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CONNES' ANALOGUE FOR CROSSED PRODUCTS OF THE THOM ISOMORPHISM

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The purpose of this note is to give a variant of the proof of Connes' analogue for crossed products of the Thom isomorphim [1]. This variant has the advantage of taking place in a somewhat more traditional setting, in that the isomorphisms arise as the usual connecting homomorphisms for the K-theory exact sequence of a certain short exact sequence. This short exact sequence is itself just an analogue of the Toeplitz extension of Pimsner and Voiculescu [4] in which the group of integers is replaced by the group of real numbers, and which it thus seems appropriate to call the Wiener-Hopf extension. On the other hand, we have nothing new to contribute to the proof of the key lemma for Connes' theorem, which associates to projections certain cocycles, and so we will simply quote this result where needed.

We recall that if $(A, \mathbb{R}, \alpha)$ is a C^*-dynamical system [3] with the real line acting, and with crossed product algebra $A\times_\alpha \mathbb{R}$, then Connes' theorem asserts that there are natural isomorphisms

$$K_0(A\times_\alpha \mathbb{R}) \cong K_1(A), \qquad K_1(A\times_\alpha \mathbb{R}) \cong K_0(A) .$$

We begin by describing the Wiener-Hopf extension for $A\times_\alpha \mathbb{R}$. Let $\mathbb{R}\cup\{+\infty\}$ denote $\mathbb{R}$ with a point adjoined at $+\infty$ (but not at $-\infty$). Let C (for "cone") denote the C^*-algebra $C_\infty(\mathbb{R}\cup\{+\infty\})$ of continuous functions on $\mathbb{R}\cup\{+\infty\}$ which vanish at $-\infty$, and let S (for "suspension") denote the

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ideal of C consisting of functions which also vanish at $+\infty$. Then we have the evident short exact sequence

$$0 \to S \to C \to \mathbb{C} \to 0,$$

where $\mathbb{C}$ denotes the complex numbers. Let τ denote the action of $\mathbb{R}$ on C which comes from translation on $\mathbb{R} \cup \{+\infty\}$, leaving the point $+\infty$ fixed. Note that S is carried into itself by τ, so that if we consider $\mathbb{R}$ to have the trivial action on $\mathbb{C}$ then the above exact sequence is equivariant for the actions on the three algebras.

Now let CA denote the tensor product of C and A, realized as the C^*-algebra of A-valued functions on $\mathbb{R} \cup \{+\infty\}$ which vanish at $-\infty$, so that CA is one of the usual descriptions of the cone over A. We define SA similarly, so that it is one of the usual descriptions of the suspension of A. Then we have the evident short exact sequence

$$0 \to SA \to CA \to A \to 0 .$$

If we still view CA as a tensor product, it is clear that we have the inner tensor product action $\tau \otimes \alpha$ of $\mathbb{R}$. Specifically, if f is an A-valued function on $\mathbb{R} \cup \{+\infty\}$, then

$$((\tau \otimes \alpha)_r f)(t) = \alpha_r(f(t - r))$$

for $r, t \in \mathbb{R}$. It is clear that SA is a $\tau \otimes \alpha$-invariant ideal of CA and that the quotient action on A is just α. It is easily seen (lemma 1 of [1]) that the corresponding sequence of crossed product algebras is exact. That is, we obtain the short exact sequence

$$0 \to SA \times_{\tau \otimes \alpha} \mathbb{R} \to CA \times_{\tau \otimes \alpha} \mathbb{R} \to A \times_\alpha \mathbb{R} \to 0.$$

We call this the Wiener-Hopf extension for $A \times_\alpha \mathbb{R}$, since the corresponding sequence in which $\mathbb{R}$ is replaced by the integers is essentially the Toeplitz extension of Pimsner and Voiculescu [4]. Strictly speaking, we should compress these sequences by the characteristic function of $[0, +\infty]$, but we find it more convenient not to do this.

