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Compact Quantum Groups associated with Toral Subgroups

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ABSTRACT. We construct a class of compact quantum groups, by strict deformation quantization of compact Lie groups within the C*-algebra setting. These deformations come from Poisson brackets associated with actions of toral subgroups of the compact Lie groups. A notable feature of our construction is that we are able to deform the entire algebra of smooth functions on the compact Lie group, not just the algebra of regular functions. We show that the C*-algebra of our quantum groups often has a quantum torus as quotient algebra, and so is often not of type I.

Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$, and let H be a maximal torus in G with Lie algebra $\mathfrak{h}$. Results of Belavin and Drinfeld (see [**So1, So2, LS**]) imply that, up to inner automorphisms, the compatible Poisson brackets [**Dr2**] on G are determined by elements of $\mathfrak{g} \wedge \mathfrak{g}$ of the form $ar + u$, where a is a real number, r is a special element of $\mathfrak{g} \wedge \mathfrak{g}$ associated with $\mathfrak{h}$, and u is an arbitrary element of $\mathfrak{h} \wedge \mathfrak{h}$. The compatible Poisson brackets are, intuitively, the directions in which one can try to deform the algebra of functions on G to obtain one-parameter families of compact quantum groups. Most of the literature on quantum groups has concentrated on the element ar (so setting $u = 0$), but Levendorskii and Soibelman [**LS**] have given a beautiful treatment of the general case, in the framework of C*-algebras.

In the present paper we will concentrate on the element u (so setting $ar = 0$). In a recent monograph [**Rf3**] I gave a general construction for deformation quantization for the case of Poisson brackets coming from actions of $\mathbf{R}^d$. We will see here that this construction applies in a natural way to provide a construction of compact quantum groups corresponding to the element u. In fact, we will have no need to assume that G is simple — we can let G be any compact

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(connected) Lie group — and we will not need the detailed structure theory of simple Lie algebras and groups which is used in [**LS**]. The quantum groups which are constructed in [**LS**] as quantizations of the Poisson-Lie groups $K(0, u)$ discussed in section 1.5 of [**LS**] will be a special case of our construction, but we will also obtain quantum groups which do not seem to have been discussed in the literature before — see section 6.

Actually, our construction also works for non-compact Lie groups under suitable hypotheses. This will be discussed in a paper under preparation [**Rf5**]. In the present paper we will essentially specialize the results of [**Rf5**] to the compact case, taking advantage of the substantial simplifications which compactness permits. Accordingly, we will not repeat here all of the background discussion included in [**Rf5**], and our treatment will be somewhat more condensed than in [**Rf5**].

The construction given here (and in [**Rf5**]) can be viewed as an analytical version of the dual to a special case of the twisting construction first introduced by Drinfeld ([**D3, D4**] with antecedents in [**D1**]). The twisting construction in a form closest to what we do here appears in equation 2.5.2 of [**LS**]. See also the discussion in section 3 of [**T1**] and lecture 3 of [**T2**], [**R**], section 14 of [**GS1**] and section 4 of [**GS2**], [**AE**], and [**GM**]. Our construction is also related to the ideas in [**X**], notably example 2.1. Heuristically, the quantum groups which we construct here are dual triangular Hopf algebras [**D2**], and it is entertaining to make aspects of this precise, but only hints of this will appear here. As would be expected of a dual to the twisting construction, the deformations which our construction provides here are "preferred deformations" as defined in [**G, GS1, GS2**], that is, the comultiplication of the original Hopf algebra (i.e of the algebra of functions on the original Lie group) is strictly unchanged, and it is only the multiplication which is deformed from a commutative one to a non-commutative one. (The deformations in [**Rf2**] are also preferred ones, but arise from a quite different construction.)

Although the quantum groups which we construct here are relatively tame ones, we emphasize that the present paper seems to be the first place in which the pointwise product on the algebra of *all* smooth functions on a compact Lie group is deformed into a non-commutative product. In previous treatments of compact quantum groups only the product on the algebra of "regular functions" (the linear span of coordinate functions of finite-dimensional representations) is deformed; also, the relation with functions is not usually kept track of (though [**Sh**] is a notable exception). One advantage of our being able to deform the algebra of all smooth functions is that we obtain a deformed smooth algebra which is closer to the spirit of Connes' non-commutative differential geometry. We also remark that by the main theorem of [**Rf4**] the K-groups of the deformed C*-algebras will be the same as those of the original compact Lie group. It is an interesting challange to discover how to deform the entire algebra of smooth functions on a simple Lie group in the cases when $ar \neq 0$.

