [24]) under which the main theorem remains true are gathered together in Section 7. Finally, in Section 8 we sketch for the reader's convenience for remaining part of Mackey's analysis, which does not involve the inducing process, but is concerned with projective representations. We make a few minor improvements of the proofs, but essentially all the material of this last section already exists in various places in the literature.

## 1. The Statement of the Main Theorem

In order to be able to state our main theorem we need to know what it means for a representation of a $C^*$-algebra to live on a given subset of the primitive ideal space of the algebra, at least when the subset is nice. This can be done in terms of the projection-valued measure introduced by Glimm [25] (see also the appendix of [23]), but since we will only need to consider subsets which are locally closed (equivalently, open in their closure, see p. 41 of [7]), it is easier to work directly with the ideals involved.

Let $A$ be a $C^*$-algebra. We will let $\text{Prim}(A)$ denote the primitive ideal space of $A$ with the hull–kernel topology [14]. By a Hermitian $A$-module we will mean (as in [36, 37]) the Hilbert space of a nondegenerate $*$-representation of $A$. We will usually work with Hermitian modules rather than the corresponding representations. Let $V$ be a Hermitian $A$-module, let $C$ be a closed subset of $\text{Prim}(A)$, and let $\ker(C)$ denote the intersection of the primitive ideals in $C$. In analogy with the case of a commutative $C^*$-algebra and representations thereof defined by measures, we say that $V$ *lives on* $C$ if $\ker(C)V = \{0\}$, so that $V$ can be viewed as a Hermitian module over

$A/\ker(C)$. In fact, this is the starting point for the definition of Glimm's projection-valued measure. Suppose now that $E$ is a subset of Prim$(A)$ which is locally closed, so that $E = \bar{E} - E'$, where $\bar{E}$ is the closure of $E$, $E' = \bar{E} - E$, and $E'$ is a closed set. We would like to say what it means for a Hermitian $A$-module $V$ to live on $E$. For this it must certainly live on $\bar{E}$, so that it can be viewed as a module over $A/\ker(\bar{E})$. Now by Proposition 3.2.1 of [14], Prim$(A/\ker(\bar{E}))$ is naturally homeomorphic to $\bar{E}$. Thus we can view $E$ as an open subset of Prim$(A/\ker(\bar{E}))$, with complement $E'$. Again, working in analogy with the commutative case, we see that to say that $V$ lives on $E$ should mean that as a module over $\ker(E')$ it is nondegenerate (and so Hermitian), that is, $\ker(E')V = V$. (Throughout this chapter whenever we juxtapose a subset of an algebra with a subset of a module this will denote the closed linear span of the products of corresponding elements.)

DEFINITION 1.1. Let $A$ be a $C^*$-algebra, let $V$ be a Hermitian $A$-module, and let $E$ be a locally closed subset of Prim$(A)$. Let $\bar{E}$ denote the closure of $E$ in Prim$(A)$ and let $E' = \bar{E} - E$. Then we will say that $V$ *lives* on $E$, or that $E$ *carries* $V$, if $\ker(\bar{E})V = \{0\}$ and $V$ is nondegenerate as a $\ker(E')$-module, so that $V$ is a Hermitian module over $\ker(E')/\ker(\bar{E})$.

Thus the category of all Hermitian $A$-modules that live on $E$ is naturally isomorphic with the category of all Hermitian modules over $\ker(E')/\ker(\bar{E})$.

If $H$ is any locally compact group, we let $C^*(H)$ denote its group $C^*$-algebra [14], and we will write Prim$(H)$ instead of Prim$(C^*(H))$. By a unitary $H$-module we will mean the Hilbert space of a strongly continuous unitary representation of $G$, as in [36]. Suppose now that $G$ is a locally compact group and that $N$ is a normal subgroup of $G$. (Subgroups will always be assumed to be closed.) Then the inner automorphisms of $G$ carry $N$ into itself, and so $G$ acts as a group of automorphisms of $N$, and so of all structures naturally associated to $N$. In particular, $G$ acts as a group of $*$-automorphisms of $C^*(N)$, and this action is easily seen to be strong operator-continuous. As a result, $G$ acts as a topological transformation group on Prim$(N)$ (see [20, 25]). In particular, if $J \in$ Prim$(N)$, we can let $GJ$ denote the orbit of $J$ in Prim$(N)$ under the action of $G$ and we can let $G_J$ denote the stability subgroup of $G$ at $J$. Since the primitive ideal space of any $C^*$-algebra is a $T_0$ topological space [14], $G_J$ will be a closed subgroup of $G$ by Lemma 1 of [6]. There is, of course, a natural bijection of $G/G_J$ onto $GJ$, which will be continuous, but is not, in general, a homeomorphism.

We now state our version of the main theorem of Mackey's analysis [32] for group extensions.

THEOREM 1.1. (The Main Theorem). *Let $G$ be a locally compact group and let $N$ be a normal subgroup of $G$. Let $J \in$ Prim$(N)$, let $GJ$ denote the orbit* *of $J$ in* Prim$(N)$ *under the action of $G$, and let $G_J$ denote the stability subgroup of $J$. We make the following hypotheses:*

(1) $\{J\}$ *is locally closed in* Prim$(N)$.

(2) $GJ$ *is locally closed in* Prim$(N)$, *and the canonical map of $G/G_J$ onto $GJ$ is a homeomorphism.*

*Then the process of inducing representations from $G_J$ to $G$ establishes an equivalence of the category of unitary $G_J$-modules whose restrictions to $N$ live on $\{J\}$ with the category of unitary $G$-modules whose restrictions to $N$ live on $GJ$. This equivalence will preserve weak containment of representations.*

We remark that Mackey stated and proved his original version of this theorem for projective representations, but the theorem for projective representations can be easily deduced from that for ordinary representations, as discussed, for example, in [3], and so we concern ourselves only with ordinary representations. In Section 7 we shall examine various other conditions which ensure that the hypotheses of Theorem 1.1 are satisfied.

We also remark that the statement about weak containment has as a special case one of the main results (Proposition 5) of [16]. This result was also known to Fell (as yet unpublished, but see 6.4 of [2]). See also [39].

Our method of proving Theorem 1.1 will be to find (in the next section) $C^*$-algebras whose categories of Hermitian modules are isomorphic with the categories of unitary modules mentioned in the theorem, and then to show (in Sections 3–6) that from the inducing process one obtains an imprimitivity bimodule [36] between these $C^*$-algebras, so that their categories of modules are equivalent by Theorem 6.23 of [36].

## 2. The $C^*$-Algebras

For any locally compact group $H$ let $C_c(H)$ denote the space of continuous functions of compact support on $H$. Then on $C_c(H)$ we have the inductive limit topology, the supremum norm $\| \ \|_\infty$, and the $L^1$-norm, $\| \ \|_1$ (for left Haar measure on $H$). In addition, $C_c(H)$ with convolution and its usual involution [14] is a (dense) subalgebra of $C^*(H)$, and so can be equipped with the norm from that algebra.

Suppose now that $H$ is a subgroup of the locally compact group $G$. Then, as discussed in Section 4 of [36], both $C_c(G)$ and $C_c(H)$ can be viewed as subalgebras of the algebra $M(G)$ of finite measures on $G$ (by viewing their elements as densities against the left Haar measures of each group), and in $M(G)$ the convolution of an element of $C_c(G)$ on the left or right by an element of $C_c(H)$ will again be an element of $C_c(G)$. For future use we state the formulas for these convolutions in terms of left Haar measures, where $\Delta$ denotes the modular function for $G$. For $f \in C_c(G)$ and $\varphi \in C_c(H)$ we have

$$\varphi * f(y) = \int_H \varphi(t) f(t^{-1}y)\,dt, \qquad f * \varphi(y) = \int_H f(yt^{-1})\,\Delta(t^{-1})\varphi(t)\,dt$$

for all $y \in G$.

For any $C^*$-algebra $A$ we will let $M(A)$ denote the double centralizer algebra of $A$ [9]. Then, according to Proposition 4.1 of [36], the left and right actions of $C_c(H)$ on $C_c(G)$ previously defined extend to give a $*$-homomorphism of $C^*(H)$ into $M(C^*(G))$. It is not known whether this homomorphism is injective in general (see the comments after Proposition 4.1 of [36]). However, in the case of interest to us, in which $H$ is normal, it is an easy consequence of Theorem 4.5 of [21] that this homomorphism is injective if $G$ is separable. Moreover, J. M. G. Fell has pointed out (personal communication) that this is also true in the nonseparable case as well. Actually, a new proof of this fact will emerge later from our general theory (Proposition 4.1).

Let $H$ still be a subgroup of $G$ which need not be normal in $G$. We will not introduce any notation for the homomorphism of $C^*(H)$ into $M(C^*(G))$, but for $d \in C^*(H)$ and $a \in C^*(G)$ we will simply write $da$ or $ad$ for the action of $d$ on the left or right of $a$ under this homomorphism. We remark that since $C_c(H)$ contains an approximate identity for the inductive limit topology for its action on $C_c(G)$, $C^*(G)$ will be an essential (i.e., nondegenerate) $C^*(H)$-module for both the left and the right actions of $C^*(H)$ on $C^*(G)$.

Suppose now that $E$ is a locally closed subset of $\mathrm{Prim}(H)$. Let $E' = \bar{E} - E$, so that $E'$ is closed. Let $W$ be a unitary $G$-module. We would like to reformulate the statement that the restriction $W_H$ of $W$ to $H$ lives on $E$. To begin, we must have $\ker(\bar{E})W_H = \{0\}$. This will hold if and only if

$$C^*(G)\ker(\bar{E})C^*(G)W = \{0\}.$$

In addition, we must have $\ker(E')W_H = W_H$. This will hold if and only if

$$C^*(G)\ker(E')C^*(G)W = W.$$

Notice that $C^*(G)\ker(E')C^*(G)$ and $C^*(G)\ker(\bar{E})C^*(G)$ are ideals in $C^*(G)$. (When we say an "ideal" in a $C^*$-algebra we will always mean a two-sided ideal, as these are the only ones with which we shall need to deal.) Furthermore the first ideal contains the second. However, in general, the inclusion need not be proper, so that the quotient algebra may be the zero-dimensional $C^*$-algebra. This will happen if $E$ is too small to carry the restriction to $H$ of any representation of $G$. How this can easily happen when $H$ is normal will become clear immediately after Proposition 2.2. Anyway, if we admit the zero-dimensional representation as a unitary representation of a group, and as the only Hermitian module over the zero-dimensional $C^*$-algebra, then these arguments give the proof of:

PROPOSITION 2.1. *Let $H$ be a subgroup of a locally compact group $G$ and let $E$ be a locally closed subset of* $\mathrm{Prim}(H)$. *Let $\bar{E}$ denote the closure of $E$ and let $E' = \bar{E} - E$. Then the category of unitary $G$-modules whose restriction to $H$ lives on $E$ is isomorphic to the category of Hermitian modules over the $C^*$-algebra*

$$[C^*(G)\ker(E')C^*(G)]/[C^*(G)\ker(\bar{E})C^*(G)]$$

*(which may be zero-dimensional).*

If $N$ is a normal subgroup of $G$, if $J \in \mathrm{Prim}(N)$, and if the hypotheses of Theorem 1.1 are met, then it is the corresponding $C^*$-algebras defined as in Proposition 2.1 for $J$ and $C^*(G_J)$ and for $GJ$ and $C^*(G)$ which we shall show have an imprimitivity bimodule.

We pause now to obtain a bit of information concerning what kinds of sets can carry the restriction of a representation to a normal subgroup. This will clarify the remark preceding Proposition 2.1. If $N$ is a normal subgroup of the locally compact group $G$, then, as mentioned, $G$ acts as a group of automorphisms of $C^*(N)$. For $d \in C^*(N)$ and $y \in G$ we will let $d^y$ denote the result of applying to $d$ the automorphism corresponding to $y$. If $I$ is an ideal in $C^*(N)$ and if $y \in G$, we let $I^y$ be the ideal $\{d^y : d \in I\}$. If $I^y = I$ for all $y \in G$, we say that $I$ is a *$G$-invariant ideal.* It is clear that $I$ is a $G$-invariant ideal if and only if $\mathrm{hull}(I)$ is a $G$-invariant subset of $\mathrm{Prim}(N)$.

PROPOSITION 2.2. *Let $N$ be a normal subgroup of a locally compact group $G$, let $W$ be a unitary $G$-module, and let $W_N$ denote the restriction of $W$ to $N$. Then the kernel $K$ of (the representation corresponding to) $W_N$ is $G$-invariant. Thus the smallest closed subset of* $\mathrm{Prim}(N)$ *which carries $W_N$, namely the hull of $K$, is $G$-invariant.*

*Proof.* For any $d \in C^*(N)$, $y \in G$, and $w \in W$, it is easily seen that

$$d^y(w) = y(d(y^{-1}w)).$$

Now if $d \in K$ and if $y \in G$, $w \in W$, it follows that $d^y(w) = 0$, so that $d^y \in K$. Thus $K$ is $G$-invariant. Q.E.D.