If we apply the Bott periodicity theorem to the Wiener-Hopf extension we obtain

$$\begin{array}{ccccc} K_1(SA\times_{\tau\otimes\alpha}\mathbb{R}) & \to & K_1(CA\times_{\tau\otimes\alpha}\mathbb{R}) & \to & K_1(A\times_\alpha\mathbb{R}) \\ \text{exp} \uparrow & & & & \downarrow \text{index} \\ K_0(A\times_\alpha\mathbb{R}) & \leftarrow & K_0(CA\times_{\tau\otimes\alpha}\mathbb{R}) & \leftarrow & K_0(SA\times_{\tau\otimes\alpha}\mathbb{R}) . \end{array}$$

Now it is known that

$$SA\times_{\tau\otimes\alpha}\mathbb{R} \cong SA\times_{\tau\otimes\iota}\mathbb{R} \cong A\otimes(S\times_\tau\mathbb{R}) \cong A\otimes K(L^2(\mathbb{R}))$$

where ι denotes the trivial action of $\mathbb{R}$ on A, and where $K(L^2(\mathbb{R}))$ denotes the algebra of compact operators on $L^2(\mathbb{R})$. (The first isomorphism is slightly buried in 7.7.12 and 7.9.2 of [3], but we describe it below.) It follows that $K_*(SA_{\tau\otimes\alpha}\mathbb{R}) \cong K_*(A)$. Suppose we already have Connes' theorem available. Then

$$K_*(CA\times_{\tau\otimes\alpha}\mathbb{R}) \cong K_*(CA) = 0$$

since CA is contractible. It follows that, whatever the isomorphisms in Connes' theorem may be, the index and exponential maps for the Wiener-Hopf extension will themselves give isomorphisms. This encourages us to try to prove Connes' theorem by showing directly that $K_*(CA\times_{\tau\otimes\alpha}\mathbb{R}) = 0$. The path we take in doing this is indicated by:

LEMMA 1. Suppose that it is known that for all A and α the index map for the Wiener-Hopf extension is surjective. Then it follows that for all A and α we have $K_*(CA\times_{\tau\otimes\alpha}\mathbb{R}) = 0$.

Proof. If the index map is always surjective, then whenever $K_1(A\times_\alpha\mathbb{R})=0$ it follows that $K_0(A) = 0$. Let $\hat{\alpha}$ denote the dual action of $\mathbb{R}$ on $A\times_\alpha\mathbb{R}$

so that by Takai duality (theorem 7.9.3 of [3]) $(A\times_\alpha \mathbb{R})\times_{\hat{\alpha}}\mathbb{R} \cong A\times K(L^2(\mathbb{R}))$. Then it follows that if $K_1(A) = 0$, so that $K_1((A\times_\alpha\mathbb{R})\times_{\hat{\alpha}}\mathbb{R}) = 0$, then $K_0(A\times_\alpha\mathbb{R}) = 0$. If we apply this with (A, α) replaced by $(CA, \tau\otimes\alpha)$, we find that $K_0(CA\times_{\tau\otimes\alpha}\mathbb{R}) = 0$. Since taking suspensions (with trivial action) commutes with taking crossed products, we can apply this with A replaced by SA to conclude by Bott periodicity that $K_1(CA\times_{\tau\otimes\alpha}\mathbb{R}) = 0$ also. Q.E.D.

Thus the crux of our argument is to show that the index map is always surjective. We show this first for the case in which A has an identity element, since we will show later by traditional arguments that the non-unital case follows from this. So assume now that A has an identity element. We will display for any projection p in any matrix algebra $M_n(A)$, an isometry V in $M_n(CA\times_{\tau\otimes\alpha}\mathbb{R})^{\sim}$ whose image in $M_n(A\times_\alpha\mathbb{R})^{\sim}$ is a unitary whose index represents the same element of $K_0(A)$ as does (the negative of) p. To do this we must examine carefully the isomorphisms in

$$K_0(A) \cong K_0(A\otimes K(L^2(\mathbb{R}))) \cong K_0(SA\times_{\tau\otimes\alpha}\mathbb{R}).$$