As would be suggested by the papers cited earlier, it seems highly likely that our analytical twisting construction can be used to deform a quantum group corresponding to an element ar such as in [LS], into one for an element $ar + u$. But investigation of this must await a later time.

With a bit more effort the results of this paper could be carried out for compact groups which are not Lie groups (e.g. infinite products of compact Lie groups). But for clarity of exposition I have chosen to treat here only the case of Lie groups.

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1. The deformed algebra

Let n be the dimension of the toral subgroup H of the Lie group G. We nowhere need that H be maximal, so we assume only that H is some toral subgroup of G (that is, a connected closed Abelian subgroup of G, so isomorphic with $\mathbf{T}^n$ where $\mathbf{T}$ is the circle group). The Lie algebra of H then looks like $\mathbf{R}^n$, and the exponential map is a homomorphism of this vector group onto H. We don't explicitly need that this homomorphism comes from the exponential map, so we will assume only that we have a homomorphism, η, from $\mathbf{R}^n$ onto H. For convenience we will assume that its kernel is the integer lattice $\mathbf{Z}^n$ of $\mathbf{R}^n$; and when convenient we will simply identify H with the quotient of $\mathbf{R}^n$ by $\mathbf{Z}^n$.

Let $V = \mathbf{R}^n \times \mathbf{R}^n = \mathbf{R}^d$, where $d = 2n$. Let $A = C(G)$, the C*-algebra of continuous complex-valued functions on G with the supremum norm, $\| \ \|_\infty$. We define an action, α, of V on A by

$$(\alpha_{(s,u)}f)(x) = f(\eta(-s)x\eta(u))$$

for $s, u \in \mathbf{R}^n$. (The minus sign in $-s$ is not significant here — we include it only in anticipation of perhaps someday being able to treat cases in which H is not Abelian.) Let J be a skew-symmetric operator on V (for the standard inner product on $V = \mathbf{R}^d$). We are then exactly in position to apply the main construction of [**Rf3**] to produce the deformed C*-algebra A_J.

In general there is no reason to expect that A_J will relate well to the usual comultiplication on A, since, in particular, the Poisson bracket on G coming from α and J will in general not be compatible with the group structure on G (in the sense [**Dr2**] that the multiplication $G \times G \to G$ be a Poisson map for the corresponding product Poisson bracket on $G \times G$). But it is easily checked (see [**Rf5**]) that this Poisson bracket *will* be compatible if J is of the special form $J = K \oplus (-K)$ where K is a skew-symmetric operator on $\mathbf{R}^n$. (Here the minus sign in $-K$ is crucial. The operator K is the incarnation in the present context of the element u discussed in the introduction.) Accordingly, we will hereafter always assume that J is of this special form. (This would suggest that we denote our deformed algebra by A_K, but we will continue to write A_J instead so as to remain in conformity with [**Rf3**].)

When we apply the general definition of the deformed product from [**Rf3**] for a J of our special form, we find that the deformed product is given by

$$f \times_J g = \int (\alpha_{(Ks,-Ku)} f)(\alpha_{(t,v)} g) e(s \cdot t + u \cdot v),$$

where the variables of integration range over $\mathbf{R}^n$, and where as in [**Rf3**] we omit ds etc., and $e(t) = exp(2\pi i t)$. For the moment we will be vague about what kind of functions are involved. But we note that the action α factors through the evident action of the torus $D = H \times H$, and so is a compact action in the sense discussed towards the end of chapter 2 of [**Rf3**]. As discussed there, the formula for the deformed product can then be put in a different form. Let $\widehat{H}$ denote the dual group of H, so that $\widehat{H} \cong \mathbf{Z}^n$. We identify $\widehat{H}$ with the integer lattice in $\mathbf{R}^n$, so that for $p \in \widehat{H}$ the corresponding character on $H = \mathbf{R}^n/\mathbf{Z}^n$ is $v \mapsto e(p \cdot v)$. When we then apply proposition 2.21 of [**Rf3**], we find that the deformed product is given by

$$f \times_J g = \sum_{\widehat{H} \times \widehat{H}} \int_{H \times H} (\alpha_{(Kp,-Kq)} f)(\alpha_{(t,v)} g) e(p \cdot t + q \cdot v).$$

However, we will not have much use here for this way of expressing the deformed product. We remark that proposition 2.22 of [**Rf3**] can be applied to give a simple expression for the product of any two elements in spectral subspaces of $C(G)$ for the action of $H \times H$ on both sides of G, much as in [**L**] and 2.13 of [**SL**].

