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Deformation Quantization and Operator Algebras

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Although no one seems yet to have formulated a satisfactory definition of what is meant by a noncommutative differentiable manifold, a number of classes of examples are now known which will undoubtedly be included when such a definition is finally given. In the meantime, it is interesting to develop further classes of examples. Intuitively, one natural way to try to do this is to take ordinary differentiable manifolds and then to try to "deform" them in some way. Suppose for the moment that the manifold M is compact. Then we can take the algebra $A = C^\infty(M)$ of smooth complex-valued functions with pointwise multiplication, and, for a deformation parameter $\hbar$ running over some open interval I of real numbers with 0 as center, we can try to deform the product to obtain new associative products $*_\hbar$ on A, which need not be commutative, but which vary smoothly in some sense, and are such that $*_0$ is the usual pointwise product. We can also try to deform the complex-conjugation involution to a family $*_\hbar$ of involutions, and to deform the supremum norm to a family $\| \ \|_\hbar$ of C^*-norms. Then it would be natural to expect that under favorable circumstances the completed C^*-algebras, $\overline{A}_\hbar$, should be examples of noncommutative differentiable manifolds.

Since we want the deformed product to have some smoothness, we should be able to express it as

$$f *_\hbar g = fg + \hbar P(f, g) + \mathcal{O}(\hbar^2)$$

as $\hbar$ goes to 0, where P is a bilinear map from $A \times A$ to A. One then checks easily that the requirement that $*_\hbar$ be associative forces P to be a Hochschild 2-cocycle.

In the usual setting for deformation quantization one assumes that P is actually a special kind of 2-cocycle, namely, essentially a Poisson bracket. The reason for this is that in the application to quantum mechanics **[BFF]**

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M will be the phase space of a classical mechanical system (so, noncompact), equipped with its usual symplectic structure and corresponding Poisson bracket. As a generalization, one then also considers Poisson brackets, $\{\ ,\ \}$, which need not come from symplectic structures. Thus one only assumes that $\{\ ,\ \}$ is a Lie algebra structure on A giving derivations for the pointwise product, that is,

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

for $f, g, h, \in A$. Since Poisson brackets are skew-symmetric, we will assume the same of P. Now because we are assuming the presence of involutions, we must have

$$(f *_\hbar g)^{*_\hbar} = g^{*_\hbar} *_\hbar f^{*_\hbar}.$$

This implies that $P(f, g)^{*_\hbar} = P(g^{*_\hbar}, f^{*_\hbar})$. Suppose that for all $\hbar$ the involution $^{*_\hbar}$ is just complex conjugation (which often, but not always, will be the case). Then for f, g real we find that $P(f, g)^- = P(g, f) = -P(f, g)$, so that $P(f, g)$ is pure imaginary. Thus it is natural to take P to be a pure imaginary multiple of the Poisson bracket. The usual convention is

$$P(f, g) = (i/2)\{f, g\}.$$

Then

$$f *_\hbar g - g *_\hbar f = \hbar i\{f, g\} + \mathscr{O}(\hbar^2),$$

so that

$$\lim_{\hbar\to 0}(f *_\hbar g - g *_\hbar f)/i\hbar = \{f, g\}.$$

This is the property characterizing how a deformation quantization is related to a given Poisson bracket. Notice that it is only an infinitesimal condition at $\hbar = 0$, so one does not expect deformations for a given Poisson bracket to be unique. Also, we will need to be more precise about what kind of convergence is involved in the limit above.

Most of the extensive literature on deformation quantization is concerned with formal deformations, in which $f *_\hbar g$ is not actually a function on M, but rather a formal power series in $\hbar$ whose coefficients are functions. Investigation of this aspect was launched about a decade ago by Vey and Flato, Fronsdal, and Lichernowicz. (See **[BL, Mr]** and the references they contain.) However, as indicated above, here we are interested in the case in which $f *_\hbar g$ is a function, and in which matters can be placed in a C^*-algebra framework. This aspect has been investigated very little so far, in part because it is difficult to construct many classes of examples. Also, it seems likely that several definitions with decreasingly tight requirements will probably prove useful (see our Example 8). What we will do here is to state what will probably prove to be the definition with the tightest useful requirements and then to describe a series of examples, giving references to the papers containing the proofs. (I would like to express here my appreciation to Alan Weinstein for having brought the subject of deformation quantization to my attention by suggesting that the noncommutative tori which I had been studying should be examples (see Example 2), and for stimulating discussions and many references to the literature.)