Now the first isomorphism comes from picking any projection, E, of rank one in $K(L^2(\mathbb{R}))$ and then using the homomorphism $a \mapsto a\otimes E$ from $M_n(A)$ into $M_n(A)\otimes K \equiv M_n(A\otimes K)$. We choose E in a very specific way, namely, we let E be the rank one projection on the unit vector f in $L^2(\mathbb{R})$ defined by $f(t) = e^{-t/2}\chi(t)$, where here and in the sequel χ denotes the characteristic function of the interval $[0, \infty)$. We recall that f is the first Laguerre function in $L^2(0, \infty)$. We will always let kernel functions $F(r, t)$ act on elements g of $L^2(\mathbb{R})$ by the formula

$$(Fg)(t) = \int_{\mathbb{R}} F(r, t)g(t - r)dr ,$$

so that composition of operators corresponds to the usual product in $S\times_\tau\mathbb{R}$. With this convention it is easily seen that the kernel function $E(r, t)$ for E is given by

$$E(r, t) = e^{r/2}e^{-t}\chi(t)\chi(t - r) .$$

Now the isomorphism of $SA\times_{\tau\otimes\alpha}\mathbb{R}$ with $SA\times_{\tau\otimes\iota}\mathbb{R}$ is given by sending $G\in L^1(\mathbb{R}, SA)$ to $\hat{G}$ where $\hat{G}$ is defined by

$$\hat{G}(r, t) = \alpha_{-t}(G(r, t)).$$

This is seen by a one line computation once it is recalled that the product in $SA\times_{\tau\otimes\alpha}\mathbb{R}$ is defined by

$$(G*H)(r, t) = \int G(s, t)\alpha_s(H(r - s, t - s))ds$$

and similarly in $SA\times_{\tau\otimes\iota}\mathbb{R}$. Thus $p\otimes E$ as an element of $M_n(SA\times_{\tau\otimes\alpha}\mathbb{R})$ is given by the kernel P defined by

$$P(r, t) = \alpha_t(p)E(r, t) .$$

We wish to construct an isometry in $M_n(CA\times_{\tau\otimes\alpha}\mathbb{R})^{\sim}$ whose image in $M_n(A\times_{\alpha}\mathbb{R})^{\sim}$ is a unitary whose index is essentially (the negative of) the above P, where $\sim$ denotes adjunction of an identity element.

To define the isometry we need Connes' cocycle for p. We have no improvement to offer for his proof [1] of the existence of this cocycle, but we have one remark to make. On the face of it Connes only shows that C^∞ projections have a cocycle. This suffices since any projection can be approximated by C^∞ projections. But if q is a C^∞ projection very close to p, then there is a unitary which conjugates q to p. A simple calculation then shows that this unitary can be used to adjust the cocycle for q to give a cocycle for p. Thus every projection, C^∞ or not, has a cocycle, and for this reason we do not need to assume in the sequel that p is a C^∞ projection.

To fix our notation, we let $t\mapsto u_t$ be the cocycle for p, so that it is a norm-continuous family of unitaries in $M_n(A)$ such that

(1) $\alpha_t(p) = u_t{}^*pu_t$ for all $t \in \mathbb{R}$,

(2) $u_{s+t} = u_s\alpha_s(u_t)$ for all $s, t \in \mathbb{R}$.

We remark that since $t \mapsto \alpha_t(p)$ is a continuous family of projections, elementary arguments show that a continuous family of unitaries can be found satisfying (1). Thus it is the fact that this family can be adjusted so as to satisfy the cocycle identity (2) which is the deep part of Connes' lemma.

For ease of notation we will from now on denote $M_n(A)$ by B, but we will still denote the action of $\mathbb{R}$ on B by α. We note that $M_n(CA) \cong CB$ naturally, and even more, that

$$M_n(CA \times_{\tau\otimes\alpha} \mathbb{R}) \cong CB \times_{\tau\otimes\alpha} \mathbb{R}$$

naturally, while analogous identifications hold for the suspension functor S.

Our choice of the isometry V (and of the projection E earlier) is motivated by the proof of lemma 6 of [2], in which an isometry in $(C\times_\tau \mathbb{R})^\sim$ is explicitly constructed whose corange projection is exactly the projection on the first Laguerre function. The relevance of this to the present situation comes from:

LEMMA 2. For any $F \in L^1(\mathbb{R}, C, \tau)$ define $F_p \in L^1(\mathbb{R}, CB, \tau\otimes\alpha)$ by

$$F_p(r, t) = pu_rF(r, t).$$

Then the mapping $F \mapsto F_p$ is an (isometric) *-homomorphism, and so extends to a *-homomorphism of $C\times_\tau \mathbb{R}$ into $CB\times_{\tau\otimes\alpha}\mathbb{R}$, which carries $S\times_\tau \mathbb{R}$ into $SB\times_{\tau\otimes\alpha}\mathbb{R}$.