If we write out the expression for the deformed product more fully, we obtain

$$(f \times_J g)(x) = \int f(\eta(-Ks) x \eta(-Ku)) g(\eta(-t) x \eta(v)) e(s \cdot t + u \cdot v).$$

If one has in mind the twisting construction mentioned in the introduction, this suggests that we define an operator, T_L, on $C^\infty(G \times G)$ by

$$(T_L F)(x,y) = \iint F(\eta(-Ks)x, \eta(-t)y) e(s \cdot t).$$

It is easily verified that the inverse of T_L is obtained just by replacing K in the above expression by $-K$. If we let T_R be the corresponding operator involving the right action of H on G, we see that the deformed product can be written as

$$f \times_J g = m(T_L T_R^{-1}(f \otimes g)),$$

where m here denotes the undeformed (pointwise) product extended to a map from $C^\infty(G \times G)$ to $C^\infty(G)$. This fits in well with the ideas of the twisting construction. We will see more evidence of this in section 5.

We now turn our attention to the question of which functions we will use. From the point of view of [**Rf3**], we only need to require that our functions are differentiable in the directions of the action α. But from the point of view of working on Lie groups, it is far more natural to try to work with the algebra

$C^\infty(G)$ of functions which are differentiable in all directions. We now show that this is easily arranged.

As prelude to this, we make the following remarks. Let β be an isometric action of the compact Lie group G on a Banach space B, and let B^∞ be the dense subspace of smooth vectors. For $X \in \mathfrak{g}$ we let β_X denote the corresponding infinitesimal action on B^∞. More generally, for X an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$, we let β_X denote the corresponding operator on B^∞. It is usual **[Sc, Ty]** to define a Fréchet topology on B^∞ by the semi-norms $\|b\|_X = \|\beta_X(b)\|$ for $X \in \mathcal{U}(\mathfrak{g})$, where the second norm is that of B. Now these semi-norms are not in general isometric for the action β on B^∞. In fact, for $x \in G$ a quick calculation shows that

$$\beta_X\beta_x(b) = \beta_x\beta_{Ad_x^{-1}(X)}(b),$$

so that

$$\|\beta_x(b)\|_X = \|\beta_X\beta_x(b)\| = \|\beta_{Ad_x^{-1}(X)}(b)\|.$$

But if we view Ad_x as a matrix with continuous functions on G as entries, it is clear that $\|\beta_x(b)\|_X$ is continuous in x, and that we in fact have a bound of form

$$\|\beta_x(b)\|_X \leq M \sum \|b\|_{Y_k}$$

where the Y_k's are a finite number of elements of $\mathcal{U}(\mathfrak{g})$ and M is a constant, independent of b. Let us define new semi-norms, $\| \ \|'_X$, by

$$\|b\|'_X = \sup\{\|\beta_x(b)\|_X : x \in G\}.$$

Then it is clear that these new semi-norms will be invariant under β, and that they will define the same topology as the original semi-norms. Thus we have shown that we can always find a family of semi-norms which define the usual topology of B^∞ and for which β is isometric.

We need this fact for the case of $G \times G$ acting on $C^\infty(G)$ simultaneously on left and right. We could apply the above remarks directly, but actually the situation here simplifies somewhat. Let λ and ρ denote the left and right actions of G on $C^\infty(G)$. Let us first apply the above remarks only to λ. Thus we define semi-norms by

$$\|f\|'_X = \sup\{\|\lambda_{Ad_x(X)}f\|_\infty : x \in G\}.$$

These define the usual topology of $C^\infty(G)$ and are λ-invariant. Notice also that $\|f\|'_{Ad_x(X)} = \|f\|'_X$ for any x. Let us now consider ρ. A quick calculation shows that for any $x \in G$ and $X \in \mathcal{U}(\mathfrak{g})$ we have