In preparation for the main definition, we remark that in the case of a manifold M which is not compact, it is less clear what algebra of C^∞-functions one should employ. It should be a subalgebra of $C_\infty(M)$, the algebra of functions vanishing at infinity, and should contain $C_c^\infty(M)$, the algebra of smooth functions of compact support. But as we will see, it often is best to use Schwartz functions if that makes sense, or at least functions which are Schwartz in certain directions. Thus we will formulate our definition in terms of any $*$-subalgebra A of $C_\infty(M)$ which consists of smooth functions, contains $C_c^\infty(M)$, and is carried into itself by the Poisson bracket.

DEFINITION 1. Let M be a manifold with Poisson bracket $\{\ ,\ \}$, and let A be as just above. By a *strict deformation quantization* of M in the direction of $\{\ ,\ \}$ we will mean an open interval I of real numbers with 0 as center, together with, for each $\hbar \in I$, an associative product $*_\hbar$, an involution $^{*_\hbar}$, and a C^*-norm $\|\ \|_\hbar$ (for $*_\hbar$ and $^{*_\hbar}$) on A, which for $\hbar = 0$ are the original pointwise product, complex conjugation involution, and supremum norm, such that

(1) For every $f \in A$ the function $\hbar \to \|f\|_\hbar$ is continuous.
(2) For every $f, g \in A$,

$$\|(f *_\hbar g - g *_\hbar f)/i\hbar - \{f, g\}\|_\hbar$$

converges to 0 as $\hbar$ goes to 0.

If we let $\overline{A}_\hbar$ denote the C^*-completion of A for $\|\ \|_\hbar$, then condition (1) means exactly that $\{\overline{A}_\hbar\}$ is a continuous field of C^*-algebras, as discussed in **[Dx]**. A similar definition, but with looser requirements on A and concerning convergence, has been given by Berezin **[Br]**.

In much of the literature on deformation quantization one assumes also the presence of a Lie group G acting as diffeomorphisms of M preserving the Poisson bracket, and one seeks deformation quantizations which respect this action of G. Within our present context the appropriate definition is:

DEFINITION 2. Let G be a Lie group, and let α be an action of G as a group of diffeomorphisms of M which preserve the Poisson structure. Assume further that the corresponding action α of G on $C^\infty(M)$ carries A into itself. We will say that a strict deformation quantization of A, as defined above, is *invariant* under the action α if

(1) For every $\hbar \in I$ and $x \in G$, the operator α_x on A is an isometric $*$-automorphism for $*_\hbar$, $^{*_\hbar}$, and $\|\ \|_\hbar$.
(2) For every $f \in A$ and $\hbar \in I$, the map $x \mapsto \alpha_x(f)$ is a C^∞ function on G, for the norm $\|\ \|_\hbar$.
(3) There is an action, α, of the Lie algebra L of G on A which for each $\hbar \in I$ is by $*$-derivations of A for $*_\hbar$ and $^{*_\hbar}$, such that for $X \in L$ and

$f \in A$

$$\alpha_X(f) = \left.\frac{d}{dt}\right|_{t=0} \alpha_{\exp(tX)}(f)$$

with respect to $\| \ \|_\hbar$.

This definition is appropriate for the examples we discuss here, but there is evidence in **[Ar]** and **[Pd]** that in other circumstances one may have to weaken it.

The main technique used in constructing the examples described below is the Fourier transform. (The lack of a good Fourier transform in other situations is thus an obstacle to generalizing these examples.) For use in the Fourier transform we will let e denote the function on the real line R defined by $e(r) = e^{2\pi i r}$. Our convention for the Fourier transform will then be that

$$\hat{f}(r) = \int \bar{e}(rx) f(x)\, dx,$$

and similarly on other Abelian groups.

