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Dentable Subsets of Banach Spaces, with Application to a Radon-Nikodym Theorem

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This work originated as an attempt to give a new proof of Phillips' Radon-Nikodym theorem [5, p. 130] using techniques analogous to those developed in [6]. Instead, we have obtained a new Radon-Nikodym theorem—one which is not a consequence of Phillips' theorem. However, we have not been able to show that our theorem implies Phillips' theorem in its full generality.

DEFINITION 1. A subset, K, of a Banach space will be called *dentable* if for every $\epsilon > 0$ there is a $b \in K$ such that $b \notin \bar{c}(K - \text{ball}(b, \epsilon))$ (where $\bar{c}$ denotes "closed convex hull").

We shall discuss the question of which subsets of Banach spaces are dentable after we have stated and proved the Radon-Nikodym theorem referred to above.

THEOREM 1. Let (X, S, μ) be a σ-finite positive measure space, and let B be a Banach space. Let m be a B-valued measure on S. Then there is a B-valued Bochner integrable function, f, on X such that

$$m(E) = \int_E f\, d\mu$$

for all $E \in S$, if and only if

1. m is μ-continuous, that is, $m(E) = 0$ whenever $\mu(E) = 0$, $E \in S$.
2. the total variation, $|m|$, of m is a finite measure.
3. locally m almost has dentable average range, that is, given $E \in S$, $\mu(E) < \infty$, and given $\epsilon > 0$, there is an $F \subseteq E$, such that $\mu(E - F) < \epsilon$ and

$$A_F = \{m(F')/\mu(F') \colon F' \subseteq F, \mu(F') > 0\}$$

is dentable.

is clear that any set whose closed convex hull has a denting point is dentable.

Now Lindenstrauss has shown in [3] that if B is a Banach space that can be given an equivalent norm which is locally uniformly convex, then any weakly compact convex subset of B is the closed convex hull of its strongly exposed points. In particular, Question 1 has an affirmative answer in all such Banach spaces. Furthermore, Kadec has shown in [1, 2] that any separable Banach space can be given an equivalent norm which is locally uniformly convex. Thus Question 1 has an affirmative answer for all separable Banach spaces.[1]

To describe a result in the opposite direction, let us call a subset K of a Banach space *subset-dentable* if every subset of K is dentable. Then it is natural to ask whether every subset-dentable subset of a Banach space is relatively weakly compact. That this need not be the case is shown by the following result (it is a pleasure to thank B. Kripke for several stimulating conversations which led us to this result).

THEOREM 3. Let X be any (possibly uncountable) set. Then any bounded subset of $l^1(X)$ is dentable.

Proof. In view of Proposition 2 it suffices to show that any bounded closed convex subset of $l^1(X)$ is dentable. Let K be such a subset, and let $\epsilon > 0$ be given. Let

$$s = \sup\{\|a\| : a \in K\}.$$

There need not be an element of K whose norm attains the value s, but we can choose an element $a \in K$ such that $\|a\| > s - \epsilon/6$. Then there is a finite subset, F, of X such that $\sum_{x \in F} |a(x)| > s - \epsilon/6$.

For any element, b, of $l^1(X)$ let b_F denote its natural projection into the subspace $l^1(F)$. Then K_F, the projection of K into $l^1(F)$, is bounded and convex, though it need not be closed. Let $\bar{K}_F$ denote its closure. Since $l^1(F)$ is finite dimensional, $\bar{K}_F$ is norm compact. Since $a_F \in K_F$ and $\|a_F\| > s - \epsilon/6$, $\bar{K}_F$ must have an extreme point, e, such that $\|e\| > s - \epsilon/6$. By the Krein-Milman theorem, as in Proposition 1, $e \notin \bar{c}(K_F - \text{ball}(e, \epsilon/6))$ and so there is an f in $l^\infty(F)$ which separates e from $\bar{c}(K_F - \text{ball}(e, \epsilon/6))$, that is, such that there is a constant, r, for which $f(e) > r$ but $f(b_F) < r$ for any b_F in $K_F - \text{ball}(e, \epsilon/6)$.