$$(\rho_X f)(x) = (\lambda_{Ad_x(X)}f)(x).$$

Thus

$$\|\rho_X f\|_\infty \leq \|f\|'_X.$$

This inequality persists when we take the supremum, $\|f\|''_X$, of the $\|\rho_X(\rho_x f)\|$'s for $x \in G$, since λ and ρ commute. But the situation between λ and ρ is symmetric, so we must have $\|f\|''_X = \|f\|'_X$. Thus $\| \ \|'_X$ is also ρ-invariant. As a result, it suffices to work with these latter semi-norms. Of course, by choosing a basis for $\mathfrak{g}$, we can find a countable family of these semi-norms which already defines the usual Fréchet topology of $C^\infty(G)$, and by taking sums of such semi-norms, we can find an increasing sequence of invariant semi-norms, if that is desired.

We now return to our original situation, and let α denote also the restriction of α to an action on $C^\infty(G)$. Clearly α is isometric for the semi-norms $\| \|'_X$, and differentiable for them. Furthermore, the inclusion of $C^\infty(G)$ into $A = C(G)$ is continuous and α-equivariant. Let A^∞ denote the space of smooth vectors in A for α, so $C^\infty(G) \subseteq A^\infty$. By the functoriality of our deformation process (proposition 2.10 of [**Rf3**]), we then obtain a continuous homomorphism of Fréchet algebras from $C^\infty(G)_J$ to A^∞_J. The inclusion of A^∞_J into the C*-algebra A_J is continuous, and so we then obtain a continuous inclusion of $C^\infty(G)_J$ into A_J. Note, in particular, that this means that if we take $f, g \in C^\infty(G)$ and view them as elements of A^∞, and if we then form their deformed product $f \times_J g \in A^\infty$, then we find that in fact $f \times_J g \in C^\infty(G)$.

We must check that $C^\infty(G)_J$ is dense in A_J. To show this it suffices to show that it is dense in A^∞_J with its Fréchet topology. But the Fréchet topologies of $C^\infty(G)_J$ and A^∞_J are those of the undeformed algebras $C^\infty(G)$ and A^∞. Thus it suffices to show that $C^\infty(G)$ is dense in A^∞. Now $C^\infty(G)$ is clearly dense in A. Thus it suffices to demonstrate the following proposition, which is essentially lemma 2.1.7 of [**Sc**], though we give a somewhat more elementary proof.

1.1 PROPOSITION. *Let A and B be Fréchet spaces and let α and β be actions of some Lie group G on A and B respectively. Let σ be a G-equivariant continuous map from B to A which has dense range. Then the image of B^∞ in A^∞ under σ is dense in A^∞ for the Fréchet topology of A^∞.*

PROOF. Let $X \in \mathcal{U}(\mathfrak{g})$ and let $\| \|$ be one of the semi-norms defining the topology of A. Then one of the semi-norms defining the topology of A^∞ is $\|a\|_X = \|\alpha_X(a)\|$. Now suppose given $a \in A^\infty$ and $\epsilon > 0$. By the strong operator continuity of α on A^∞ [**Sc**] we can find $\varphi \in C^\infty_c(G)$ such that $\|a - \alpha_\varphi(a)\|_X < \epsilon/2$, where α_φ is the integrated form of α. Then for any $b \in B$ we have

$$\|\alpha_\varphi(a) - \sigma(\beta_\varphi(b))\|_X = \|\alpha_\varphi(a - \sigma(b))\|_X$$

$$= \|\alpha_{X\varphi}(a - \sigma(b))\| \le c\|X\varphi\|_1 \|a - \sigma(b)\|,$$

by 2.33 of chapter 0 of [**Ty**], where $\| \|_1$ is the usual L^1-norm, and c is the supremum of the $\|\alpha_x\|$'s over the support of φ (so we don't need to assume that α and β are isometric here). Since the range of σ is dense, we can choose b such that the whole right-hand side above is $\le \epsilon/2$. Thus $\|a - \sigma(\beta_\varphi(b))\|_X \le \epsilon$. But $\beta_\varphi(b) \in B^\infty$. The above argument can be carried out simultaneously for any finite set of semi-norms defining the topology of A^∞. $\square$

1.2 COROLLARY. *Let G be a compact Lie group and let H be a toral subgroup of G. Let $A = C(G)$, and let α and J be as defined earlier. Then $C^\infty(G)_J$ is dense in A_J^∞ and A_J.*

This justifies the fact that in much of what follows we will work with $C^\infty(G)$ and $C^\infty(G)_J$.