It follows that if $C$ is a closed subset of $\mathrm{Prim}(N)$ that contains no non-empty $G$-invariant subset, then the restriction of no unitary $G$-module to $C$ will live on $C$, and so the algebra of Proposition 2.1 will be zero-dimensional.

As mentioned following Proposition 2.1, our aim is to show that there is an imprimitivity bimodule between the $C^*$-algebra for $J$ and $C^*(G_J)$ and that for $G_J$ and $C^*(G)$. (We shall be more precise shortly.) To show this we shall need to use facts that depend on the normality of $N$. The main such fact relates the right and left actions of $C^*(N)$ on the various objects involved. This does not depend on $G_J$ being a stability subgroup, and so for a while now we will assume only that we have a locally compact group $G$, a normal subgroup $N$, and a subgroup $H$ containing $N$. Then, according to results in Section 4 of [36], particularly Proposition 4.10, $C_c(G)$ can be viewed as a Hermitian $C_c(H)$-rigged module over both $C_c(G)$ and $C_c(N)$.

Since we need to study ideals in the group $C^*$-algebras, and these are not in general contained in their dense subalgebras of continuous functions of compact support, we need to extend the preceding situation to the completions. Now if $A$ and $B$ are pre-$C^*$-algebras and if $X$ is a Hermitian $B$-rigged $A$-module, then it is natural to equip $X$ with the norm

$$\|x\|_X = \|\langle x, x\rangle_B\|^{1/2}$$

for $x \in X$. For a proof that this is in fact a norm see Proposition 2.10 of [36]. Since the action of the pre-$C^*$-algebra $C_c(H)$ on the right of $C_c(G)$ and the $C_c(H)$-valued inner product on $C_c(G)$ are easily seen to be continuous for this norm, as are the left actions of the pre-$C^*$-algebras $C_c(G)$ and $C_c(N)$ by Proposition 4.10 of [36], these actions all extend to the completions. We thus obtain:

PROPOSITION 2.3. *If $X$ denotes the completion of $C_c(G)$ for the norm just defined, then $X$ is a Hermitian $C^*(H)$-rigged module over both $C^*(G)$ and $C^*(N)$.*

It is easily seen that the inductive limit topology on $C_c(G)$ is finer than the topology from the norm defined on $X$. (The argument is given in the proof of Proposition 4.11 of [36].)

Now $X$ is an essential right $C^*(H)$-module, and so, by applying the right-handed versions of either Proposition 3.9 of [36] or the proof of Theorem 21 of [29], $X$ becomes an essential right module over $M(C^*(H))$. Since $C^*(N)$ is mapped into $M(C^*(H))$ by Proposition 4.1 of [36], $X$ becomes a right $C^*(N)$-module, where the action of $\varphi \in C_c(N)$ on any element of $X$ of the form $f \cdot \psi$ for $f \in C_c(G)$ and $\psi \in C_c(H)$ (where $\cdot$ is defined in 4.5 of [36]) is, by definition, given by

$$(f \cdot \psi)\varphi = f \cdot (\psi * \varphi).$$

Since $N$ is normal in both $G$ and $H$, the modular functions, $\Delta_G$ and $\Delta_H$, of $G$ and $H$, respectively, must coincide on $N$ with the modular function of $N$ (15.23 of [27]) and so $\gamma(s) = 1$ for all $s \in N$, where $\gamma(r) = (\Delta_G(r)/\Delta_H(r))^{1/2}$ for $r \in H$, as in 4.2 of [36]. From this fact it is easily seen that $\gamma(\psi * \varphi) = (\gamma\psi) * \varphi$

so that $(f \cdot \psi)\varphi = (f \cdot \psi) * \varphi$. It follows that the right action of $C^*(N)$ on $X$ is just given by ordinary right convolution of $C_c(N)$ on $C_c(G)$.

As we saw previously, $C^*(N)$ also acts on the left as bounded operators on the space $X$, this action just coming from left convolution of $C_c(N)$ on $C_c(G)$. We wish to show that the left and right actions of $C^*(N)$ on $X$ are related in terms of the action of $G$ as automorphisms of $N$, and so of $C^*(N)$. To do this we now examine more specifically how these automorphisms are defined at the level of functions. Since $G$ acts as a group of automorphisms of $N$, it will act as a group of automorphisms of the measure algebra of $N$. For any finite complex measure $m$ on $N$ and any $y \in G$ we will let $m^y$ denote the measure obtained by applying to $m$ the automorphism corresponding to $y$. If $\varphi \in C_c(N)$, we can view $\varphi$ as a measure on $N$, as mentioned, and so $\varphi^y$ is defined, as a measure. A simple computation shows that actually $\varphi^y$ is defined by a density from $C_c(N)$, which we also denote by $\varphi^y$, and that in fact $\varphi^y(t) = \Delta_y \varphi(y^{-1}ty)$ for $t \in N$ and $y \in G$, where $\Delta_y$ is the factor by which the automorphism corresponding to $y$ changes the left Haar measure on $N$ (see the bottom of page 1103 of [6]).

The relation between the right and left actions of $C^*(N)$ on $X$ is suggested by the easily verified relation

$$f * \varphi(y) = (\varphi\Delta)^y * f(y)$$

for $f \in C_c(G)$, $\varphi \in C_c(N)$, and $y \in G$, although this relation can have no meaning at the level of the group $C^*$-algebras. This relation is related to Lemma 2 of [6] or Lemma 16.1 of [23], as is the next lemma. Rather than stating the full relation between the left and right actions immediately, the next lemma is stated as a preliminary result (although it is the crux of the matter), because we will need this result in exactly this form later (Proposition 5.2 and Lemma 6.3).

LEMMA 2.1 (Main Lemma). *Let $G$ be a locally compact group, $N$ a normal subgroup of $G$, and $H$ a subgroup of $G$ containing $N$. Let $X$ be the completion of $C_c(G)$ as a $C_c(H)$-rigged space, so that $X$ is a Hermitian $C^*(H)$-rigged module over $C^*(N)$, on which $C^*(N)$ also acts on the right. Let $d \in C^*(N)$ and $f \in C_c(G)$, with $f$ viewed as an element of $X$. Then*

$$fd \in [d^y : y \in \operatorname{support}(f)]X. \qquad df \in X[d^y : y^{-1} \in \operatorname{support}(f)],$$

*(where the sets on the right are, as usual, taken to be closed linear spans in $X$).*

*Proof.* For any $z \in G$ let $\delta_z$ denote the positive measure of mass one at $z$. Then for $\varphi \in C_c(N)$ it is easily seen that, as measures on $G$,

$$\delta_z * \varphi = \varphi^z * \delta_z,$$

in analogy with the relation previously mentioned. Let $h \in C_c(G)$. We begin by establishing the formulas

$$f * \varphi * h = \int_G (\varphi^z * \delta_z * h) f(z)\, dz, \qquad h * \varphi * f = \int_G (h * \delta_z * \varphi^{z^{-1}}) f(z)\, dz,$$

for the inductive limit topology on $C_c(G)$. Both integrands are easily seen to be continuous functions of compact support from $G$ into $C_c(G)$ for the inductive limit topology, so, in particular, with values in $C(K)$ where $K$ is an appropriate compact set in $G$ depending on $f$, $\varphi$, and $h$. The integrals are taken as Bochner integrals for the supremum norm in $C(K)$. Now point evaluations are continuous linear functionals on $C(K)$, and the collection of point evaluations separates $C(K)$. But if either formula is evaluated at points, the two sides are seen to be equal. Thus the formulas are established.

Since the inductive limit topology on $C_c(G)$ is finer than the norm topology from $X$, so that the injection of $C(K)$ into $X$ is continuous, it follows that the two previous formulas are also valid when the integrals are taken as Bochner integrals for the norm on $X$.

We would now like to replace $\varphi$ with $d$ in the formulas. Clearly the function $z \mapsto \delta_z * h$ is continuous for the inductive limit topology, and also for the norm topology from $X$. Since the action of $G$ on $C^*(N)$ is strongly continuous, it follows that the function $z \mapsto [d^z(\delta_z * h)] f(z)$ is a continuous function of compact support with values in $X$. A similar statement holds for the other integrand. But routine estimates show that if $\varphi$ is chosen close to $d$, then the corresponding integrals are close. In this way we establish the formulas

$$f(dh) = \int [d^z(\delta_z * h)] f(z)\, dz, \qquad h(df) = \int [(h * \delta_z) d^{z^{-1}}] f(z)\, dz,$$

in which $\delta_z * h$ and $h * \delta_z$ are regarded as elements of $X$.

Let $\epsilon > 0$ be given, and choose a partition of unity $\{k_i\}$ in $C_c(G)$ fine enough (p. 65 of [8]) so that if $y, z \in \text{support}(fk_i)$ for some $i$, then

$$\|d^y - d^z\|_{C^*(N)} \leq \epsilon/(\|f\|_1 M),$$

where $M = \|h\|_X$ for the first formula, but

$$M = \sup\{\|h * \delta_z\|_X : z \in \text{support}(f)\}$$

for the second. Now only a finite number of the $k_i$ will be nonzero at some point of the support of $f$. For each of these choose $z^i \in \text{support}(fk_i)$ and discard the rest of the $k_i$. Then, from the formulas previously established and the fact that $\sum fk_i = f$ (where all such sums will be finite sums over the

$i$ for which $k_i$ has not been discarded), we have

$$f(dh) = \sum \int [d^z(\delta_z * h)] f(z) k_i(z)\, dz.$$

Then from this we have

$$\begin{aligned} &\left\| f(dh) - \sum \int [d^{z_i}(\delta_z * h)] f(z) k_i(z)\, dz \right\|_X \\ &\qquad \leqslant \sum \int \| d^z - d^{z_i} \|_{C^*(N)} \|\delta_z * h\|_X |f(z)| k_i(z)\, dz \leqslant \epsilon. \end{aligned}$$

But it is easily seen that

$$\int [d^{z_i}(\delta_z * h)] f(z) k_i(z)\, dz = d^{z_i}(fk_i * h).$$

Since $\epsilon$ is arbitrary, it follows that

$$f(dh) \in [d^y : y \in \operatorname{support}(f)]X.$$

In a similar way it is established that

$$h(df) \in X[d^y : y^{-1} \in \operatorname{support}(f)].$$

Now let $h$ run through an approximate identity consisting of positive elements of $C_c(G)$ whose supports shrink to the identity of $G$, with $\|h\|_1 = 1$. From the properties of the action of $C^*(G)$ on the left of $X$ it is clear that $h(df)$ converges to $df$ in $X$, and so we have completed the proof of the first relation of the lemma. However, $C^*(G)$ does not act on the right on $X$, and so we must argue directly that $f(dh)$ converges to $fd$. But

$$\begin{aligned} \|fd - f(dh)\|_X &= \left\| fd \int h(z)\, dz - \int [(f * \delta_z) d^{z^{-1}}] h(z)\, dz \right\|_X \\ &\leqslant \int \| fd - (f * \delta_z) d^{z^{-1}} \|_X h(z)\, dz, \end{aligned}$$

which can be made as small as desired by choosing the support of $h$ sufficiently small, since the rest of the integrand is a continuous function with value zero at the identity element of $G$. Q.E.D.

A simple approximation argument lifts this result to all elements of $X$, although we can no longer say anything about supports.

COROLLARY 2.1. *Let $d \in C^*(N)$ and $x \in X$. Then*

$$xd \in [d^G]X \qquad \text{and} \qquad dx \in X[d^G],$$

*where $[d^G]$ denotes the set of elements in $C^*(N)$ of form $d^y$ for $y \in G$.*

If we let $G = H$, then it is easily seen that $X = C^*(H)$. We thus obtain:

COROLLARY 2.2. *Let $N$ be a normal subgroup of the locally compact group $H$, let $a \in C^*(H)$, and let $d \in C^*(N)$. Then*

$$da \in C^*(H)[d^H] \quad \text{and} \quad ad \in [d^H]C^*(H).$$

COROLLARY 2.3. *Let $N$ be a normal subgroup of the locally compact group $H$ and let $I$ be an ideal in $C^*(N)$ which is $H$-invariant. Then*

$$IC^*(H) = C^*(H)I.$$

COROLLARY 2.4. *Let $N$ be a normal subgroup of the locally compact group $H$, and let $C$ be a closed $H$-invariant subset of* $\mathrm{Prim}(N)$. *Then* $\ker(C)$ *is a $H$-invariant ideal, so that*

$$\ker(C)C^*(H) = C^*(H)\ker(C).$$

Suppose now that $N$ is a normal subgroup of $G$ and that $J \in \mathrm{Prim}(N)$ with $\{J\}$ locally closed. Let $G_J$ be the stability subgroup of $J$. Let $\{J\}^c$ denote the closure of $\{J\}$, and note that $\{J\}^c$ is a $G_J$-invariant closed subset of $\mathrm{Prim}(N)$ whose kernel is just the $G_J$-invariant ideal $J$ itself, and that $(\{J\}^c - \{J\})$ is also a closed $G_J$-invariant subset. Let $J' = \ker(\{J\}^c - \{J\})$, which is equal to $C^*(N)$ if $\{J\}$ is already closed. Then $J'$ is a $G_J$-invariant ideal which contains $J$. It follows from Corollary 2.4 that

$$C^*(G_J)J'C^*(G_J) = C^*(G_J)J', \qquad C^*(G_J)JC^*(G_J) = C^*(G_J)J.$$

Applying Proposition 2.1, we see that the category of unitary $G_J$-modules whose restriction to $N$ lives on $\{J\}$ is naturally isomorphic to the category of Hermitian modules over the $C^*$-algebra $C^*(G_J)J'/C^*(G_J)J$. Furthermore, let $GJ$ be the orbit of $J$ under $G$, and assume that $GJ$ is locally closed. Let

$$I = \ker((GJ)^c), \qquad I' = \ker((GJ)^c - GJ).$$

Then in the same way

$$C^*(G)I'C^*(G) = C^*(G)I', \qquad C^*(G)IC^*(G) = C^*(G)I$$

and the category of unitary $G$-modules whose restriction to $N$ lives on $GJ$ is naturally isomorphic to the category of Hermitian modules over the $C^*$-algebra $C^*(G)I'/C^*(G)I$. It is this pair of $C^*$-algebras that we shall show have an imprimitivity bimodule.