Choose $b \in K$ such that $f(b_F) > r$ (so that necessarily $\|b_F - e\| \leq \epsilon/6$). We show that $b \notin \bar{c}(K - \text{ball}(b, \epsilon))$. Suppose that $c \in K$ and $f(c_F) \geq r$. Then $\|c_F - e\| \leq \epsilon/6$ and so $\|c_F - b_F\| \leq \epsilon/3$. Also, since

[1] Added in proof: A short, elegant proof of this fact, independent of the work of Lindenstrauss and Kadec, has recently been given by I. Namioka and E. Asplund in "A geometric proof of Ryll-Nardzewski's fixed point theorem" Bull. A.M.S. 73 (1967) 443–445.

that A_{E_d} is dentable. Choose $b \in A_{E_d}$ such that $b \notin Q$, where $Q = \bar{c}(A_{E_d} - \text{ball}(b, \epsilon))$. Let $b = m(F_0)/\mu(F_0)$ where $F_0 \subseteq E_d$ and $0 < \mu(F_0) < \infty$. Suppose that F_0 is not (b, ϵ)-pure. Let k_1 be the smallest integer ≥ 2 for which there is an $E_1 \subseteq F_0$ such that $\mu(E_1) \geq 1/k_1$ and $m(E_1)/\mu(E_1)$ is in Q. Let $F_1 = F_0 - E_1$, and suppose that F_1 is not (b, ϵ)-pure. Let k_2 be the smallest integer ≥ 2 for which there is an $E_2 \subseteq F_1$ such that $\mu(E_2) \geq 1/k_2$ and $m(E_2)/\mu(E_2)$ is in Q. Let $F_2 = F_1 - E_2$. Continuing in this way we obtain a sequence $\{E_i\}$ of disjoint subsets of F_0, and a nondecreasing sequence $\{k_i\}$ of integers with the property that $m(E_i)/\mu(E_i)$ is in Q and $\mu(E_i) \geq 1/k_i$ for each i, and if $E' \subseteq F_0 - \bigcup_{i=1}^{n} E_i$ and $m(E')/\mu(E')$ is in Q, then $\mu(E') < 1/(k_n - 1)$. Since F_0 has finite measure and the E_i are disjoint, k_i must converge to ∞.

Let $E_0 = \bigcup E_i$, and let $F = F_0 - E_0$. Then F is (b, ϵ)-pure, for if $F' \subseteq F$, $\mu(F') > 0$ and $m(F')/\mu(F')$ is in Q, then, since $F' \subseteq F_0 - \bigcup_{i=1}^{n} E_i$ for each n, it follows that $\mu(F') < 1/(k_n - 1)$ for each n. But, k_n converges to ∞, and so $\mu(F') = 0$.

Finally, we show that $\mu(F) > 0$. Suppose that $\mu(F) = 0$. Then since m is μ-continuous, $m(F) = 0$, and so $m(F_0)/\mu(F_0) = m(E_0)/\mu(E_0)$. Now

$$m(E_0)/\mu(E_0) = \sum_i (m(E_i)/\mu(E_i))(\mu(E_i)/\mu(E_0)).$$

Then, since $\sum_i \mu(E_i)/\mu(E_0) = 1$, and $m(E_i)/\mu(E_i)$ is in Q for each i, and since Q is closed and convex, it follows that $m(E_0)/\mu(E_0)$ is in Q. Thus $m(F_0)/\mu(F_0)$ is in Q, contradicting the way in which F_0 was chosen.

Proof of Theorem 2. Since X is the union of a countable number of subsets of finite measure, it suffices to prove Theorem 2 under the assumption that X has finite measure.