Proof. For $F, G \in L^1(\mathbb{R}, C, \tau)$ we have

$$\begin{aligned}(F_p * G_p)(r, t) &= \int F_p(s, t)\alpha_s(G_p(r - s, t - s))ds \\ &= \int pu_sF(s, t)\alpha_s(pu_{r-s}G(r - s, t - s))ds \\ &= \int pu_su_s{}^*pu_s\alpha_s(u_{r-s})F(s, t)G(r - s, t - s)ds.\end{aligned}$$

But $\alpha_s(u_{r-s}) = u_s{}^*u_r$ by the cocycle identity, so that

$$(F_p{}^*G_p)(r, t) = pu_r(F^*G)(r, t)$$

as desired. Also

$$(F_p)^*(r, t) = \alpha_r(F_p(-r, t - r)^*)$$

$$= (\alpha_r(pu_{-r}))^*F^*(r, t) = (u_r{}^*pu_r\alpha_r(u_{-r}))^*F^*(r, t).$$

But $u_r\alpha_r(u_{-r}) = 1$ by the cocycle identity, so that

$$(F_p)^*(r, t) = pu_rF^*(r, t)$$

as desired. That the correspondence carries $S\times_\tau \mathbb{R}$ into $SB\times_{\tau\otimes\alpha}\mathbb{R}$ is clear.

Q.E.D.

We now define F specifically by

$$F(r, t) = e^{-r/2}\chi(r)\chi(t - r).$$

This is the $V_L(k)M_L(f)$ of lemma 6 of [2]. As explained there, if I denotes the identity operator adjoined to $C\times_\tau\mathbb{R}$ (or later to $CA\times_{\tau\otimes\alpha}\mathbb{R}$), then $I - F$ acts under the natural representation on $L^2(\mathbb{R})$ as the identity operator on $L^2(-\infty, 0)$ and as the unilateral shift on $L^2(0, \infty)$ with respect to the Laguerre functions. In particular one calculates easily that

$$(\#) \qquad F + F^* - F^**F = 0, \qquad F + F^* - F*F^* = E.$$

Lemma 6 of [2] also contains the simple estimates which show that $F \in C\times_\tau\mathbb{R}$ even though F is not continuous in the t variable. These amount to the following. Let $g \in L^1(\mathbb{R})$ and for any $\varepsilon > 0$ let

$$m_\varepsilon(s) = \sup\{\|g(s - r)\|: 0 \leq r \leq \varepsilon\}.$$

Let T be the operator with kernel

$$T(r, t) = g(r)\chi(t - r).$$

For any $\varepsilon > 0$ let χ_ε be the continuous function which agrees with χ off the interval $[0, \varepsilon]$ and is linear on this interval, and let T_ε be the operator defined by replacing χ by χ_ε in the expression for T. It is clear that $T_\varepsilon \in L^1(\mathbb{R}, C, \tau)$. Then one calculates that

$$\|T - T_\varepsilon\|^2 \leq \varepsilon\int m_\varepsilon(s)^2 ds.$$

Now it is clear that m_ε does not increase as ε decreases. Thus if $\int m_\varepsilon(s)^2 ds$ is finite for some ε, then T_ε approaches T as ε approaches zero. In particular, it is clear that all of this holds for $g(r) = f(r) = e^{-r/2}\chi(r)$, so that $F \in C\times_\tau \mathbb{R}$.

With F defined specifically as above, we now let $V = I - F_p$. It follows from the above discussion that F_p is in $CB\times_{\tau\otimes\alpha}\mathbb{R}$ even though it is not continuous in the t variable. The equations (#) above together with Lemma 2 make it clear that V is an isometry in $(CB\times_{\tau\otimes\alpha}\mathbb{R})^\sim$ whose corange projection is E_p. Note that E_p is in $SB\times_{\tau\otimes\alpha}\mathbb{R}$ since $E \in S\times_\tau\mathbb{R}$.