Now it follows from Section 7 of [36] that the process of inducing representations from $G_J$ up to $G$ establishes an equivalence between the category of Hermitian $C_c(G_J)$-modules and the category of Hermitian modules over the imprimitivity algebra $C_c(G \times G/G_J)$, with $C_c(G)$ serving as an imprimitivity bimodule for these two algebras. We need to study how this imprimitivity bimodule is related to the ideals in $C^*(G_J)$ and $C^*(G)$ defined previously, and to the corresponding quotient algebras.

## 3. Imprimitivity Bimodules for Ideals and Quotients

Let $B$ and $E$ be pre-$C^*$-algebras and let $X$ be an $E$–$B$-imprimitivity bimodule (Definition 6.10 of [36]). We wish to relate $X$ to ideals in the completions of $B$ and $E$ and to corresponding quotient algebras. For this reason we will need to consider the completion of $X$. Now both the $B$-valued and the $E$-valued inner products give norms with respect to which we can complete $X$. Thus it is reasuring that these norms agree.

Proposition 3.1. *Let $E$ and $B$ be pre-$C^*$-algebras and let $X$ be an $E$–$B$-imprimitivity bimodule. Then for any $x \in X$,*

$$\|\langle x, x\rangle_E\| = \|\langle x, x\rangle_B\|.$$

*Proof*

$$\begin{aligned}\|\langle x, x\rangle_B\|^2 &= \|\langle x, x\rangle_B\langle x, x\rangle_B\| = \|\langle x, x\langle x, x\rangle_B\rangle_B\| = \|\langle x, \langle x, x\rangle_E x\rangle_B\| \\ &\leqslant \|\langle x, x\rangle_B\|^{1/2}\,\|\langle\langle x, x\rangle_E x, \langle x, x\rangle_E x\rangle_B\|^{1/2} \leqslant \|\langle x, x\rangle_B\|\,\|\langle x, x\rangle_E\|,\end{aligned}$$

where the first inequality is an application of the generalized Cauchy–Schwartz inequality in Proposition 2.9 of [36] and the second inequality is an application of condition 2 in the definition of an imprimitivity bimodule (6.10 of [36]). Dividing by $\|\langle x, x\rangle_B\|$ we obtain inequality in one direction. The opposite inequality is shown by a similar calculation. Q.E.D.

It follows that the completions of $X$ with respect to the two norms are the same, and that we can simultaneously extend the actions of $E$ and $B$ to actions of their completions, $\bar{E}$ and $\bar{B}$, on the completion $\bar{X}$, so that $\bar{X}$ become an $\bar{E}$–$\bar{B}$-imprimitivity bimodule. Consequently we will usually, from now on, work with $C^*$-algebras rather than pre-$C^*$-algebras, although it is not usually important that $X$ be complete as long as actions of the (complete) $C^*$-algebras can be defined on it.

The following result is the analog for $C^*$-algebras of part of Theorem 3.5(5) on p. 65 of [4] for algebras and Proposition 8.4 of [37] for $W^*$-algebras.

Theorem 3.1. *Let $E$ and $B$ be $C^*$-algebras and let $X$ be an $E$–$B$-imprimitivity bimodule. Then there are natural isomorphisms among*

(1) *the lattice of (closed two-sided) ideals of $B$,*
(2) *the lattice of closed $E$–$B$-submodules of $X$, and*
(3) *the lattice of ideals of $E$.*

*where ideals and submodules are ordered by inclusion. If $J$ is an ideal in $B$, then the $E$–$B$-subspace of $X$ corresponding to $J$ under the first isomorphism is*

$$X_J = \{y \in X : \langle x, y\rangle_B \in J \text{ for all } x \in X\}.$$

*If $Y$ is a closed $E$–$B$-subspace of $X$, then the ideal of $B$ corresponding to $Y$ under the isomorphism is*

$$I_Y = \text{closed span of } \{\langle x, y\rangle_B : x \in X,\ y \in Y\}.$$

*Similar statements hold for $E$.*

LEMMA 3.1. *With the previously defined notation we have*

$$XJ = \{y \in X : \langle x, y\rangle_B \in J \text{ for all } x \in X\} = \{y \in X : \langle y, y\rangle_B \in J\}.$$

*Proof.* It is clear that the first space is contained in the second, which in turn is contained in the third, so we need to show that the third is contained in the first. Let $x \in X$ with $\langle x, x\rangle_B \in J$. Let $\{i_k\}$ denote a self-adjoint approximate identity for $J$ (1.7.2 of [14]). Then, expanding $\langle x - xi_k, x - xi_k\rangle_B$, we obtain a sum of products of $i_k$ with $\langle x, x\rangle_B$ which converges to zero. It follows that $xi_k$ converges to $x$ in the norm of $X$, so that $x \in XJ$. Q.E.D.

*Proof of Theorem* 3.1. Let $J$ be an ideal in $B$. It is clear that $J$ contains $I_{X_J}$. But by Lemma 3.1, $X_J = XJ$, and so $I_{X_J}$ contains $\langle XJ, XJ\rangle_B = J\langle X, X\rangle_B J$, which spans a dense subset of $J$ since $\langle X, X\rangle_B$ is assumed to span a dense subset of $B$. Thus $I_{X_J} = J$.

Conversely, let $Y$ be a closed $E$–$B$-submodule of $X$. Then $X_{I_Y}$ clearly contains $Y$. Now by Lemma 3.1, $X_{I_Y} = XI_Y$. Let $x, z \in X$ and $y \in Y$. Then

$$x\langle z, y\rangle_B = \langle x, z\rangle_E y,$$

which is in $Y$ since $Y$ is assumed to be $E$-invariant. But every element of $XI_Y$ is a limit of linear combinations of elements of the form $x\langle z, y\rangle_B$ for $x, z \in X$, $y \in Y$. It follows that $X_{I_Y}$ is contained in $Y$.

The fact that this correspondence between ideals and submodules preserves inclusion is clear. Q.E.D.

COROLLARY 3.1. *Let $E$ and $B$ be $C^*$-algebras and let $X$ be an $E$–$B$-imprimitivity bimodule. Let $J$ be an ideal in $B$ and let $K$ be the ideal in $E$ which corresponds to $X_J$ according to Theorem* 3.1. *Then both $\langle X, X_J\rangle_E$ and $\langle X_J, X_J\rangle_E$ are contained in and generate $K$. In particular, $X_J$ is a $K$–$J$-imprimitivity bimodule.*

COROLLARY 3.2. *Let E and B be $C^*$-algebras and let X be an E–B-imprimitivity bimodule. Let I and J be ideals in B with $I \supset J$ and let $K(I)$ and $K(J)$ denote the ideals in E which correspond to I and J according to Theorem 3.1. Then the B-valued inner product on X drops to an $(I/J)$-valued inner product on $X_I/X_J$, and the E-valued inner product drops to a $(K(I)/K(J))$-valued inner product on $X_I/X_J$, so that $X_I/X_J$ becomes a $(K(I)/K(J))$–$I/J$-imprimitivity bimodule.*

We now show that the process previously described of passing to ideals and quotients behaves well with respect to the inducing process. Since for the inducing process we do not need an imprimitivity bimodule, but only a Hermitian rigged module, we work with these instead.

PROPOSITION 3.2. *Let A and B be $C^*$-algebras, let X be a Hermitian B-rigged A-module, and let I and J be ideals in B with $I \supset J$. Then $XI$ and $XJ$ are Hermitian I- and J-rigged A-modules, and $XI/XJ$ is a Hermitian $(I/J)$-rigged A-module, which we will denote by Z. Let $F_Z$ denote the functor consisting of inducing Hermitian $(I/J)$-modules up to A. Let $D = \mathrm{hull}(J) - \mathrm{hull}(I)$, so that the category of Hermitian $(I/J)$-modules is isomorphic to the category of those Hermitian B-modules which live on the locally closed set D. Let $F_X$ denote the functor of inducing Hermitian B-modules up to A via X. Then $F_Z$ is naturally unitarily equivalent to the restriction of $F_X$ to the category of Hermitian B-modules which live on D (viewed as Hermitian $(I/J)$-modules).*

*Proof.* The assertions concerning $XI$, $XJ$, and $X$ are easily verified. We must show that the indicated functors are unitarily equivalent. Let $V$ be a Hermitian $(I/J)$-module, which we can view as a Hermitian $B$-module. Then the algebraic span $IV$ (now not closed) is a dense submodule of $V$, and, as algebraic tensor products,

$$X \otimes_B IV = XI \otimes_B V.$$

Define a map $t_V$ from $XI \times V$ into $Z \otimes_{I/J} V$ by

$$t_V(x, v) = (x + XJ) \otimes v$$

for $x \in XI$ and $v \in V$. It is easily verified that $t_V$ is $B$-balanced and bilinear, so that it lifts to a linear map, also denoted by $t_V$, of $XI \otimes_B V$ into $Z \otimes_{I/J} V$. It is clear that $t_V$ is in fact surjective, and it is easily seen that $t_V$ is an $A$-module homomorphism. Furthermore, a simple calculation shows that $t_V$ is an isometry with respect to the pre-inner products on these spaces (where that on $XI \otimes_B V$ comes from $X \otimes_B V$). But the image of $X \otimes_B IV$ is dense in $F_X(V)$, and so $t_V$ extends to an isometry of $F_X(V)$ into $F_Z(V)$, which is surjective since the image of $Z \otimes_{I/J} V$ is dense in $F_Z(V)$. The collection of the unitary maps $t_V$ gives the desired unitary natural equivalence of the restriction of $F_X$ with $F_Z$. Q.E.D.

We conclude this section by showing that the kernels of representations induced by an imprimitivity bimodule behave well with respect to the bijection of Theorem 3.1, so that this bijection preserves primitive ideals.

PROPOSITION 3.3 *Let $E$ and $B$ be $C^*$-algebras and let $X$ be an $E$–$B$-imprimitivity bimodule. Let $V$ be a Hermitian $B$-module and let $J$ be the kernel of (the representation corresponding to) $V$. Let $K$ be the ideal in $E$ corresponding to $J$ according to Theorem 3.1 and let ${}^EV$ be the Hermitian $E$-module obtained by inducing $V$ to $E$ via $X$. Then the kernel of (the representation corresponding to) ${}^EV$ is $K$.*

*Proof.* We use the notation of Theorem 3.1. Let $x, y \in X_J$, so that, in particular, $\langle y, z\rangle_B \in J$ for all $z \in X$. Let $z \in X$, $v \in V$. Then

$$\langle x, y\rangle_E(z \otimes v) = x\langle y, z\rangle_B \otimes v = x \otimes \langle y, z\rangle_B v = 0,$$

so that $\langle x, y\rangle_E$ is in the kernel of ${}^EV$. Since the elements of this form span $K$, it follows that $K$ is contained in the kernel of ${}^EV$. Suppose conversely that $e$ is an element of the kernel of ${}^EV$. Then for every $x \in X$, $v \in V$ we have $e(x \otimes v) = 0$, and so for every $y \in X$, $w \in V$ we have

$$0 = \langle e(x \otimes v), y \otimes w\rangle = \langle\langle y, ex\rangle_B v, w\rangle.$$

It follows that $\langle y, ex\rangle_B$ is in $J$ for all $x, y \in X$, so that $ex \in X_J$ for all $x \in X$. Now $\langle X, X\rangle_E$ is assumed to span a dense subset of $E$, and $E$ has an approximate identity, so $e$ can be approximated by linear combinations of elements of the form

$$e\langle x, y\rangle_E = \langle ex, y\rangle_E.$$

But by definition each such element is in $K$, since $ex \in X_J$. Thus $e \in K$. Q.E.D.