EXAMPLE 1 (The Moyal Product). This example, essentially going back to Moyal **[Ml]** in 1949, has been the original inspiration for the study of deformation quantization. The setting is R^{2n} with its standard Poisson structure given by

$$\{f, g\} = \sum_{k=1}^{n} (\partial f/\partial x_k)(\partial g/\partial x_{n+k}) - (\partial f/\partial x_{n+k})(\partial g/\partial x_k).$$

As the algebra A, one takes the algebra $S(R^{2n})$ of Schwartz functions on R^{2n}, and as Lie group G we will take R^{2n} acting on itself by translation, though we could take the affine symplectic group. The Fourier transform carries $S(R^{2n})$ with pointwise multiplication to $S(\hat{R}^{2n})$ with convolution, where $\hat{R}^{2n} = R^{2n}$ but we write $\hat{R}^{2n}$ to emphasize that we are on the dual group. The Fourier transform also takes the Poisson bracket to the operation

$$\{\phi, \psi\}(t) = -4\pi^2 \int \phi(r)\psi(t-r)\gamma(r, t-r)\, dr$$

where γ is the skew bilinear form on $\hat{R}^{2n}$ defined by

$$\gamma(r, s) = \sum (r_k s_{n+k} - r_{n+k} s_k)$$

for $r, s \in \hat{R}^{2n}$. This new $\{\ ,\ \}$ is a Poisson bracket for convolution on $S(\hat{R}^{2n})$. For any $\hbar \in R$ we define a skew bicharacter, $\sigma_\hbar$, on $\hat{R}^{2n}$ by

$$\sigma_\hbar(r, s) = e(-\pi\hbar\gamma(r, s)),$$

and then we define a product $*_\hbar$ on $S(\hat{R}^{2n})$ by

$$(\phi *_\hbar \psi)(t) = \int \phi(r)\psi(t-r)\sigma_\hbar(r, t-r)\, dr.$$

The involution we use on $S(R^{2n})$ for all $\hbar$ will be complex conjugation, and so on $S(\hat{R}^{2n})$ it is given by $\phi^*(r) = \bar{\phi}(-r)$. For any $\hbar$ we define the norm,

$\| \|_\hbar$, on $S(\hat{R}^{2n})$ to be the operator norm for $S(\hat{R}^{2n})$ acting on $L^2(\hat{R}^{2n})$ by the same formula as defines $*_\hbar$. By means of the Fourier transform we carry the products and norms back to $S(R^{2n})$, so that, for example,

$$f *_\hbar g = (\hat{f} *_\hbar \hat{g})^\vee$$

for any $f, g \in S(R^{2n})$, where $^\vee$ denotes the inverse Fourier transform. It is not difficult to verify directly property 2 of Definition 1 (see the arguments in **[Rf5]**) while property 1 follows from corollary 2.7 of **[Rf4]**. Thus we have a strict deformation quantization. For $\hbar \neq 0$ the algebras involved are closely related to the Heisenberg commutation relations and representations of the Heisenberg Lie group, and it is well known (see, e.g., **[Rf1]**) that their completions, $\overline{A}_\hbar$, are isomorphic to the algebra of compact operators. For $\hbar = 0$ the completion is, of course, just $C_\infty(R^{2n})$. Invariance under $G = R^{2n}$ is easily verified.

EXAMPLE 2 (Noncommutative tori). Let T^n be an ordinary n-torus, and let θ be a real skew symmetric $n \times n$ matrix. Let $A = C^\infty(T^n)$. If we view T as R/Z (where Z denotes the integers), then we can define a Poisson bracket on A by

$$\{f, g\} = \sum \theta_{jk}(\partial f/\partial x_j)(\partial g/\partial x_k).$$