Let k_1 be the smallest integer ≥ 2 for which there is a $b_1 \in B$ and an $E_1 \subseteq X$ such that E_1 is (b_1, ϵ)-pure and $\mu(E_1) \geq 1/k_1$. Let k_2 be the smallest integer ≥ 2 such that there is a $b_2 \in B$ and an $E_2 \subseteq X - E_1$ such that E_2 is (b_2, ϵ)-pure and $\mu(E_2) \geq 1/k_2$. Continuing in this way we obtain a sequence $\{E_i\}$ of disjoint subsets of X, a sequence $\{b_i\}$ of elements of B, and a nondecreasing sequence $\{k_i\}$ of integers with the property that E_i is (b_i, ϵ)-pure and $\mu(E_i) \geq 1/k_i$ for each i, and if $F \subseteq X - \bigcup_{i=1}^{n} E_i$ and F is (b, ϵ)-pure for some $b \in B$ then $\mu(F) < 1/(k_n - 1)$. Since X is assumed to have finite measure, k_i must converge to ∞.

The slight generalization of Phillips' theorem for which we tried to find a new proof is obtained by replacing the word "dentable" in hypothesis 3 of Theorem 1 by "relatively weakly compact."

The necessity of hypotheses 1 and 2 is well known. In [6, Prop. 1.12] it was shown that an additional necessary condition is obtained if in hypothesis 3 the word "dentable" is replaced by "relatively norm compact." But,

PROPOSITION 1. Any relatively norm compact subset of a Banach space is dentable.

Proof. First, suppose that K is a norm compact convex subset of a Banach space. Let b be any extreme point of K. Then, since the norm closure of $K - \text{ball}(b, \epsilon)$ does not contain b for any $\epsilon > 0$, it follows from part of the Krein-Milman theorem that $b \notin \overline{c}(K - \text{ball}(b, \epsilon))$.

The general case follows from

PROPOSITION 2. Let K be any subset of a Banach space. If $\overline{c}(K)$ is dentable, then so is K.

Proof. Given $\epsilon > 0$, choose $b' \in \overline{c}(K)$ such that $b' \notin Q$, where

$$Q = \overline{c}(\overline{c}(K) - \text{ball}(b', \epsilon/2)).$$

Since $b' \in \overline{c}(K) - Q$, and Q is closed and convex, Q can not contain K. Choose $b \in K - Q$. Then $b \in \text{ball}(b', \epsilon/2)$, and so $(K - \text{ball}(b, \epsilon)) \subseteq Q$. Thus $b \notin \overline{c}(K - \text{ball}(b, \epsilon))$.

We now turn to the proof of the sufficiency of the hypotheses of Theorem 1. Since finite measures are carried on measurable sets, it suffices to prove sufficiency under the assumption that (X, S, μ) is totally σ-finite. From now on we restrict our attention to this case.

As in [6], the key step in the proof of sufficiency is a decomposition theorem analogous to the classical Hahn decomposition theorem. We recall the following definition from [6].

DEFINITION 2. Given $b \in B$ and $\epsilon > 0$ we say that a set $E \in S$ is (b, ϵ)-pure (for m with respect to μ) if $m(F)/\mu(F)$ is in $\text{ball}(b, \epsilon)$ for all $F \subseteq E$ such that $0 < \mu(F) < \infty$.

The appropriate decomposition theorem for the present situation is

THEOREM 2. Let (X, S, μ) be a totally σ-finite measure space, and let m be a B-valued measure on S which satisfies hypotheses 1, 2 and 3 of Theorem 1. Then, given $\epsilon > 0$, there are (possibly finite) sequences $\{b_i\}$ and $\{E_i\}$ of elements of B and S respectively such that E_i is (b_i, ϵ)-pure for each i, and $X = \bigcup E_i$.

LEMMA 1. Let the hypotheses of Theorem 2 be satisfied, and let $E \in S$ and $\epsilon > 0$ be given, with $\mu(E) > 0$. Then there is an $F \subseteq E$ and a $b \in B$ such that $\mu(F) > 0$ and F is (b, ϵ)-pure.