COROLLARY 3.3. *Let $E$ and $B$ be $C^*$-algebras and let $X$ be an $E$–$B$-imprimitivity bimodule. Then the bijection of Theorem 3.1 between ideals of $B$ and ideals of $E$ restricts to a bijection between the primitive ideals of $B$ and the primitive ideals of $E$. In particular,* $\mathrm{Prim}(B)$ *and* $\mathrm{Prim}(E)$ *are homeomorphic.*

*Proof.* By corollary 6.25 of [36], ${}^EV$ is irreducible if and only if $V$ is. Q.E.D.

From Proposition 3.3 we obtain the following generalization of Theorem 6.1 of [26].

COROLLARY 3.4. *Let $A$ and $B$ be $C^*$-algebras and let $X$ be a Hermitian $B$-rigged $A$-module. Let $V_i$, $i = 1, 2$, be Hermitian $B$-modules, let $J_i = \ker(V_i)$, and let $L_i = \ker({}^A V_i)$. If $J_1 \supseteq J_2$, then $L_1 \supseteq L_2$.*

*Proof.* Let $K_i = \ker({}^E V_i)$, so that $K_1 \supseteq K_2$ by Proposition 3.3. Then $L_1 \supseteq L_2$ follows from the fact that $L_i = \{a \in A : aE \subseteq K_i\}$. This is a special case of the easily verified fact that if $C$ is any $C^*$-algebra, if $E$ is any two-sided ideal of $C$, if $A$ is a subalgebra of $C$, if $W$ is a Hermitian $C$-module which is non degenerate as an $E$-module, and if $K$ is the kernel of $W$ as an $E$-module (so $K$ is an ideal in $C$ also), then the kernel of $W$ as an $A$-module is $\{a \in A : aE \subseteq K\}$. Q.E.D.

## 4. THE HERMITIAN RIGGED MODULE OF AN ORBIT

In this section we apply the results of the last section to the situation of Theorem 1.1. Actually, for the first part of this section we still do not need to know that $G_J$ is a stability subgroup, and so we begin by assuming only that we have a locally compact group $G$, a normal subgroup $N$, and a subgroup $H$ of $G$ which contains $N$. As in Section 2 we let $X$ denote the completion of $C_c(G)$ as a $C_c(H)$-rigged space. Then from Corollary 2.1 we immediately conclude, in analogy with Corollary 2.3, the following:

COROLLARY 4.1. *If $I$ is an ideal in $C^*(N)$ which is $G$-invariant, then $IX = XI$.*

Setting $H = N$, we are now in a position to prove that for a normal subgroup $N$ of a locally compact group $G$, the $*$-homomorphism of $C^*(N)$ into $M(C^*(G))$ is injective.

PROPOSITION 4.1. *Let $N$ be a normal subgroup of the locally compact group $G$ and let $V$ be a unitary $N$-module. Let $({}^G V)_N$ denote the restriction to $N$ of the unitary $G$-module ${}^G V$ obtained by inducing $V$ to $G$. Then $({}^G V)_N$ weakly contains $V$. In particular, the homomorphism of $C^*(N)$ into $M(C^*(G))$ is injective. Furthermore, if $E$ is any locally closed subset of* $\mathrm{Prim}(N)$ *that carries $V$, then $({}^G V)_N$ is carried on any locally closed $G$-invariant subset of* $\mathrm{Prim}(N)$ *that contains $E$.*

*Proof.* Let $X$ be as previously defined, but for $H = N$. Let $K$ denote the kernel of (the representation on) $({}^G V)_N$. Then $K$ is $G$-invariant according to Proposition 2.2, and so $KX = XK$ by Corollary 4.1. We need to show that $KV = \{0\}$. Working symbolically, with $B = C^*(N)$, we have

$$\{0\} = K(({}^G V)_N) = K(X \otimes_B V) = KX \otimes_B V = XK \otimes_B V = X \otimes_B KV.$$

This says that if the Hermitian $B$-module $KV$ is induced to $G$, one obtains the zero-dimensional $G$-module. But this can only happen if $KV = \{0\}$ (as can be seen from Theorem 5.1 of [36]).

Now the kernel of the map of $C^*(N)$ into $M(C^*(G))$ must be contained in the kernel of the restriction to $N$ of every unitary representation of $G$, and, in particular, in the kernel of $({}^G V)_N$ for every unitary $N$-module $V$. But according to the previous paragraph this kernel must then be in the kernel of every unitary representation of $N$, and so is zero.

Finally, suppose $V$ lives on the locally closed subset $E$ of $\mathrm{Prim}(N)$, and let $D$ be a $G$-invariant locally closed subset of $\mathrm{Prim}(N)$ containing $E$, so that $V$ also lives on $D$. Let $D' = \bar{D} - D$ and let $I = \ker(\bar{D})$, $I' = \ker(D')$, so that $V$ is a Hermitian $I'/I$-module. Now $I$ and $I'$ are $G$-invariant, so that Corollary 4.1 is applicable. Working symbolically, we have

$$I(({}^G V)_N) = IX \otimes_B V = XI \otimes_B V = X \otimes_B IV = \{0\},$$
$$I'(({}^G V)_N) = I'X \otimes_B V = XI' \otimes_B V = X \otimes_B I'V = X \otimes_B V = ({}^G V)_N.$$

Thus $({}^G V)_N$ is a Hermitian $I'/I$-module, and so lives on $D$. Q.E.D.

We remark that the first part of Proposition 4.1 is Theorem 4.5 of [21] or Proposition 5.3 of [22], but without separability assumptions. J. M. G. Fell had shown (personal communication) that the separability assumption could be dropped, but this proof is somewhat different from his.

COROLLARY 4.2. *Let $N$ be a normal subgroup of the locally compact group $G$ and let $E$ be a nonempty locally closed subset of* $\mathrm{Prim}(N)$ *which is $G$-invariant. Let $I = \ker(\bar{E})$ and $I' = \ker(\bar{E} - E)$, so that $I$ and $I'$ are $G$-invariant ideals. Then the algebra*

$$C^*(G)I'/C^*(G)I$$

*is not the zero-dimensional algebra.*

*Proof.* It is clear that $I'/I$ cannot be the zero-dimensional algebra, and so there is at least one unitary $N$-module $V$ which lives on $E$ (use 2.10.4 of [14]). Then $({}^G V)_N$ will also live on $E$ by Proposition 4.1. But ${}^G V$ is then a nonzero Hermitian module over $C^*(G)I'/C^*(G)I$ by Proposition 2.1, so this algebra cannot be the zero algebra. Q.E.D.

Suppose now that, as in Theorem 1.1, $J \in \mathrm{Prim}(N)$ and $\{J\}$ is locally closed. Let $GJ$ be the orbit of $J$, and suppose that $GJ$ is locally closed. Define the ideals $J'$, $I$, and $I'$ as in the paragraph after Corollary 2.4. Then from Corollary 4.2 the algebras $C^*(G_J)J'/C^*(G_J)J$ and $C^*(G)I'/C^*(G)I$ are nonzero. In particular, $C^*(G_J)J' \neq C^*(G_J)J$. It follows from Theorem 3.2 that

$XC^*(G_J)J' \neq XC^*(G_J)J$. It is clear that $XC^*(G_J)J' = XJ'$, and $XC^*(G_J)J = XJ$. Thus $XJ' \neq XJ$. Furthermore, $I \subseteq J$, so $IX = XI \subseteq XJ$. Combining this observation with the results at the end of Section 2 and in Section 3, we obtain:

COROLLARY 4.3. *The space $XJ'/XJ$ is a nonzero Hermitian $C^*(G_J)J'/C^*(G_J)J$-rigged module over both $C^*(N)/I$ and $C^*(G)/C^*(G)I$. The inducing functor defined by this Hermitian rigged module is equivalent to the restriction to the category of unitary $G_J$-modules supported on $\{J\}$ of the ordinary inducing functor from unitary $G_J$-modules to unitary $G$-modules.*

PROPOSITION 4.2. *As an $I'/I$-module or a $C^*(G)I'/C^*(G)I$-module $XJ'/XJ$ is still nondegenerate, and so is a Hermitian rigged module.*

*Proof.* We need to show that

$$I'(XJ'/XJ) = XJ'/XJ$$

or, since $I'$ is $G$-invariant so that $I'X = XI'$, that

$$XI'J' + XJ = XJ'.$$

To show this it sufficies to show that

$$I'J' + J = J'.$$

Now $I'J' = I' \cap J'$ by 1.9.12 of [14], and so we need to show that

$$(I' \cap J') + J = J'.$$

Working with the definitions of these ideals, one finds that this relation is equivalent to

$$[((GJ)^c - GJ) \cup (\{J\}^c - \{J\})] \cap \{J\}^c = \{J\}^c - \{J\}.$$

But this relation is easily seen to be true. Q.E.D.

PROPOSITION 4.3. *The representation of $I'/I$ on $XJ'/XJ$ is faithful.*

*Proof.* Let $K$ be the kernel of the representation of $I'$ on $XJ'/XJ$, so that $K$ consists of those elements $k$ of $I'$ for which $kXJ' \subseteq XJ$. If $k$ is such an element and if $y \in G$, then

$$k^y XJ' = y(k(y^{-1}XJ')) \subseteq XJ.$$

Thus we see that $K$ is $G$-invariant. Let $C$ be the hull of $K$. Since $I' \supseteq K \supseteq I$, it follows that

$$(GJ)^c - GJ \subseteq C \subseteq (GJ)^c.$$

Since $C$ is closed and $G$-invariant, it follows that either $C = (GJ)^c - GJ$ or $C = (GJ)^c$. In the former case we would have $K = I'$, so that every element of $I'$ would act as the zero operator on $XJ'/XJ$. But this is impossible since $XJ'/XJ$ is a nonzero Hermitian $I'/I$-module. It follows that $K = I$ and the representation is faithful. Q.E.D.

We will call $XJ'/XJ$ the Hermitian rigged module associated with the orbit $GJ$.

## 5. The System of Imprimitivity

We keep the notation used at the end of the last section. Our aim is to show that $C^*(G)I'/C^*(G)I$ is the imprimitivity algebra (Definition 6.4 of [36]) of the $C^*(G_J)J'/C^*(G_J)J$-rigged space $XJ'/XJ$. Now from the definition, the imprimitivity algebra of $XJ'/XJ$ is clearly the algebra of operators on $XJ'/XJ$ spanned by the images of the operators $\langle xc, yd \rangle_E$ on $X$, for $x$, $y \in X$ and $c$, $d \in J'$, where $E$ denotes the imprimitivity algebra of $X$. As shown in Corollary 7.13 of [36], $E$ is normdense (for the norm on operators on $X$) in $C_c(G \times G/G_J)$, which is the transformation group algebra for the action of $G$ on $G/G_J$ (see 7.5 of [36] or 3.11 of [18]). But $C_c(G \times G/G_J) = C_c(G, C_c(G/G_J))$, and one of the ingredients of its action on $X$ is the pointwise multiplication of elements of $C_c(G)$ by elements of $C_c(G/G_J)$ viewed as functions on $G$ constant on cosets. In fact, $E$ or $C_c(G \times G/G_J)$ can be viewed as a semidirect product of $C_c(G)$ with $C_c(G/G_J)$, in the sense of covariance algebras [40], and the algebra $C_c(G/G_J)$ together with its action of pointwise multiplication on $C_c(G)$ is the system of imprimitivity for the situation. Since $C_c(G/G_J)$ defines an algebra of bounded operators on $X$ (Proposition 7.2 of [36]), it will also define an algebra of operators on $XJ'/XJ$. In this section we will study this action of $C_c(G/G_J)$ on $XJ'/XJ$. For much of what we do it is actually slightly more convenient to study the larger algebra $C(G/G_J)$ of bounded continuous functions on $G/G_J$, which also acts as an algebra of bounded operators on $X$ (Proposition 7.2 of [36]), and so on $XJ'/XJ$.

PROPOSITION 5.1. *The action of $C(G/G_J)$ on $C_c(G)$ commutes with the left action of $C_c(N)$. Thus the action of $C(G/G_J)$ on $X$ commutes with the left action of $C^*(N)$.*

The proof consists of a routine computation.

Now one of the hypotheses of Theorem 1.1 is that the canonical map from $G/G_J$ to $GJ$ is a homeomorphism. As a result of this, $GJ$ is Hausdorff, and we can identify $C(G/G_J)$ with $C(GJ)$. But $GJ$ is (identified with) $\mathrm{Prim}(I'/I)$, and so $C(G/G_J)$ becomes identified with the $C^*$-algebra of bounded continuous functions on $\mathrm{Prim}(I'/I)$.

Now if $A$ is any $C^*$-algebra, Dixmier [15] has shown that the algebra of bounded continuous functions on $\mathrm{Prim}(A)$ can be identified with the center of $M(A)$. Dixmier does not state his results in terms of double centralizers, but it is an easy matter to restate them in this way, as remarked by Busby on p. 371 of [11]. Dixmier based his proof on a theorem of Dauns and Hoffmann, but he also gave an alternate proof based on a result of Stormer. This result of Stormer has more recently been given a very short proof by Bunce [9], and consequently a quite direct path to Dixmier's theorem is via Stormer's result with Bunce's proof. Another short proof has just appeared in [19].