(We could also do this on R^{2n}, in slight generalization of the previous example.) As Lie group G we will take T^n, acting on itself by translation. The Fourier transform (i.e., taking Fourier series), carries $C^\infty(T^n)$ onto $S(Z^n)$ with convolution, and takes the Poisson bracket to the operation

$$\{\phi, \psi\}(p) = -4\pi^2 \sum_q \phi(q)\psi(p-q)\gamma(q, p-q),$$

where γ is the skew bilinear form on Z^n defined by

$$\gamma(p, q) = \sum \theta_{jk} p_j q_k.$$

For any $\hbar \in R$ we define a skew bicharacter $\sigma_\hbar$ on Z^n by

$$\sigma_\hbar(p, q) = e(-\pi\hbar\gamma(p, q)),$$

and then define a product $*_\hbar$ on $S(Z^n)$ by

$$(\phi *_\hbar \psi)(p) = \sum_q \phi(q)\psi(p-q)\sigma_\hbar(q, p-q).$$

The involution we use on $C^\infty(T^n)$ will be complex conjugation for all $\hbar$, and so on $S(Z^n)$ it is given by $\phi^*(p) = \overline{\phi}(-p)$. For any $\hbar$ we define the norm $\| \|_\hbar$ to be the operator norm for $S(Z^n)$ acting on $L^2(Z^n)$ by the same formula as defines $*_\hbar$. By means of the Fourier transform we carry the products and norms back to $C^\infty(T^n)$. Again it is not difficult to verify property 2 of Definition 1 (see **[Rf5]** for details), while property 1 follows from corollary 2.7 of **[Rf4]**. Thus we have a strict deformation quantization. Invariance under $G = T^n$ is easily verified. The completed algebras $\overline{A}_\hbar$ are recognized as being exactly the noncommutative tori, as studied in **[Rf2]** and references therein. In particular, for many choices of θ they are simple C^*-algebras. Nevertheless, as seen in **[Cn1, Cn2, Cn3, Rf2]**, there are ample reasons for viewing them as noncommutative differentiable manifolds.

EXAMPLE 3. The above example admits the following generalization. Let T^n, θ, and $\{\ ,\ \}$ be as above, and let H be a Lie group with a cocompact subgroup Γ. Let β be a homomorphism of Γ into $\mathrm{SL}(n, Z)$, with corresponding action on T^n. Assume that this action preserves $\{\ ,\ \}$ (which is easily seen to mean that β carries Γ to matrices which are "symplectic" with respect to θ). Let ρ be the corresponding diagonal action of Γ on $H \times \Gamma^n$, and let $M = (H \times T^n)/\rho$, so that M is a torus bundle over G/Γ. If we let $\{\ ,\ \}$ denote also the Poisson structure on $H \times T^n$ coming from $\{\ ,\ \}$ on T^n, ignoring the coordinates of H, then ρ preserves $\{\ ,\ \}$, so that we obtain a Poisson structure on M, which we also denote by $\{\ ,\ \}$. Then we can seek a strict deformation quantization of M in the direction of $\{\ ,\ \}$. We can proceed as follows. By means of the previous example we construct a strict deformation quantization of $H \times T^n$ for which $A = C_c^\infty(H \times T^n)$. Thus we take Fourier transform just in the T^n variables, and call the corresponding space $S_c(H \times Z^n)$. With $\sigma_\hbar$ defined as in the previous example, we set

$$(\phi *_\hbar \psi)(u,p) = \sum_q \phi(u,q)\psi(u,p-q)\sigma_\hbar(q,p-q),$$

and similarly for the involution and norm. We then carry these back to $H \times T^n$ by the Fourier transform. It is easily seen that the resulting strict deformation quantization is invariant under the action of the discrete group Γ, in the evident sense.

But because the action of Γ on H is proper, it is not difficult to see that the action of Γ on the resulting algebras $A_\hbar$ using ρ is proper in the sense defined in **[Rf3]**. In particular, there will be a generalized fixed point algebra $B_\hbar$ for each $\hbar \in R$. After taking Fourier transform in the T^n variables, the elements of $B_\hbar$ will consist of smooth functions Φ on $H \times Z^n$ which satisfy (1) $\Phi(uk, \beta_k(p)) = \Phi(u,p)$ for each $u \in H, k \in \Gamma$ and $p \in Z^n$, and (2) for any polynomial P on Z^n and any finite product, $\tilde{X}$, of elements of the Lie algebra of H, the function $P(p)(\tilde{X}\Phi)(u,p)$ is uniformly bounded for u in any fixed compact subset of H and for all $p \in Z^n$ (all this independently of $\hbar$). The product and involution are given by the same formula as for $A_\hbar$, and the norm is given by the evident representation on $L^2(H \times Z^n)$ which extends the representation of $A_\hbar$. One obtains in this way the desired strict deformation quantization. The details of the proof are given in **[Rf5]**. Since, aside from keeping track of the smooth structure, the situation is contained in the setting studied in §2 of **[RW]**, the results of that paper can be used to obtain information about the structure of the resulting completed C^*-algebras.