Proof. By hypothesis 3 of Theorem 1 choose $E_d \subseteq E$, $\mu(E_d) > 0$ such

$\|e\| > s - \epsilon/6$, we have $\|c_F\| > s - \epsilon/3$, as well as $\|b_F\| > s - \epsilon/3$. But $\|b\| \le s$ and $\|c\| \le s$, and so $\|b - b_F\| < \epsilon/3$ and $\|c - c_F\| < \epsilon/3$. Thus $\|b - c\| < \epsilon$. It follows that if $c \in K - \text{ball}(b, \epsilon)$, then $f(c_F) \le r$, and so the same inequality holds for all elements of $\overline{c}(K - \text{ball}(b, \epsilon))$. But $f(b_F) > r$, and so $b \notin \overline{c}(K - \text{ball}(b, \epsilon))$.

With this result in mind it is not at all clear how one might hope to characterize the dentable subsets of Banach spaces. Since every Banach space can be isometrically embedded in $C(X)$ (the Banach space of all continuous functions on some compact Hausdorff space X), this problem is equivalent to

QUESTION 2. Which are the dentable subsets of $C(X)$?

In view of Theorem 3 one is also led to ask

QUESTION 3. Which Banach spaces have the property that all bounded subsets are dentable?[2]

We remark that it is easily checked that the unit balls of the Banach spaces c_0, $C([0, 1])$, and $L^1(m)$, where m is Lebesgue measure, are not dentable.

Another question which we have not been able to resolve is whether dentability always comes about essentially because of the presence of strongly exposed points. Put another way

QUESTION 4. Does there exist a closed, bounded, convex set which is dentable but which has no strongly exposed points?

In view of Theorem 3 the Banach space l^1 would seem to be an excellent place in which to look for such an example. But we have not been able to answer Question 4 even for l^1. We remark that in connection with this problem we asked whether there is a closed, bounded, convex subset of l^1 which has no extreme points at all. This question has now been answered by Lindenstrauss [4], who shows that every closed, bounded, convex subset of l^1 is the closed convex hull of its extreme points.

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[2] Added in proof: I. Namioka has recently shown, in "Neighborhoods of Extreme Points" (to appear), that every dual Banach space which is separable has this property.

Let $E = X - \cup E_i$. We show that $\mu(E) = 0$. Suppose that $\mu(E) > 0$. Then by Lemma 1 there is an $F \subseteq E$ and a $b \in B$ such that $\mu(F) > 0$ and F is (b, ϵ)-pure. But $F \subseteq X - \bigcup_{i=1}^{n} E_i$ for each n and so $\mu(F) < 1/(k_n - 1)$ for each n. But k_n converges to 0, and so $\mu(F) = 0$.

Since $\mu(E) = 0$, E can be adjoined to E_1 and E_1 will still be (b_1, ϵ)-pure. With this change made, $X = \cup E_i$.

Proof of Theorem 1. As in [6] let Π denote the set of all collections, π, consisting of a finite number of disjoint elements of S of strictly positive finite measure. Then Π becomes a directed set (up to null sets) if $\pi_1 \geq \pi$ is defined to mean that every element of π is, except for possible null sets, the union of elements of π_1. For each $\pi \in \Pi$ we define a simple integrable function f_π by

$$f_\pi = \sum_{E \in \pi} (m(E)/\mu(E))\chi_E$$

(where χ_E denotes the characteristic function of E). We show that the f_π form a mean Cauchy net.

Let $\epsilon > 0$ be given. We seek a $\pi_0 \in \Pi$ such that if $\pi \geq \pi_0$ then $\|f_\pi - f_{\pi_0}\| < \epsilon$. Since $|m|$ is assumed to be a finite measure, we can find $E \in S$ such that $\mu(E) < \infty$ and $|m|(X - E) < \epsilon/3$. Since m is μ-continuous, so is $|m|$, and so, since $|m|$ is a finite measure, there is a $\delta > 0$ such that if $\mu(F) < \delta$ then $|m|(F) < \epsilon/6$.