It follows from these considerations that $C(G/G_J)$ can be identified with the center of $M(I'/I)$. Since $XJ'/XJ$ is a Hermitian rigged $I'/I$-module, it is also a Hermitian rigged module over $M(I'/I)$ (by Proposition 3.9 of [36]), and so is also a Hermitian rigged module over the center of $M(I'/I)$, which we have just identified with $C(G/G_J)$. We thus have defined two actions of $C(G/G_J)$ on $XJ'/XJ$, one coming from the pointwise action of $C(G/G_J)$ on $C_c(G)$ and the other from the identification of $C(G/G_J)$ with the center of $M(I'/I)$. The main result of this section is that these two actions coincide.

To distinguish between the two actions in the proof, we will, for $F \in C(G/G_J)$, denote by $P_F$ the operator on $XJ'/XJ$ obtained from pointwise multiplication on $C_c(G)$ by $F$, while we will denote by $T_F$ the operator on $XJ'/XJ$ obtained by identifying $F$ with an element of the center of $M(I'/I)$.

PROPOSITION 5.2. *For any $F \in C(G/G_J)$, $P_F = T_F$.*

*Proof.* Let $d \in C^*(N)$ and let $f \in C_c(G)$, viewed as an element of $X$. Then according to Lemma 2.1,

$$d(P_F f) = d(Ff) \in X[d^y : y^{-1} \in \mathrm{support}(F)],$$

since $\mathrm{support}(Ff) \subseteq \mathrm{support}(F)$. It follows that for any $x \in X$ we have

$$d(P_F x) \in X[d^y : y^{-1} \in \mathrm{support}(F)].$$

Now view $F$ as a function on $GJ$, and suppose it happens that $d \in \ker(\mathrm{support}(F))$. This means that $d \in J^y$ for each $y \in \mathrm{support}(F)$, where now $F$ is viewed as a function on $G$. But then $d^y \in J$ for each $y^{-1} \in \mathrm{support}(F)$, and so $[d^y : y^{-1} \in \mathrm{support}(F)] \subseteq J$. Thus $dP_F X \subseteq XJ$, so that $dP_F$ acts as the zero operator on $XJ'/XJ$.

Let $L(XJ'/XJ)$ denote the algebra of bounded operators on the rigged space $XJ'/XJ$ (Definition 2.3 of [36]). Since the representation of $I'/I$ on $XJ'/XJ$ is faithful (Proposition 4.3), we can view $I'/I$ as a subalgebra $D$

of the $C^*$-algebra $L(XJ' XJ)$. Since $XJ' XJ$ is nondegenerate as an $I' I$-module, it suffices to show that $T_F d = P_F d$ for all $d \in D$ in order to conclude that $T_F = P_F$. (Note that $P_F d = dP_F$ by Proposition 5.1.) Thus the proof will be complete once we have proven the following proposition (in which the primitive ideal space need not be assumed to be Hausdorff).

PROPOSITION 5.3. *Let $A$ be a $C^*$-algebra with identity and let $D$ be a $C^*$-subalgebra of $A$ (possibly without identity). For $F \in C(\text{Prim}(D))$ (the algebra of bounded continuous functions on $\text{Prim}(D)$), let $T_F$ denote the corresponding double centralizer of $D$. Let $P$ be a $*$-homomorphism of $C(\text{Prim}(D))$ into $A$ such that*

(1) *$P_1$ is the identity element of $A$.*
(2) *$P_F d = dP_F$ for all $F \in C(\text{Prim}(D))$ and $d \in D$.*
(3) *If $d \in D$ and $F \in C(\text{Prim}(D))$ and if $d \in \ker(\text{support}(F))$, then $P_F d = 0$.*

*Then $P_F d = T_F d$ for all $F \in C(\text{Prim}(D))$ and $d \in D$.*

*Proof.* Let $d \in D$ and $F \in C(\text{Prim}(D))$. Suppose first that for some complex number $m$ and some $\epsilon > 0$, we have $d \in \ker(R)$ where

$$R = \{J \in \text{Prim}(D) : |F(J) - m| \geqslant \epsilon\}.$$

Let

$$S = \{J \in \text{Prim}(D) : |F(J) - m| \geqslant 2\epsilon\},$$

so that $S$ is contained in $R$. Let $h$ be a continuous function from the complex plane to the interval $[0, 1]$ such that, for any complex number $z$, $h(z) = 1$ if $|z - m| \geqslant 2\epsilon$, and $h(z) = 0$ if $|z - m| \leqslant \epsilon$. Let $F_0 = h \circ F$, so that $F_0$ is a continuous function on $\text{Prim}(D)$ with values in $[0, 1]$ and $F_0(J) = 1$ for $J \in S$, while $F_0(J) = 0$ for $J \notin R$. If we let $G = FF_0 + m(1 - F_0)$, then $F - G = (F - m)(1 - F_0)$, so that $\|F - G\|_\infty \leqslant 2\epsilon$. Furthermore $G - m$ is zero off of $R$, so that $d \in \ker(\text{support}(G - m))$. Then by hypotheses (1) and (3) we see that $P_{(G-m)} d = 0$, so that $P_G d = md$. It follows that

$$\|P_F d - md\| = \|P_{(F-G)} d\| \leqslant 2\epsilon \|d\|.$$

Now let $d \in D$ and $F \in C(\text{Prim}(A))$, and assume that $d = d^*$ and that $F$ is real-valued. Let $B$ be the $C^*$-subalgebra of $A$ generated by the $T_H d$ for all $H \in C(\text{Prim}(D))$ together with the image of all of $C(\text{Prim}(A))$ under $P$. Note that $B$ is commutative by hypothesis (2). Let $\epsilon > 0$ be given, and let $\{U_i\}$ be a finite covering of the closure of the range of $F$ by open intervals of length strictly less than $\epsilon$, and such that no $U_i$ meets more than two other of the $U_i$s. For each $i$ choose a point $m_i$ in $U_i$ and let $V_i = F^{-1}(U_i)$. Then the $V_i$ form a finite open covering of $\text{Prim}(D)$, and $|F(J) - m_i| < \epsilon$ for $J \in V_i$.

Let $\{h_i\}$ be a partition of unity on the real line which is finer than the $U_i$ together with the complement of the closure of the range of $F$. Discard the functions supported in the complement of the range of $F$, and for each remaining $i$ set $G_i = h_i \circ F$. Then each $G_i$ is a continuous function on $\mathrm{Prim}(D)$ having range in $[0, 1]$, and value 0 off of $V_i$. Furthermore $\sum G_i \equiv 1$ on $\mathrm{Prim}(D)$. In particular, $\sum T_{G_i} d = d$.

Now the relation defining how a double centralizer $T_{G_i}$ is associated to the function $G_i$ is

$$T_{G_i} d - G_i(J) d \in J$$

for every $J \in \mathrm{Prim}(D)$. Since $G_i$ vanishes off of $V_i$, it follows that $T_{G_i} d \in J$ for each $J \notin V_i$. This means that

$$T_{G_i} d \in \ker\{J \in \mathrm{Prim}(D) : |F(J) - m_i| \geqslant \epsilon\},$$

so that by the first paragraph of the proof

$$\|P_{(F-m_i)} T_{G_i} d\| = \|P_F T_{G_i} d - m_i T_{G_i} d\| < 2\epsilon \|d\|.$$

Now for any $i$ there are at most two $j$'s different from $i$ such that $h_i h_j \neq 0$, such that $T_{G_i} T_{G_j} \neq 0$, and such that

$$(P_{(F-m_i)} T_{G_i} d)(P_{(F-m_j)} T_{G_j} d) \neq 0.$$

Since all of this is happening in the commutative $C^*$-algebra $B$, it is easily seen that

$$\|\sum P_{(F-m_i)} T_{G_i} d\| < 6\epsilon \|d\|.$$

Since $P_F d = \sum P_F T_{G_i} d$, it follows that

$$\|P_F d - \sum m_i T_{G_i} d\| < 6\epsilon \|d\|.$$

However, $\|F - \sum m_i G_i\|_\infty < \epsilon$, so

$$\|T_F d - \sum m_i T_{G_i} d\| < \epsilon \|d\|.$$

Consequently

$$\|P_F d - T_F d\| \leqslant 7\epsilon \|d\|.$$

Since $\epsilon$ is arbitrary, $M_F d = T_F d$. This relation extends by linearity to arbitrary $F$ and $d$. Q.E.D.

COROLLARY 5.1. *The homomorphism $P$ of $C(G/G_J)$ into the algebra of bounded operators on $XJ'/XJ$ coming from the action on $X$ corresponding to pointwise multiplication on $C_c(G)$, is injective, and for every $d \in I'/I$ and $F \in C(G/G_J)$, we have $P_F d \in I'/I$ (viewed as a subalgebra of the algebra of bounded operators on $XJ'/XJ$).*

*Proof.* According to Proposition 4.3 the representation of $I'/I$ on $XJ'/XJ$ is faithful. If $P_F = 0$, then $T_F d = 0$ for all $d \in I'/I$, and it follows that $T_F = 0$, so that $F = 0$. Furthermore, for any $F$, $P_F d = T_F d \in I'/I$.

Q.E.D.

## 6. The Proof of the Main Theorem

We have not yet shown that the representation of $C^*(G)I'/C^*(G)I$ on $XJ'/XJ$ is injective, and in fact we will leave the proof of this fact to the end. Consequently, we will here denote by $C$ the image of $C^*(G)I'/C^*(G)I$ as operators on $XJ'/XJ$.

Lemma 6.1. *The imprimitivity algebra of $XJ'/XJ$ is contained in $C$.*

*Proof.* According to the definition of the imprimitivity algebra of $XJ'/XJ$, it is the image of the subalgebra of $E$ generated by $\langle XJ', XJ' \rangle_E$. Now it is easily seen that $I' + J \supseteq J'$. Furthermore, if $x$ or $y$ is in $XJ$, then $\langle x, y \rangle_E z \ (= x\langle y, z \rangle_{C^*(G_J)})$ is in $XJ$ for $z \in X$ (since $J$ is $G_J$-invariant), so that $\langle x, y \rangle_E$ as an operator on $XJ'/XJ$ is the zero operator. Thus the imprimitivity algebra for $XJ'/XJ$ will be contained in the image of $\langle XI', XI' \rangle_E$. But $XI' = I'X$. Furthermore, since $E$ is an ideal in the algebra of bounded operators on $X$, and elements of $E$ come out of the $E$-valued inner product, elements of $I'$ acting on the left of $X$ will also come out of the $E$-valued inner product. Thus the imprimitivity algebra for $XJ'/XJ$ will be contained in the image of $I'\langle X, X \rangle_E I'$, or $I'EI'$, and so it suffices to show that the image of this latter algebra is contained in $C$.

Now the transformation group algebra $C_c(G \times G/G_J)$ forms a dense subalgebra of $E$, and elements of the form $\Phi(y) = g(y)F$ for $g \in C_c(G)$, $F \in C_c(G/G_J)$ span a dense subspace of $C_c(G \times G/G_J)$. Furthermore, a simple calculation shows that the action of such a $\Phi$ on an $x \in X$ is given by $\Phi x = P_F(gx)$ (where $P_F$ was defined just before Proposition 5.2). Thus it suffices to show that for $d, d' \in I'$, the operator on $XJ'/XJ$ corresponding to $dP_F gd'$ is in $C$. But according to Proposition 5.2 the operator $P_F$ acts like the double centralizer $T_F$, and so the above operator is the same as the operator $T_F(d)gd'$ (viewing $d$ as an element of $I'/I$), and this operator is clearly in the image of $I'C^*(G)I'$, and so in $C$. Thus the imprimitivity algebra for $XJ'/XJ$ is contained in $C$.

Q.E.D.

Lemma 6.2. *The imprimitivity algebra of $XJ'/XJ$ is all of $C$.*

*Proof.* Since the orbit $GJ$ is Hausdorff, it is easily seen that $J' = I' + J$. Combining this fact with the discussion in the proof of Lemma 6.1, it follows that the imprimitivity algebra of $XJ'/XJ$ will be all of the image of $I'EI'$, and so, as above, of the algebra generated by the $T_F(d)gd'$. But the elements of $I'/I$, which are of the form $T_F(d)$ as $d$ ranges over $I'/I$ and $F$ ranges over $C_c(G/G_J)$, will span a dense ideal of $I'/I$, as is easily seen from Proposition 3.3.7 of [14]. Thus the image of $I'EI'$ contains a dense part of the image of $C^*(G)I'$, and so the imprimitivity algebra of $XJ'/XJ$ is all of $C$. Q.E.D.