We remark that a useful point of view is to consider the strict deformation quantization of $H \times T^n$ to be a noncommutative covering space for the strict deformation quantization of M, with Γ as the group of covering transformations, acting via ρ.

EXAMPLE 4 (Semidirect products by R^d). Let M be a smooth manifold, and let α be a smooth action of R^d on M. For $k = 1, \ldots, d$ let X_k denote the vector field on M consisting of differentiating via α in the kth direction of R^d, and let ∂_k denote differentiation on R^d by itself in the kth direction. Then we can define a Poisson structure on $M \times R^d$ by

$$\{f, g\} = \sum^d (X_k f)(\partial_k g) - (\partial_k f)(X_k g).$$

This is the semi-direct Poisson structure for α, as defined in the appendix of [**Wn2**], when M is initially given the zero Poisson structure. We let $A = S_c(M \times R^d)$, the algebra of smooth functions which are of compact support on M and Schwartz on R^d. Taking Fourier transform in the R^d variables carries A to $S_c(M \times \hat{R}^d)$, with the product being pointwise multiplication in the M variables and convolution in the $\hat{R}^d$ variables. The Poisson bracket is then carried to the operation on $S_c(M \times \hat{R}^d)$ defined by

$$\{\phi, \psi\}(m, r) =$$
$$2\pi i \sum_k \int ((X_k\phi)(m, s)(r_k - s_k)\psi(m, r - s) - s_k\phi(m, s)(X_k\psi)(m, r - s))\, ds.$$

Some playing with the formulas (see [**Rf5**]) leads one to realize that a strict deformation quantization can then be defined as follows. For any $\hbar \in R$, define a product $*_\hbar$ on $S_c(M \times \hat{R}^d)$ by

$$(\phi *_\hbar \psi)(m, r) = \int \phi(\alpha_{\pi\hbar(s-r)}(m), s)\psi(\alpha_{\pi\hbar s}(m), r - s)\, ds,$$

and an involution, independent of $\hbar$, by

$$\phi^*(m, r) = \overline{\phi}(m, -r).$$

We could also define the C^*-norms directly, but it is useful anyway to observe that the structure just defined is isomorphic to a crossed product algebra structure [**Pd**], and then we may as well use this observation to define the C^*-algebra norms. To be specific, for any $\hbar$ let $\gamma^\hbar$ be the action of R^n on M defined by $\gamma_r^\hbar = \alpha_{-2\pi\hbar r}$, and let $J_\hbar$ be the mapping from $S_c(M, \hat{R}^d)$ into the crossed product algebra $C^*(R^d, C_\infty(M), \gamma^\hbar)$ defined by

$$J_\hbar(\phi)(m, r) = \phi(\alpha_{\pi\hbar r}(m), r).$$

Then it is easily verified that $J_\hbar$ is a $*$-homomorphism (from $*_\hbar$) with dense range. We then set

$$\|\phi\|_\hbar = \|J_\hbar(\phi)\|$$

where the norm on the right is that of the crossed product. With these structures defined on $S_c(M, \hat{R}^d)$, and pulled back to $S_c(M, R^d)$ by the Fourier transform, one can verify, by a slight modification of arguments in **[Rf5]**, that one obtains a strict deformation quantization. Since the completed C^*-algebras $\overline{A}_\hbar$ will be isomorphic to the crossed product algebras, one can apply the extensive theory of the structure of crossed product algebras **[Pd]** to study the structure of the $A_\hbar$'s.