We must show that E_p defines the same element of $K_0(A)$ as does P. We do this by showing that these two projections are, in fact, homotopic in $SB\times_{\tau\otimes\alpha}\mathbb{R}$. This is most easily done by passing to $SB\times_{\tau\otimes\iota}\mathbb{R}$ under the isomorphism ^ defined earlier. Under this isomorphism E_p becomes

$$\hat{E}_p(r, t) = \alpha_{-t}(pu_r)E(r, t) = u_{-t}{}^*pu_{-t}\alpha_{-t}(u_r)E(r, t).$$

But $\alpha_{-t}(u_r) = u_{-t}{}^*u_{r-t}$ by the cocycle identity, so that

$$\hat{E}_p(r, t) = u_{-t}{}^*pu_{r-t}E(r, t) = u_{-t}{}^*\hat{P}(r, t)u_{r-t}.$$

Now for any $\lambda \in [0, 1]$ let U_λ denote the unitary double centralizer on SB consisting of pointwise multiplication by the function $t \mapsto u_{-\lambda t}$. Notice next that any double centralizer of SB can be viewed as a double centralizer of $SB\times_{\tau\otimes\iota}\mathbb{R}$ (by considering it to be the "delta function" with that value at the point 0 of $\mathbb{R}$). If we so view U_λ, then a simple calculation shows that

$$\hat{E}_p = U_1^*\hat{P}U_1 .$$

Thus $U_\lambda^*\hat{P}U_\lambda$ for $\lambda \in [0, 1]$ will provide the desired homotopy of $\hat{E}_p$ with $\hat{P}$ once we show that this family of projections is continuous (since $U_0 = I$). Now by the uniform continuity of $t \mapsto u_t$ on compact subsets of. $\mathbb{R}$ it is easily seen that for any $f \in SB$ the maps $\lambda \mapsto U_\lambda f$ and $\lambda \mapsto fU_\lambda$ are norm continuous. The desired continuity then follows from the following lemma by an $\varepsilon/2$ arguement.

LEMMA 3. Let (D, G, α) be a C*-dynamical system, and let $\lambda \to w_\lambda$ be a map of $[0, 1]$ into unitary double centralizers of D which is strictly continuous in the sense that for every $d \in D$ the maps $\lambda \mapsto w_\lambda d$ and $\lambda \mapsto dw_\lambda$ are norm continuous. View the w_λ as double centralizers of $D\times_\alpha G$. Then for every $F \in D\times_\alpha G$ the maps $\lambda \mapsto w_\lambda F$ and $\lambda \mapsto Fw_\lambda$ are norm continuous.

Proof. Since the w_λ are uniformly bounded, it suffices to verify continuity on a dense subset of $D\times_\alpha G$. For the left-hand continuity, take as such a dense subset the linear span of elements of form $t \mapsto d\phi(t)$ where $d \in D$ and $\phi \in L^1(G)$. Then for any such element

$$\|(w_\lambda - w_\mu)(d\phi)\| \leq \|(w_\lambda - w_\mu)d\|\|\phi\|_{L^1(G)},$$

so that the continuity in this case is obvious. The right-hand continuity follows by taking adjoints. Q.E.D.

This concludes the proof of the fact that the index map for the Wiener-Hopf extension is surjective in the case that A has an identity element. We now use standard arguments to deduce from this the corresponding fact for the case in which A does not have an identity element. To do this we need the following fact which is surely well-known, and which follows essentially immediately from the definition of the index map:

PROPOSITION 1. Let C and D be arbitrary C^*-algebras, let I and J be ideals in C and D respectively, and let ϕ be a homomorphism from C to D which carries I into J, so that it induces a homomorphism from C/I to D/J. That is, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & I & \to & C & \to & C/I & \to & 0 \\ & & \downarrow\phi & & \downarrow\phi & & \downarrow\phi & & \\ 0 & \to & J & \to & D & \to & D/J & \to & 0 \end{array}$$

with exact rows. Then the diagram

$$\begin{array}{ccc} K_1(C/I) & \xrightarrow{\text{index}} & K_0(I) \\ \downarrow K_1(\phi) & & \downarrow K_0(\phi) \\ K_1(D/J) & \xrightarrow{\text{index}} & K_0(J) \end{array}$$

is commutative.