It follows that the functor consisting of inducing to $G$ unitary $G_J$-modules whose restrictions to $N$ live on $\{J\}$ is a full embedding of the category of such modules into the category of unitary $G$-modules whose restrictions to $N$ live on $GJ$. We have thus proven what may be considered our version of Theorem 1 of [6].

To complete the proof of our Main Theorem (Theorem 1.1), which also includes our version of Theorem 2 of [6], we must show that the representation of $C^*(G)I'/C^*(G)I$ on $XJ'/XJ$ is faithful. That the proof of this fact should still be somewhat delicate is perhaps not too surprising in view of the fact that the representation of $C^*(G)$ on $X$ need not be faithful. It will be convenient to change our notation and hereafter let $C = C^*(G)I'/C^*(G)I$.

Now $C^*(G)I'$ is an ideal in $C^*(G)$, and so an ideal in $M(C^*(G))$ by Proposition 1.8.5 of [14]. This gives a natural homomorphism of $M(C^*(G))$ into $M(C^*(G)I')$. By Proposition 3.8 of [10] there is a natural homomorphism of $M(C^*(G)I')$ into $M(C)$ (which we do not know is surjective—see the comments after Theorem 4.2 of [1]). Thus we have a natural homomorphism of $M(C^*(G))$ into $M(C)$. Now $I'$ and $I$ can both be viewed as subalgebras of $M(C^*(G))$ by Proposition 4.1, and it is clear that the elements of $I$ act as the zero operator on $C$. Thus we have a natural homomorphism of $I'/I$ into $M(C)$. It is clear that $C$ is nondegenerate as a right or left $I'/I$-module, and so an application of the left- and right-handed versions of Proposition 3.9 of [36] gives a natural homomorphism of $M(I'/I)$ into $M(C)$. Since we have identified $C(G/G_J)$ with the center of $M(I'/I)$, it follows that we have a homomorphism from $C(G/G_J)$ into $M(C)$, which is defined by the relation

$$F(dc) = (T_F(d))c$$

for $F \in C(G/G_J)$, $d \in I'/I$, $c \in C$, and similarly for the right action, where $T_F$ is defined as in Proposition 5.2.

Each element of $G$ also defines an element of $M(C^*(G))$, and so of $M(C)$. In addition, $G$ acts by left translation on $C(G/G_J)$. We show that these two actions are related. Now $T_F$ is defined by the fact that for each $z \in G$,

$$T_F(d) - F(J^z)d \in J^z$$

for every $d \in I'/I$. If we let $yF$ denote the left translate of $F$ by $y$, then a simple calculation shows that

$$(T_F(d))^y - yF(J^z)d^y \in J^z$$

for every $z \in G$. But this is the relation that defines $T_{yF}(d^y)$. It follows that

$$(T_F(d))^y = T_{yF}(d^y).$$

Then a simple calculation shows that for $c \in C$ we have

$$y(F(dc)) = (yF)(y(dc)).$$

By continuity we obtain:

LEMMA 6.3. *For $y \in G$, $F \in C(G/G_J)$, and $c \in C$, we have*

$$y(F(c)) = (yF)(yc).$$

Thus we see that $C(G/G_J)$ acts like a system of imprimitivity (as defined in Theorem 7.18 of [36]) for the action of $G$ on $C$. Furthermore, in the proof of Lemma 6.2 we saw that $I'/I$ is nondegenerate as a $C(G/G_J)$-module, and it follows that the same is true of $C$.

Let $E$ denote the (norm-closed) imprimitivity algebra of the $C^*(G_J)$-rigged space $X$. In the proof of Theorem 7.18 of [36] it was shown that every unitary $G$-module on which $C_\infty(G/G_J)$ acts as a system of imprimitivity becomes an a natural way a Hermitian $E$-module. We would like to show here that in the same way the maps of $G$ and $C_\infty(G/G_J)$ into $M(C)$ give a homomorphism of $E$ into $M(C)$. Now from Proposition 7.11 of [36] it follows that $C_c(G \times G/G_J)$ is dense in $E$. Then for $c \in C$ and $\Phi \in C_c(G \times G/G_J)$, with $\Phi$ viewed as an element of $C_c(G, C_\infty(G/G_J))$, we can define, in analogy with 7.20 of [36],

$$\Phi c = \int \Phi(y)(yc)\, dy, \qquad c\Phi = \int ((c\Phi(y))y)\, dy. \tag{6.1}$$

Then it is easily seen that these actions define a $*$-homomorphism of $C_c(G \times G/G_J)$ into $M(C)$ (see the proof of Proposition 7.6 of [36]). We need to show that this $*$-homomorphism is continuous, so that it extends to all of $E$. Now for this purpose the most convenient information about the norm of $E$ that we have is that if $W$ is a unitary $G$-module on which $C_\infty(G/G_J)$ acts as a system of imprimitivity, then the corresponding $*$-representation of $C_c(G \times G/G_J)$ is continuous (see the proof of Theorem 7.18 of [36]). But suppose that $W$ is a faithful Hermitian $C$-module, so that $C$ can be identified with a subalgebra of the algebra $L(W)$ of the bounded operators on $W$. Then, according to Theorem 3.9 of [10], $M(C)$ can be identified with the idealizer of $C$ in $L(W)$. The homomorphisms of $G$ and $C_\infty(G/G_J)$ into $M(C)$ become homomorphisms into $L(W)$, so that $W$ becomes a unitary $G$-module on which $C_\infty(G/G_J)$ acts as a system of imprimitivity. The homomorphism of $C_c(G \times G/G_J)$ into $M(C)$ becomes the homomorphism into $L(W)$ corresponding to this system of imprimitivity, and so is continuous for the reason previously indicated. Thus we do obtain a homomorphism of $E$ into $M(C)$. Since we saw that $C$ is nondegenerate as a $C_\infty(G/G_J)$-module, it follows that $C$ is nondegenerate as an $E$-module. We thus obtain:

LEMMA 6.4. *The actions defined in* (6.1) *extend by continuity to give a homomorphism of $E$ into $M(C)$ under which $C$, as a left or right $E$-module, is nondegenerate.*

The proof of the following lemma, which is the crux of the proof of the faithfulness of the representation of $C$ on $XJ'/XJ$, was motivated by that part of the proof of Theorem 2 of [6] which appears on p. 1108.

LEMMA 6.5. *Let $a \in C^*(G)I'$, and suppose that $a(XJ') \subseteq XJ$. Then $a \in C^*(G)I$.*

*Proof.* If we let $b$ denote the image of $a$ in $C$, then it will suffice to show that $b^* = 0$. Now $b^* \in a^*C$, and $C$ is nondegenerate as a left $I'$-module, and so $b^* \in I'a^*C$. From Lemma 6.4, $C$ is nondegenerate as an $E$-module, and so $b^* \in \langle X, X\rangle_E I'a^*C$ (where we always take closed linear spans). Thus it suffices to show that $\langle y, x\rangle_E d^*a^*c = 0$ for all $x, y \in X$, $d \in I'$, and $c \in C$. Now $E$ is an ideal in $L(X)$, so by application of Proposition 3.9 of [36], we obtain a homomorphism of $L(X)$ into $M(C)$. Then if $\Phi \in C_c(G \times G/G_J)$ and $f \in C_c(G)$, it is easily seen that, for any $c \in C$,

$$\Phi(f(c)) = (\Phi * f)(c),$$

where $\Phi * f$ is the element of $C_c(G \times G/G_J)$ [corresponding to composition of operators in $L(X)$], defined [in $C_c(G, C_c(G/G_J))$] by

$$(\Phi * f)(z) = \int \Phi(zy) f(y^{-1})\, dy.$$

By continuity it follows that

$$e(a'(c)) = (ea')c$$

for any $e \in E$, $a' \in C^*(G)$, and $c \in C$. However, for any element $S$ of $L(X)$, it is easily seen that

$$\langle y, x\rangle_E S^* = \langle y, Sx\rangle_E$$

(see Proposition 6.3 of [36]). From all this it follows that

$$\langle y, x\rangle_E d^*a^*c = \langle y, adx\rangle_E c,$$

so it suffices to show that the right-hand side is 0.

Now $adx \in a(I'X) = a(XI') \subseteq a(XJ')$, which by hypothesis is contained in $XJ$. Thus it suffices to show that

$$\langle y, xd\rangle_E c = 0, \tag{6.2}$$

where now $d \in J$, while $x, y \in X$, $c \in C$.

It is easily seen that if $U$ is a compact neighborhood of the identity element $e_G$ of $G$, then $U\{J\}^c$ is a closed subset of Prim$(N)$, and $\{J\}^c$ is the intersection of the $U\{J\}^c$ as the $U$ shrink down to $e_G$. It follows that $d \in J$ can be approximated in norm by elements from the various $\ker(U\{J\}^c)$. Thus it suffices to show that (6.2) holds for every $d \in \ker(U\{J\}^c)$ for every compact neighborhood $U$ of $e_G$.

Let us now fix a compact neighborhood $U$ of $e_G$ and a $d \in \ker(U\{J\}^c)$. It suffices to show that

$$\langle y, xd^*\rangle_E c = 0$$

for all $x, y \in X$, $c \in C$. Now for any $g \in C_c(G)$, viewed as an element of $X$, we have

$$\langle y, g\rangle_E \langle g, xd^*\rangle_E c = \langle y\langle g, g\rangle_B, xd^*\rangle_E c, \tag{6.3}$$

where $B = C^*(G_J)$. But as $g$ ranges over positive elements of $C_c(G)$ whose supports shrink to $e_G$, it is easily seen that $\langle g, g\rangle_B$ ranges over positive elements of $C_c(G_J)$ whose supports shrink to $e_G$. Thus if the $g$ are suitably normalized, the $\langle g, g\rangle_B$ will form an approximate identity for $B$, so that the right-hand side of (6.3) will converge to $\langle y, xd^*\rangle_E c$. It thus suffices to show that

$$\langle g, xd^*\rangle_E c = 0 \tag{6.4}$$

for all $x \in X$, $c \in C$, and all $g \in C_c(G)$ which are supported in a suitably small neighborhood of $e_G$.

In preparation for this we need to relate the action of $C(G/G_J)$ on $C$ with its action on $C_c(G)$ by pointwise multiplication. For $F \in C(G/G_J)$ we will let $P_F$ denote the operator on $C_c(G)$ consisting of pointwise multiplication by $F$. Then for $f, g \in C_c(G)$ and $y \in G$ it is easily verified that

$$F(\langle f, g\rangle_E(y)) = \langle P_F f, g\rangle_E(y),$$

where we identify $\langle f, g\rangle_E$ with its integral kernel defined in 7.8 of [36], and view it as an element of $C_c(G, C_c(G/G_J))$. Then a straightforward calculation using the preceding identity shows that, for $c \in C$,

$$F(\langle f, g\rangle_E c) = \langle P_F f, g\rangle_E c.$$

We return now to the situation in which we have a compact neighborhood $U$ of $e_G$, $d \in \ker(U\{J\}^c)$, $x \in X$, and $c \in C$. We wish to show that (6.4) holds for all $g \in C_c(G)$ which are supported in a suitably small neighborhood of $e_G$. As that suitably small neighborhood we choose a compact symmetric neighborhood $V$ of $e_G$ such that $V^3 \subseteq U$, and assume from now on that $g$ is supported in $V$. Choose $F \in C_c(CJ)$ such that $F \equiv 1$ on $VJ$ while $F \equiv 0$ outside $V^2J$. Viewing $F$ as a function in $C_c(G/G_J)$ and on $G$, we see that $P_F g = g$. Then

$$\langle g, xd^* \rangle_E c = \langle P_F g, xd^* \rangle_E c = F(\langle gd, x \rangle_E c). \tag{6.5}$$

According to the Main Lemma (Lemma 2.1),

$$gd \in [d^Y : y \in V]X.$$

Furthermore, for the reasons given in the proof of Lemma 6.1, the $d^Y$ will come outside the $E$-valued inner product. But any element of the support of $F$ is of the form $J^z$ with $z \in V^2$, and if $y \in V$, then $y^{-1}z \in V^3 \subseteq U$ so that $d \in J^{y^{-1}z}$ and $d^y \in J^z$. It follows that $Fd^y = 0$ for $y \in V$, and so (6.5) is equal to 0. Q.E.D.

The fact that the representation of $C^*(G)I'/C^*(G)I$ on $XJ'/XJ$ is faithful is an immediate consequence of Lemma 6.5. Thus we have concluded the proof of the Main Theorem, once we have remarked that the statement about weak containment follows easily from Proposition 6.26 of [36].

We remark that examples showing how the Main Theorem can fail when its hypotheses are weakened can be found in [6, 32].

## 7. Alternative Hypotheses

In this section we collect various conditions which imply that the hypotheses of the Main Theorem are satisfied. These conditions were suggested in part by results appearing in [17] and [3], and there are certainly additional conditions which can be concocted from the results appearing in these papers. We recall now several definitions.

DEFINITION 7.1. A topological space is said to be *almost Hausdorff* if every nonempty closed subset contains a nonempty open Hausdorff subset, or equivalently, if every closed subset contains an open dense Hausdorff subset.