EXAMPLE 5. The ideas of the previous example can be used equally well on $M \times T^d$, with Poisson bracket defined by the same formula. One just has to replace $S_c(M, \hat{R}^d)$ by $S_c(M, Z^d)$. The details are contained in **[Rf5]**.

EXAMPLE 6 (Heisenberg manifolds). Let G be the Heisenberg Lie group of upper triangular 3×3 matrices with ones on the diagonal. For any positive integer n we can parametrize G as R^3 with product given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + nyx').$$

Let D denote the discrete subgroup of G consisting of elements with integer entries, and let $M_n = R^3/D$ for the parametrization using n. This is the corresponding Heisenberg manifold, on which G acts on the left. For $k = 1, 2, 3$ let ∂_k denote the partial derivative on $G = R^3$ in the kth direction. Then it is easily checked that the only skew 2-vector fields on G which are invariant under both left translation by G and right translation by D are of the form

$$(\mu\partial_1 + \nu\partial_2) \wedge \partial_3$$

for some $\mu, \nu \in R$, and that these do define Poisson brackets on G, and so define G-invariant Poisson brackets on M_n. It is easily seen that M_n can be identified with the quotient of $R \times T^2$ by the action ρ of Z defined on functions by

$$(\rho_k f)(x, y, z) = f(x + k, y, z + nky),$$

where T is viewed as R/Z. Fix μ and ν, not both zero, and let $\{\ ,\ \}$ denote the corresponding Poisson bracket on $R \times T \times R$. Then it is of the form considered in Example 4 for the action α of R on $R \times T$ defined by

$$\alpha_r(x, y) = (x - r\mu, y - r\nu).$$

Thus we can construct a deformation quantization for $(R \times T) \times T$ along the lines indicated in Example 5, where the appropriate algebra of functions, after Fourier transform in the last variable, is $S_c(R \times T \times Z)$. For each $\hbar \in R$ let $A_\hbar$ denote the corresponding pre-C^*-algebra. It is easily seen that ρ gives an action by automorphism of each $A_\hbar$. Furthermore, this action can be seen to be proper in the sense defined in **[Rf3]**, and so we can form the corresponding generalized fixed-point algebra, $D_\hbar$, as defined there. Then $D_\hbar$ will consist of the collection of C^∞ functions Φ on $R \times T \times Z$ which satisfy

(1) $\Phi(x + k, y, p) = e(ckpy)\Phi(x, y, p)$ for all $k \in Z$.
(2) For every polynomial P on Z and every partial differential operator $\tilde{X} = \partial^{m+n}/\partial x^m \partial y^n$ on $R \times T$ the function $P(p)(\tilde{X}\Phi)(x, y, p)$ is bounded on $K \times Z$ for any compact subset K of $R \times T$.

The product and involution on $D_\hbar$ are defined by

$$(\Phi *_\hbar \Psi)(x,y,p) = \sum \Phi(x - \pi\hbar(q-p)\mu, y - \pi\hbar(q-p)\nu, q)\Psi(x - \pi\hbar q\mu, y - \pi\hbar q\nu, p-q),$$

and

$$\Phi^*(x,y,p) = \overline{\Phi}(x,y,-p),$$

while the C^*-norm comes from the representation on $L^2(R \times T \times Z)$, for Lebesgue measure, defined by

$$(\Phi\xi)(x,y,p) = \sum \Phi(x - \pi\hbar(q-2p)\mu, y - \pi\hbar(q-2p)\nu, q)\xi(x,y,p-q).$$

The algebras $D_\hbar$ then provide a strict deformation quantization of $C^\infty(M_n)$ in the direction of $\{\ ,\ \}$. Verification of property 2 is fairly straightforward but tedious. The details are given in **[Rf5]**. On the other hand, verification of property 1 is somewhat subtle. The details are given in **[Rf3]** and **[Rf5]**. This strict deformation quantization is invariant under the action of G, and the action of G is ergodic on each $D_\hbar$ in the sense that the only invariant elements are the scalar multiples of the identity operator. One can show that when $\{1, \pi\hbar\mu, \pi\hbar\nu\}$ is independent over the rationals, $\overline{D}_\hbar$ is a simple C^*-algebra.