By Theorem 2 we can find (possibly finite) sequences $\{E_i\}$ and $\{b_i\}$ of elements of S and B respectively such that E_i is $(b_i, \epsilon/6\mu(E))$-pure for each i, $\mu(E_i) > 0$, and $E = \cup E_i$, the union being disjoint. Since E has finite measure, there is an integer n such that if $E_0 = E - \bigcup_{i=1}^{n} E_i$ then $\mu(E_0) < \delta$. Let $\pi_0 = \{E_i: 0 \leq i \leq n\}$ unless $\mu(E_0) = 0$, in which case let $\pi_0 = \{E_i: 1 \leq i \leq n\}$. Then a routine calculation, as given in [6], shows that $\|f_\pi - f_{\pi_0}\| < \epsilon$ whenever $\pi \geq \pi_0$.

Since the f_π form a mean Cauchy net, there is an integrable function, f, to which f_π converges in mean. In particular,

$$\int_E f\,d\mu = \lim_\pi \int_E f_\pi\,d\mu$$

for all $E \in S$. We show that

$$m(E) = \int_E f\,d\mu$$

for all $E \in S$. If $\mu(E) = 0$, the result follows from the μ-continuity of m.

If $0 < \mu(E) < \infty$, let $\pi_0 = \{E\}$. It is easily checked that $\int_E f_\pi\,d\mu = m(E)$ whenever $\pi \geq \pi_0$, and so

$$m(E) = \lim_\pi \int_E f_\pi\,d\mu = \int_E f\,d\mu.$$

The case $\mu(E) = \infty$ then follows easily using the σ-finiteness of μ.

We remark that the proof of Theorem 1 is essentially measure theoretic. All the geometric difficulties involved in obtaining Radon-Nikodym theorems for the Bochner integral with values in some particular Banach space are contained in the problem of determining which subsets of the Banach space are dentable.

In particular, in view of the fact that we should like to obtain Phillips' Radon-Nikodym theorem as a consequence of Theorem 1, we are led to ask

QUESTION 1. Is every relatively weakly compact subset of a Banach space dentable?

The proof of Proposition 1 does not work in this case, since an extreme point b of a closed convex set K may well be in the *weak* closure of K − ball(b, ϵ). This is illustrated by the following well known example.

EXAMPLE 1. Let H be an infinite dimensional Hilbert space, and let $\{e_i\}$ be an infinite orthonormal sequence in H. Let $K = \overline{c}(\{e_i\})$. Then $0 \in K$, since the e_i converge weakly to 0, and norm closed convex sets are weakly closed. It is also easily checked that 0 is an extreme point of K. But it is clear that $K = \overline{c}(K - \text{ball}(0, \epsilon))$ for any $\epsilon < 1$.

We have not been able to answer Question 1 completely, but the answer is affirmative in many cases. To begin with, it can be shown by a routine argument that any bounded subset of a uniformly convex Banach space is dentable. Since uniformly convex spaces are reflexive, the bounded subsets coincide with relatively weakly compact subsets, and so Question 1 is answered affirmatively in this case.

To describe a much deeper result, we first recall from [3]

DEFINITION 3. A point b of a set K is called *strongly exposed* if there is a continuous linear functional f such that $f(b) > f(b')$ for all $b' \in K$, $b' \neq b$, and such that if $f(b_n) \to f(b)$ for $\{b_n\} \subseteq K$ then $b_n \to b$.

Also, in view of Definition 1, it is natural to make

DEFINITION 4. A point b of a set K is called a *denting point* if $b \notin \overline{c}(K\text{-ball}(b, \epsilon))$ for all $\epsilon > 0$.

It is easily checked that any strongly exposed point is a denting point. Namioka has pointed out to us that the converse is not true, as there are norm compact sets which have extreme points which are not exposed. In Example 1 the point 0 is an extreme point which is not a denting point. It