The equivalence of the two formulations in this definition is shown by a simple argument (found on p. 125 of [24]) using Zorn's lemma. It is easily seen that any subset of an almost Hausdorff space is almost Hausdorff, and that any subset consisting of only one element of an almost Hausdorff space will be open in its closure.

DEFINITION 7.2. Let $N$ be a normal subgroup of the locally compact group $G$, so that $G$ acts on $\mathrm{Prim}(N)$. Then $N$ is said to be *regularly embedded* in $G$ if the orbit space $\mathrm{Prim}(N)/G$ with the quotient topology is almost Hausdorff.

This definition was introduced by Blattner [6]. If $N$ is regularly embedded in $G$, then from the fact that points of $\mathrm{Prim}(N)/G$ will be open in their closures, it follows that the orbits in $\mathrm{Prim}(N)$ will be open in their closures, that is, locally closed.

We recall further that if $N$ is a normal subgroup of the locally compact group $G$, then the Borel structure on the orbit space $\mathrm{Prim}(N)/G$ is defined to be the quotient of the Borel structure on $\mathrm{Prim}(N)$ generated by the topology of $\mathrm{Prim}(N)$, and not the Borel structure generated by the quotient topology on $\mathrm{Prim}(N)/G$. A Borel structure is said to be *countably separated* if it contains a countable collection of subsets which separate the points of the space. Finally, a subset of a topological space is a $G_\delta$ if it is the intersection of a countable number of open sets.

ALTERNATE HYPOTHESES 7.1. *In the statement of the Main Theorem the following alternate sets of hypotheses can be used.*

A. *Hypothesis 2 can be replaced by:*
   (2) *$GJ$ is locally closed in* $\mathrm{Prim}(N)$ *and is almost Hausdorff, and $G/G_J$ is $\sigma$-compact.*
B. *Hypotheses 1 and 2 can be replaced by:*
   (1) $\mathrm{Prim}(N)$ *is almost Hausdorff.*
   (2) *$N$ is regularly embedded in $G$ and $G/G_J$ is $\sigma$-compact.*
C. *Hypothesis 2 can be replaced by:*
   (2) *$\overline{GJ}$ is almost Hausdorff (which will be true if* $\mathrm{Prim}(N)$ *is almost Hausdorff), $GJ$ is a $G_\delta$-subset of $\overline{GJ}$, and $G/G_J$ is $\sigma$-compact.*
D. *Hypotheses 1 and 2 can be replaced by:*
   (1) $\mathrm{Prim}(N)$ *is almost Hausdorff and second countable.*
   (2) $\mathrm{Prim}(N)/G$ *is a $T_0$ topological space and $G/G_J$ is $\sigma$-compact.*
E. *Hypotheses 1 and 2 can be replaced by:*
   (1) *$G$ is second countable and* $\mathrm{Prim}(N)$ *is almost Hausdorff.*
   (2) $\mathrm{Prim}(N)/G$ *has countably separated Borel structure.*

We now give the proofs that these alternate hypotheses imply that the hypotheses of the Main Theorem are satisfied.

*Proof for A.* The primitive ideal space of any $C^*$-algebra is a $T_0$ locally quasi-compact space (see 3.3.8 of [14]). Furthermore it is easily seen that any subset of a locally quasi-compact space which is locally closed is itself locally quasi-compact, so that $GJ$ is locally quasi-compact. But a simple argument shows that any almost Hausdorff locally quasi-compact space is a Baire space. (For facts about Baire spaces see p. 109 of [8]. The condition of almost Hausdorffness cannot be dropped, as can be seen by considering a countable set whose closed subsets are the finite subsets.) Thus $GJ$ is a Baire space. The proof now follows from the following proposition.

PROPOSITION 7.1. *Let $G$ be a locally compact group which acts as a transformation group on the $T_0$ topological space $M$. Let $m \in M$, and let $G_m$ denote the stability subgroup of $m$. Suppose that the orbit $Gm$ is a Baire space and almost Hausdorff in the relative topology, and that $G/G_m$ is $\sigma$-compact. Then the canonical mapping $q$ of $G/G_m$ onto $Gm$ is a homeomorphism.*

*Proof.* Note that $G_m$ is closed by Lemma 1 of [6], so that $G/G_m$ is Hausdorff. We begin by showing that the orbit $Gm$ is Hausdorff. Since $Gm$ is assumed to be almost Hausdorff, we can find a dense relatively open Hausdorff subset $H$ of $Gm$. Then the preimage of $H$ under $q$ will be an open subset of $G/G_m$ and since $G/G_m$ is assumed $\sigma$-compact, a countable number of translates of this preimage will cover $G/G_m$. It follows that a countable number of translates of $H$ will cover $Gm$, but each translate of $H$ is again open and dense, and so, since $Gm$ is assumed to be a Baire space, the intersection of such a countable number of translates will be nonempty. But a point in this intersection can be separated by open neighborhoods from any other point in $Gm$. Since $Gm$ is a homogeneous space, it follows that $Gm$ is Hausdorff.

The rest of the proof follows by a quite standard argument using the fact that $Gm$ is a Baire space (see the proof on p. 65 of [33]). Q.E.D.

We remark that this argument provides a direct proof of "(1) implies (6)" in Theorem 1 of [24]. Phil Green has pointed out (personal communication) that in Proposition 7.1 and in Alternate Hypothesis A we can replace "almost Hausdorff" by "almost $T_1$".

*Proof for B.* As mentioned, any subset consisting of a single point in an almost Hausdorff space will be locally closed. (In particular, the space will be $T_0$.) Thus hypothesis 1 of the Main Theorem will be satisfied when Prim($N$) is almost Hausdorff. We also mentioned before that if $N$ is regularly embedded, then orbits in Prim($N$) will be locally closed. Thus the hypotheses of alternate hypotheses A are satisfied.

*Proof for C.* We recall from exercise 12 on p. 115 of [8] that a topological space is said to be *totally inexhaustible* (*totalement inepuisable*) if no nonempty closed subset of it is meager in itself. It is then easily verified that a locally quasi-compact almost Hausdorff space will be totally inexhaustible. According to part c of the exercise 6 in [8], any $G_\delta$ subset of a totally inexhaustible space is itself totally inexhaustible, and hence Baire. Thus $GJ$ will be a Baire subset of Prim($N$). Thus we can apply Proposition 7.1 to conclude that the map from $G/G_J$ to $GJ$ is a homeomorphism. In particular, $GJ$ will be Hausdorff and locally compact in the relative topology. The proof will then be completed once we have shown:

PROPOSITION 7.2. *Let $G$ be a group that acts as a transformation group on an almost Hausdorff topological space $M$. Let $m \in M$ and suppose that the orbit $Gm$ of $m$ is Hausdorff and locally compact in the relative topology. Then $Gm$ is locally closed in $M$.*

*Proof.* We may assume that $M$ is in fact the closure of $Gm$. Let $H$ be an open dense Hausdorff subset of $M$. Then $H$ must contain some point, say $n$, of $Gm$. Then $H \cap Gm$ is a neighborhood of $n$ in $Gm$ and so, since $Gm$ is locally compact, there must be a compact neighborhood $V$ of $n$ in $Gm$ contained in $H \cap Gm$. Let $W$ be an open subset of $H$ whose intersection with $Gm$ is the interior of $V$. Then $W \cap Gm$ must be closed in $W$, for any net of elements in $W \cap Gm$ which converges to an element of $W$ must have a subnet which also converges to an element of $V$, since $V$ is compact. But $H$ is Hausdorff so that limits in $H$ are unique, so the limit of such a net must be in $W \cap V$ which is contained in $W \cap Gm$. Thus $W \cap Gm$ is closed. We can now translate this situation to any other point of $Gm$, and we find in this way that $Gm$ is locally closed. Q.E.D.

We remark that Lemma 17.2 of [23] shows that the conditions of Proposition 7.2 are always satisfied if $G$ is compact.

*Proof for D.* If a group $G$ acts as a transformation group on a space $M$ which is second countable, then it is easily seen that the orbit space is second countable. (See Lemma 2.3 of [17]—Hausdorffness of $M$ and the topology of $G$ are not used here.) In a second countable $T_0$ space, every point is a $G_\delta$ in its closure, for by $T_0$-ness it must be the intersection of the intersections with its closure of those members of a countable base for the topology which contain it. Thus $GJ$ is a $G_\delta$-subset of its closure, and so alternate hypotheses $C$ are satisfied.

We remark that this argument provides a direct proof of "(2) implies (1) and (6)" in Theorem 1 of [24].

*Proof for E.* This is essentially the statement "(3) implies (1) and (6)" in Theorem 1 of [24]. We have found no direct proof of this implication, and so the proof seems to depend on the very deep arguments of [24].

## 8. $G$-STABLE REPRESENTATIONS

When $N$ is a normal subgroup of a locally compact group $G$ which is regularly embedded in $G$, the Main Theorem provides a major step in trying to describe the unitary representations of $G$ in terms of those of $N$ and of the $G_J/N$. In this section we will, under suitable hypotheses, complete that description. Our treatment has the feature that it involves neither cocycles nor measure theory. However, almost all of the results of this section already appear in one form or another in the literature, especially in [6] and [23], and so we will allow our discussion to be quite sketchy in places.

Suppose now that $N$ is a regularly embedded normal subgroup of the locally compact group $G$, so that orbits in Prim$(N)$ are locally closed. Then according to Lemma 9 of [6] the restriction to $N$ of any primary (i.e., factor) representation of $G$ will live on an orbit of Prim$(N)$. We indicate here a proof of this fact which does not use Glimm's projection-valued measure.

PROPOSITION 8.1. *Let $N$ be a regularly embedded normal subgroup of the locally compact group $G$ and let $W$ be a primary unitary $G$-module. Then there is a $G$-orbit in* Prim$(N)$ *(necessarily locally closed) on which $W_N$ lives.*

*Proof.* Let $D$ be any closed $G$-invariant subset of Prim$(N)$ and let $I$ be the kernel of $D$. Then $I$ is a $G$-invariant ideal, so $IW$ is a $G$-invariant subspace of $W$. But it is easily seen that $IW$ is also invariant under the commutant of the action of $G$ on $W$. Since $W$ is assumed to be primary, it follows that $IW$ is either all of $W$ or $\{0\}$.

Let $C$ be the intersection of all the closed $G$-invariant subsets of Prim$(N)$ on which $W_N$ lives. Then it is easily verified that $W_N$ lives on $C$, so that $C$ is the smallest closed $G$-invariant set on which $W_N$ lives. In particular, if $D$ is a proper closed $G$-invariant subset of $C$, then $(\ker(D))W = W$.

Since $C$ is closed and $G$-invariant, its image in Prim$(N)/G$ is closed and, by assumption, contains a relatively open dense Hausdorff subset. Suppose this subset contains more than one point. Then, since it is Hausdorff, it must contain two disjoint nonempty relatively open subsets. Let the preimages in Prim$(N)$ of these two subsets be denoted by $E_1$ and $E_2$, so that $E_1$ and $E_2$ are relatively open disjoint nonempty $G$-invariant subsets of $C$. Then $C - E_1$ and $C - E_2$ are proper $G$-invariant closed subsets of $C$ whose union is $C$.

By the previous paragraph, $(\ker(C - E_i))W = W$ for $i = 1, 2$. Since

$$\ker(C) = (\ker(C - E_1))(\ker(C - E_2)),$$

it follows that $(\ker(C))W = W$, which contradicts the fact that $W_N$ lives on $C$. Thus $C$ contains a dense relatively open orbit, say $GJ$. Since $C - GJ$ is a proper closed $G$-invariant subset of $C$,

$$(\ker(C - GJ))W = W,$$

so that $W$ lives on $GJ$ as desired. Q.E.D.

Thus, in the case in which $N$ is regularly embedded in $G$ and $\mathrm{Prim}(N)$ is almost Hausdorff, the Main Theorem says that in order to classify the irreducible representations of $G$ it suffices to look at each $J \in \mathrm{Prim}(N)$ and classify the representations of $G_J$ whose restrictions to $N$ live on $\{J\}$, at least as long as the $G/G_J$ are $\sigma$-compact. Thus, letting $G = G_J$, we need to analyze the case in which $J$ is $G$-invariant and find all representations of $G$ whose restriction to $N$ lives on $\{J\}$.

We recall from [6] that an irreducible representation of a $C^*$-algebra is called *semicompact* if its range contains at least one nonzero compact operator, and hence all compact operators, and that a primitive ideal is called semicompact if it is the kernel of a semicompact representation. If $J$ is a semicompact primitive ideal which is the kernel of the semicompact representation $R$, then it is easily seen that the preimage under $R$ of the ideal of all compact operators will be the smallest ideal properly containing $J$, and so will be the kernel of $\{J\}^c - \{J\}$. In particular, $\{J\}$ will be open in its closure. Then, from standard facts about the algebra of compact operators (4.1 of [14]), one immediately obtains the following special case of Lemma 8 of [6].