A generalization of this example is contained in **[Rf5]**.

EXAMPLE 7 (Nilpotent Lie algebras). Let L be any finite dimensional Lie algebra over R, and let L^* be its dual vector space. It is well known **[Wn1]** that the Lie algebra structure on L defines a natural Poisson structure on L^*. These are what are commonly called linear Poisson structures **[Wn1]**. The definition is as follows. Given $f \in C^\infty(L^*)$, its differential, $df(\mu)$, at $\mu \in L^*$ is a linear functional on the tangent space to L^* at μ, i.e., on L^*, and so $df(\mu)$ can be viewed as an element of L. Then the Poisson bracket is defined by

$$\{f,g\}(\mu) = \langle [df(\mu), dg(\mu)], \mu\rangle,$$

where $[\ ,\ ]$ denotes the Lie product in L. This extends in the evident way to complex-valued functions.

We seek a strict deformation quantization of L^* in the direction of $\{\ ,\ \}$. For L nilpotent this works out quite smoothly, and so we assume from now on that L is nilpotent. We take as the algebra A of functions on L^* just the Schwartz functions, $S(L^*)$. The Fourier transform then carries A to $S(L)$ with convolution. It is not difficult to verify that the Poisson bracket is carried by the Fourier transform to the operation

$$\{\phi, \psi\}(X) = 2\pi i \int_L \phi(Y)\langle [X,Y], (d\psi)(X-Y)\rangle\, dY.$$

For $\hbar \in R$ let $L_\hbar$ denote L with Lie bracket $\hbar[\ ,\ ]$. For nilpotent Lie algebras, the exponential map is a bijection with the corresponding simply connected Lie group G **[Br, Vr]**. Thus via the exponential map we can identify L with its simply connected Lie group. For emphasis we will denote the resulting group law on L by $*$. In the same way, for each $\hbar$ we identify $L_\hbar$

via the exponential map with its corresponding simply connected Lie group $G_\hbar$, and denote the resulting group law on $L_\hbar = L$ by $*_\hbar$. (The context will distinguish between this $*_\hbar$ and the one for functions.) It is not difficult to verify that

$$X *_\hbar Y = \hbar^{-1}((\hbar X) * (\hbar Y)),$$

where for $\hbar = 0$ we take this to mean $X + Y$. On L we take the Plancherel Lebesgue measure from the Fourier transform. It is well known that because L is nilpotent, this will be a Haar measure for $*$, and so for all the $*_\hbar$. For each $\hbar \in R$ we let $A_\hbar$ denote $S(L)$ equipped with the corresponding convolution

$$(\phi *_\hbar \psi)(X) = \int \phi(Y)\psi(Y^{-1} *_\hbar X)\,dY,$$

involution $\phi^*(X) = \overline{\phi}(-X)$, and norm from the group C^*-algebra $C^*(G_\hbar)$ **[Pd]**. Then one can verify that, pulling this structure back to $S(L^*)$ via the Fourier transform, one obtains a strict deformation quantization in the direction of $\{\ ,\ \}$ (once one puts in a factor of 2π), which is, in fact, invariant for the coadjoint representation of G on L^*. The details of the proof are given in **[Rf6]**.

EXAMPLE 8 (General Lie algebras). If L in the above example is not nilpotent, it is not clear to me whether it is possible to construct a strict deformation quantization, and especially one which is invariant for the coadjoint representation. But if we try to imitate the steps of the above example, we obtain a setup which can still be considered a deformation quantization, although not a strict one, in the following way. To begin with, when the Lie group for L is not unimodular, the Fourier transform of the Poisson bracket has an additional term, becoming

$$\{\phi, \psi\}(X) = 2\pi i \int \phi(Y)(\langle [X, Y], d\psi(X - Y)\rangle - \psi(X - Y)tr(ad_Y))\,dY.$$