Suppose now that the algebra A of our C^*-dynamical system $(A, \mathbb{R}, \alpha)$, does not have an identity element, and let $\tilde{A}$ denote the algebra with identity element adjoined. We let α denote also the action of $\mathbb{R}$ extended to $\tilde{A}$. It is then easily seen (e.g. lemmas 1 and 2 of [1]) that we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & SA\times_{\tau\otimes\alpha}\mathbb{R} & \to & CA\times_{\tau\otimes\alpha}\mathbb{R} & \to & A\times_{\alpha}\mathbb{R} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S\tilde{A}\times_{\tau\otimes\alpha}\mathbb{R} & \to & C\tilde{A}\times_{\tau\otimes\alpha}\mathbb{R} & \to & \tilde{A}\times_{\alpha}\mathbb{R} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S\mathbb{C}\times_{\tau}\mathbb{R} & \to & C\mathbb{C}\times_{\tau}\mathbb{R} & \to & C^*(\mathbb{R}) & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Applying the above proposition twice, and also the K-theory exact sequence, to this diagram, we obtain the following commutative diagram with exact columns:

$$\begin{array}{ccc}
 & & K_1(SC\times_\tau \mathbb{R}) \\
 & & \downarrow \\
K_1(A\times_\alpha \mathbb{R}) & \xrightarrow{\text{index}} & K_0(SA\times_{\tau\otimes\alpha}\mathbb{R}) \\
\downarrow & & \downarrow \\
K_1(\tilde{A}\times_\alpha \mathbb{R}) & \xrightarrow{\text{index}} & K_0(S\tilde{A}\times_{\tau\otimes\alpha}\mathbb{R}) \\
\downarrow & & \downarrow \\
K_1(C^*\mathbb{R}) & \xrightarrow{\text{index}} & K_0(S\mathbb{C}\times_\tau \mathbb{R}) \quad .
\end{array}$$

Now the second and third rows are index maps for Wiener-Hopf extensions involving algebras with identity elements, and we have shown above that these must be surjective. Furthermore, the two groups of the third row are both well known to be isomorphic to the group of integers (since $S\mathbb{C}\times_\tau \mathbb{R}$ is isomorphic to the compact operators), and so the surjection of the third row must in fact be an isomorphism. Also $K_1(S\mathbb{C}\times_\tau \mathbb{R}) = 0$, so the middle map of the right-hand column is injective. Simple diagram chasing then shows that the top index map must also be surjective, as desired. This concludes our proof of Connes' theorem.

We make two concluding remarks. The first is that it is, of course, nice to give an axiomatic characterization of the isomorphisms, as Connes does. In the present setting the proposition given above yields almost immediately the naturality of the index isomorphism, which is part of Connes' characterization. The second remark is that from the proof given above that the index is surjective, one obtains, in fact, an explicit form for the inverse isomorphism to the index. Specifically, if A has an identity element and if p is a projection in some $M_n(A)$, then the corresponding element of $K_1(A\times_\alpha \mathbb{R})$ must be represented by the image in $(M_n(A)\times_\alpha \mathbb{R})^\sim$ of the isometry V defined earlier. But this image is easily computed just by evaluating at $t = +\infty$ the kernel E_p. One finds in this way that the corresponding unitary in

$M_n(A\times_\alpha \mathbb{R})^\sim$ is

$$I - pu_r e^{-r/2}\chi(r).$$

This is related to the b of proposition 6 of [1]. Notice that the Fourier transform of

$$I - e^{-r/2}\chi(r)$$

is, up to a reparametrization, $(t - i)/(t + i)$.

REFERENCES

1. A. Connes, An analogue of the Thom isomorphism for crossed products of a C*-algebra by an action of IR, Adv. in Math. 39 (1981), 31-55.

2. P. Green, C*-algebras of transformation groups with smooth orbit space, Pacific J. Math. 72 (1977), 71-97.

3. G. K. Pedersen, C*-algebras and their Automorphism Groups, Academic Press, London, 1979.

4. M. Pimsner and D. Voiculescu, Exact sequences for K-groups and Ext-groups of certain crossed-product C*-algebras, J. Operator Theory 4 (1980), 93-118.

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