PROPOSITION 8.2. *Let $J$ be a semicompact ideal of a $C^*$-algebra. Then any representation which lives on $\{J\}$ is a direct sum of copies of the unique (within unitary equivalence) irreducible representation whose kernel is $J$.*

Note that a primitive ideal can easily be open in its closure without being semicompact. For example, this will be true of the zero ideal in any simple $C^*$-algebra, such as the Calkin algebra. In fact, the requirement that every primitive ideal of a $C^*$-algebra be semicompact is equivalent to the requirement that the algebra be GCR (i.e., postliminaire; see 4.3.1 of [14], and so type I by [38]), in which case the primitive ideal space is automatically almost Hausdorff by 4.4.5 of [14].

Anyway, we see from Proposition 8.2 that if $J$ is a semicompact $G$-invariant element of $\mathrm{Prim}(N)$, then the restriction to $N$ of an irreducible unitary $G$-module will live on $\{J\}$ if and only if this restriction is a direct sum of copies of the irreducible unitary $N$-module whose kernel is $J$. We now discuss how to classify the latter.

Let $J$ be a semicompact $G$-invariant primitive ideal and let $V$ be the corresponding irreducible unitary $N$-module. Since $J$ is $G$-invariant, $V^x$ also must have $J$ as kernel for every $x \in G$, so, since $J$ is semicompact, $V^x$ must be equivalent to $V$. Thus for each $x \in G$ we can find a unitary operator $P_x$ on $V$ such that

$$(xsx^{-1})v = P_x s P_x^{-1} v \tag{8.1}$$

for all $s \in N$, $v \in V$. Since $V$ is irreducible, $P_x$ must be unique up to a scalar multiple of modulus one. Let $U(V)$ denote the group of unitary operators on the Hilbert space $V$ and let $PU(V)$ denote the corresponding projective unitary group, which is the quotient of $U(V)$ by the subgroup consisting of the scalar multiples of the identity operator. On $U(V)$ the strong and weak operator topologies agree, and with them $U(V)$ is a topological group. We always consider $PU(V)$ to have the corresponding quotient topology. For each $x \in G$ let $P_x$ also denote its image in $PU(V)$, which now is uniquely determined. Then it is easily verified that $P$ is a group homomorphism from $G$ into $PU(V)$, but in general we do not know whether it is continuous. However, if $G$ is second countable, then one can use Polish topology to show that $P$ is continuous, as was done, for example, in Proposition 17.2 of [23]. Alternatively, if $V$ is semicompact as in the case we are considering, then $P$ will again be continuous, as shown in Proposition 17.1 of [23]. We include the argument here, since we can give a somewhat simpler proof of the lemma on which it depends (Lemma 17.1 of [23]).

LEMMA 8.1. *Let $V$ be a Hilbert space, let $P \in U(V)$, and let $\{P_i\}$ be a net of elements of $U(V)$. Then the (images of the) $P_i$ converge in $PU(V)$ to $P$ if and only if $P_i T P_i^{-1}$ converges strongly to $PTP^{-1}$ for every rank one operator $T$ on $V$.*

*Proof.* Routine arguments show that the direct implication actually holds for all bounded operators $T$. We show the converse. Choose a vector $v_0$ of unit length in $V$ and let $T_0$ denote the projection on the one-dimensional subspace spanned by $v_0$. Then $P_i T_0 P_i^{-1} P v_0$ converges to $PT_0P^{-1}Pv_0 = Pv_0$. Let $v_i$ denote $P_i T_0 P_i^{-1} P v_0$ divided by its length when its length is nonzero, and eliminate the terms of length zero, so that $\{v_i\}$ is a net in $V$ that converges to $Pv_0$. Then each $v_i$ is a unit vector in the range of the rank one operator $P_i T_0$. It follows that there is a complex number of modulus one $c_i$ such that $v_i = c_i P_i v_0$. Let $Q_i = c_i P_i$. Then $Q_i v_0 = c_i P_i v_0 = v_i$, which converges to $Pv_0$.

We show now that the $Q_i$ (which have the same image in $PU(V)$ as the $P_i$), converge in $U(V)$ to $P$. Note that

$$Q_i T Q_i^{-1} = P_i T P_i^{-1},$$

so that $Q_i T Q_i^{-1}$ converges to $PTP^{-1}$ for any rank one operator $T$. Let $v$ be any element of $V$, and chose as $T$ any rank one operator such that $Tv_0 = v$. Then

$$Q_i v = Q_i T v_0 = Q_i T Q_i^{-1} Q_i v_0 = Q_i T Q_i^{-1}(Q_i v_0 - P v_0) + Q_i T Q_i P v_0.$$

The first term converges to zero, while the second term converges to $PTP^{-1}Pv_0 = Pv$ as desired. Q.E.D.

Let us return to the notation used in the paragraph preceding the lemma. Then relation (8.1) lifts to elements of $C^*(N)$, so if $V$ is semicompact, the hypotheses of Lemma 8.1 will be satisfied, and the continuity of $P$ follows. If $V$ is a Hilbert space and $G$ is a topological group, then a continuous homomorphism of $G$ into $PU(V)$ will be called a *projective representation* of $G$ on $V$.

Let $V$ be a Hilbert space, $G$ a locally compact group, and $P$ a projective representation of $G$ on $V$. Let $p$ denote the canonical projection of $U(V)$ onto $PU(V)$, and form the pullback of $P$ and $p$, that is,

$$\tilde{G} = \{(x, U) \in G \times U(V) : P(x) = p(U)\}.$$

Then it is easily verified that $\tilde{G}$ is a central topological group extension of $G$ by the group $T$ of complex numbers of modulus one. As such, it is locally compact (p. 52 of [33]). The isomorphism classes of central extensions of $G$ by $T$ form an Abelian group under Baer multiplication (see [28, 34] and p. 147 of [23]), which in the separable case is isomorphic with the second cohomology group (defined in terms of cocycles) of $G$ with coefficients in $T$ [31, 34, 35]. If each element $(x, U)$ of $\tilde{G}$ is sent to the corresponding operator $U$ on $V$, one obtains an ordinary representation of $\tilde{G}$ whose restriction to $T$ is a multiple of the standard representation of $T$, that is, is of the form $t \mapsto tI_V$ for $t \in T$, where $I_V$ is the identity operator on $V$. Conversely, every ordinary representation of a central extension of $G$ by $T$ whose restriction to $T$ is a multiple of the standard representation defines in an obvious way a projective representation of $G$. A projective representation of $G$ lifts to an ordinary representation of $G$ [that is, factors through $p: U(V) \to PU(V)$] if and only if the central extension $\tilde{G}$ as previously constructed from it splits. But even then it need not come from a unique ordinary representation. Two ordinary irreducible representations of $G$ define the same projective representation if and only if each is the inner tensor product of the other with some one-dimensional representation (character) of $G$.

Let us now return to the situation in which we have a normal subgroup $N$ of $G$ and a semicompact $G$-invariant primitive ideal $J$ of $C^*(N)$, with corresponding representation $R$ on $V$. Let $P$ be the projective representation of $G$ which implements the equivalence of $V^x$ with $V$ for $x \in G$, as defined earlier. Then for $s \in N$ the equivalence of $V^s$ with $V$ can be implemented by $R_s$, so for $s \in N$ we can assume that $P_s = R_s$, so that $P$ is an extension of the ordinary representation of $N$ to a projective representation of $G$. This result can be summarized as follows.

Proposition 8.3. *Let $N$ be a normal subgroup of the locally compact group $G$. Then any irreducible semicompact $G$-invariant ordinary representation of $N$ can be extended to a projective representation of $G$.*

With the previous notation, $R$ can be viewed as a projective representation of $N$ (coinciding with the restriction of $P$ to $N$) and we can form the corresponding central extension $\tilde{N}$ of $N$. This extension splits via the map $s \mapsto (s, R_s)$, but $\tilde{N}$ can be viewed as a normal subgroup of $\tilde{G}$. We then obtain the following diagram of exact sequences, which is commutative in rows and columns, but not with its diagonal maps.

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
 & 1 \arrow[d] & 1 \arrow[d] & T \arrow[d] & 1 \\
 & T \arrow[d] \arrow[r, phantom, "\cong"] & T \arrow[d] & \tilde{G}/N \arrow[d] \arrow[ur] & \\
1 \arrow[r] & \tilde{N} \arrow[r] \arrow[d] & \tilde{G} \arrow[r] \arrow[d] \arrow[ur] & \tilde{G}/\tilde{N} \arrow[r] \arrow[d, "\cong"] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[d] \arrow[u, bend left] \arrow[ur] & G \arrow[r] \arrow[d] & G/N \arrow[r] \arrow[d] & 1 \\
1 \arrow[ur] & 1 & 1 & 1 &
\end{tikzcd}$$

The diagonal map from $N$ to $\tilde{G}$ is defined by composing the map from $\tilde{N}$ to $\tilde{G}$ with the splitting map defined previously from $N$ to $\tilde{N}$. The isomorphism class of the extension $\tilde{G}/N$ of $G/N$ is called the *Mackey obstruction* of $R$ or $J$.

We are looking for the ordinary representations of $G$ whose restriction to $N$ is a multiple of $R$. Now any such representation of $G$ clearly defines (see the diagram) an ordinary representation of $\tilde{G}$ whose restriction to $N$ is $R$. Recall that $R$ extends to an ordinary representation, namely $P$, of $\tilde{G}$. Then we can find the desired representations of $G$ by the two-step process of first finding all ordinary representations of $\tilde{G}$ whose restriction to $N$ is $R$, and then determining which of these drop to $G$. The first step is accomplished by the following proposition, in which $H$ plays the role of $\tilde{G}$.

PROPOSITION 8.4. *Let $N$ be a normal subgroup of the locally compact group $H$ and let $R$ be an ordinary irreducible representation of $N$. Suppose that $R$ extends to an ordinary representation $P$ of $H$. For any ordinary representation $Q$ of $H/N$, viewed as a representation of $H$, let $P \mathbin{\#} Q$ denote the inner tensor product of $P$ with $Q$. Then the functor $Q \mapsto P \mathbin{\#} Q$ is an equivalence between the category of ordinary representations of $H/N$ and the category of ordinary representations of $H$ whose restriction to $N$ is a multiple of $R$.*

*Proof.* It is clear that the indicated functor gives representations of the desired type. Suppose, conversely, that $S$ is a representation of $H$ whose restriction to $N$ is a multiple of $R$. If we let $V$ denote the space for $R$ and $P$, then the space for $S$ can be written in the form $V \otimes W$ where $W$ is a Hilbert space of dimension equal to the multiplicity of $R$ in the restriction of $S$ to $N$. We can also assume that for each $s \in N$,

$$S_s = R_s \otimes I_W.$$

For each $x \in H$ let

$$T_x = (P_x^{-1} \otimes I_W)S_x.$$

Then a simple calculation shows that each $T_x$ commutes with all operators of the form $R_x \otimes I_W$. Since $R$ is irreducible, it follows from the double commutant theorem (p. 42 of [13]) that each $T_x$ commutes with all operators of the form $M \otimes I_W$, where $M$ is any bounded operator on $V$. It follows (p. 24 of [13]) that $T_x = I_V \otimes Q_x$ for some operator $Q_x$ on $W$. It is then easily verified that $Q$ is a unitary representation of $H$ which is trivial on $N$, so that it gives a representation of $H/N$, and that $S = P \mathbin{\#} Q$. Finally, arguments similar to these show that the previously defined functor is an isomorphism between the corresponding spaces of intertwining operators.

Q.E.D.

We now carry out the second step, namely, we determine which of the ordinary representations of $\tilde{G}$ of the form $P \mathbin{\#} Q$ drop to $G$, where $Q$ is an ordinary representation of $\tilde{G}/N$. Now from the diagram it is clear that these will be exactly the representations that have the subgroup $T$ of $\tilde{G}$ in their kernel. But it is easily calculated that these will be exactly the representations in which the restriction of $Q$ to $T$ is the inverse of the standard representation, that is, $Q_t = t^{-1} I_W$. Finally, if we now form the extension of $G/N$ by $T$ which is inverse under Baer multiplication to the given extension, then it is easily verified that the preceding $Q$'s are exactly those representations of this inverse extension whose restriction to $T$ is a multiple of the standard extension. We summarize these results as follows.

PROPOSITION 8.5. *Let $N$ be a normal subgroup of the locally compact group $G$ and let $J$ be a semicompact $G$-invariant ideal in* Prim($N$). *Then the category of unitary $G$-modules whose restriction to $N$ lives on $\{J\}$ is equivalent to the category of projective representations of $G/N$ which belong to the inverse of the Mackey obstruction of $J$.*

We remark that Dixmier has shown (Proposition 6 of [16]) that this equivalence preserves weak containment, at least for irreducible representations when $G$ is second countable and $N$ is type $I$.

*Note Added in Proof*. Extensive generalizations of the results and techniques of this article, with applications, are contained in Philip Green, The local structure of twisted covariance algebras, *Acta Math.* **140** (1978), 191–250.

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