The first real obstacle is that the exponential map need no longer be a bijection. Even when it is (i.e., when one is dealing with exponential solvable Lie groups **[Br]**), functions in $S(L)$ will in general not be integrable with respect to Haar measure. To deal with this latter obstacle, it is natural to restrict attention to $C_c^\infty(L)$ (which corresponds to only a subspace of $S(L^*)$, dense for most topologies, but not containing $C_c(L^*)$.) Doing this also deals with the first obstacle, in the following way. Choose an open neighborhood U of 0 in L on which the exponential map is a diffeomorphism into G, and identify U with its image in G. Let C be a convex open neighborhood of 0 such that $C^3 \subseteq U$ in G and $C = -C$ ($= C^{-1}$ in G). Let $U_\hbar = \hbar^{-1}U$ and $C_\hbar = \hbar^{-1}C$, with $U_0 = C_0 = L$. Note that $C_\hbar$ and $U_\hbar$ increase to L as $\hbar$ goes to 0. Then, much as in the previous example, we can define a partial product from $C_\hbar$ into $U_\hbar \subseteq L$ by

$$X *_\hbar Y = \hbar^{-1}((\hbar X) * (\hbar Y)).$$

The left Haar measures on the $G_\hbar$'s can be chosen in a coherent way so that their Radon-Nikodym derivatives, $\omega_\hbar$, on $U_\hbar$ with respect to the fixed Lebesgue measure of L, converge uniformly on compact subsets of L to the constant function 1, as $\hbar$ goes to 0. Then for any $\phi, \psi \in C_c^\infty(L)$, as soon as $\hbar$ is sufficiently small, the supports of both ϕ and ψ will be contained in $C_\hbar$, and so their convolution

$$(\phi *_\hbar \psi)(X) = \int \phi(Y)\psi(Y^{-1} *_\hbar X)\omega_\hbar(Y)\,dY$$

is defined. We equip this situation with the involutions and C^*-norms from the reduced group C^*-algebras $C_r^*(G_\hbar)$ **[Pd]**. Then it makes sense to ask whether property 2 of the definition of a strict deformation quantization is true, and this can, in fact, be shown to be the case (once a factor of 2π is included). With respect to property 1, one can show that at least one has lower semi-continuity, for those $\hbar$ for which $\|\phi\|_\hbar$ is defined. This whole setup can be formalized in what we call a deformation quantization by *partial embeddings*, where we have in mind the embeddings of $C_c^\infty(C_\hbar)$ into $C^*(G_\hbar)$. See **[Rf6]** for details.

All the above suggests that in some sense the group C^*-algebras of Lie groups are deformations of $S(L^*)$, which provides further support to the well-known idea that they should be considered noncommutative differentiable manifolds.

I have some preliminary indications that the above ideas may extend to give a relation between Poisson Lie groups **[Dr1, LW]** and the quantum groups which have been introduced by Woronowicz **[Wr1, Wr2, Wr3, MM1, MM2, Mg]** and which at the infinitesimal level have been studied by Drinfeld **[Dr2]** and others as deformation quantizations of universal enveloping algebras of Lie algebras. (For their interrelations see **[Rs]**.)

EXAMPLE 9 (The sphere). Let S^2 denote the ordinary 2-sphere, with its usual action of $SO(3)$ (and so $SU(2)$). It is well known that there is a Poisson bracket on S^2 invariant under the action of $SO(3)$. However, one can show that there is no strict deformation quantization of $C^\infty(S^2)$ in the direction of this Poisson bracket which is invariant under the action of $SO(3)$. This actually has nothing to do with the Poisson bracket, but rather with the fact that each irreducible representation of $SO(3)$ occurs in $C^\infty(S^2)$ with multiplicity one, so that the resulting rigidity forces any invariant deformation to be commutative. (See **[Rf5]** for details.) That this might happen was suggested by the much more general results of Wassermann **[Ws]** showing that $SU(2)$ does not have any ergodic actions on von Neumann algebras not of type I. The question of whether S^2 admits strict deformation quantizations which are not invariant seems to be open, but it is not clear to me the precise relationship with the quantum spheres of Podles **[Po]**, coming from the quantum groups of Woronowicz. Alan Weinstein has speculated to me that there may be some connection between the quantum spheres of Podles and the Bruhat-Poisson structure discussed in [LW].

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