2. The comultiplication

At the level of functions we will let the comultiplication, Δ, on the deformed algebra still be the usual comultiplication, defined for $f \in C(G)$ by $(\Delta f)(x, x') = f(xx')$. That is, we will not need to deform the comultiplication, so our deformation is a "preferred deformation" **[GS1, GS2, Gq]**. Since it is surprising that this works, let us verify first that, at least at the level of functions, Δ is still an algebra homomorphism for the deformed product. It is for this that the special form of J is crucial. Since we will later need to show that Δ is continuous for the C*-norms, and since the proof of that fact will also show that Δ is a homomorphism (but not in as direct a way), we will not be very careful here about explicitly justifying the changes of variables involved.

So let $f, g \in C^\infty(G)$. We will, of course, take $C^\infty(G) \otimes C^\infty(G)$ to be the completed tensor product which is identified with $C^\infty(G \times G)$ in the evident way. Then

$$(\Delta f \times_J \Delta g)(x, x')$$

$$= \int (\Delta f)(\eta(-Ks)x\eta(-Ku),\ \eta(-Ks')x'\eta(-Ku'))$$
$$(\Delta g)(\eta(-t)x\eta(v),\ \eta(-t')x'\eta(v'))e(s \cdot t + u \cdot v + s' \cdot t' + u \cdot v')$$

$$= \int f(\eta(-Ks)x\eta(-K(u+s'))x'\eta(-Ku'))g(\eta(-t)x\eta(v-t')x'\eta(v'))e(\dots).$$

We make the change of variables $u \mapsto u - s'$, $t' \mapsto t' + v$, and note that because of cancellations this leaves the term $e(\dots)$ unchanged. Thus the above

$$= \int f(\eta(-Ks)x\eta(-Ku)x'\eta(-Ku'))g(\eta(-t)x\eta(-t')x'\eta(v'))e(\dots).$$

But now we see that v and s' do not appear in the integrand except in $e(\dots)$. Then by proposition 1.11 of **[Rf3]** (essentially the Fourier inversion formula), the above integral becomes

$$\int f(\eta(-Ks)xx'\eta(-Ku'))g(\eta(-t)xx'\eta(v'))e(s \cdot t + u' \cdot v')$$
$$= (\Delta(f \times_J g))(x, x').$$

This is auspicious. We proceed now to give a rigorous verification that Δ is a continuous homomorphism at the level of C*-algebras. The main difficulty is that $A_J \otimes A_J$ is defined by the action $\alpha \otimes \alpha$ of $V \times V$ on $C(G \times G)$, while A_J is defined by the action α of V on $C(G)$. Since the groups which are acting are different, we can not immediately invoke the functoriality of our deformation proceedure stated in theorem 5.7 of [**Rf3**]. We will approach the matter in several steps.

To begin with, set

$$C = \{F \in C(G \times G) : F(xw, y) = F(x, wy) \text{ for } x, y \in G, w \in H\}.$$

Note that $\Delta f \in C$ for all $f \in C(G)$, and that Δ is a unital homomorphism from A into C. Define an action, β, of V on C by

$$(\beta_{(s,u)}F)(x,y) = F(\eta(-s)x, y\eta(u))$$

for $(s, u) \in V$. It is clear that this action does carry C into itself. Most important, it is clear that Δ as a map into C is equivariant for the actions α on A and β on C. Consequently, from the functoriality expressed in theorem 5.7 of [**Rf3**], Δ determines a unital homomorphism, Δ_J, from A_J to C_J (at the C*-algebra level). Since Δ is injective, it follows from proposition 5.8 of [**Rf3**] that Δ_J is injective.

Now let $W = V \times V$, and let $\gamma = \alpha \otimes \alpha$ be the action of W on $C(G \times G)$ defined by

$$(\gamma_{(s,u,s',u')}F)(x, x') = F(\eta(-s)x\eta(u), \eta(-s')x'\eta(u')).$$

Notice that C is carried into itself by γ, exactly because H is Abelian. So let γ also denote the restriction of γ to an action on C. Let

$$L = J \oplus J = K \oplus (-K) \oplus K \oplus (-K),$$

a skew-symmetric operator on W. Then we can form the deformed algebra C_L (the action γ being implicit), and according to proposition 5.8 of [**Rf3**] it will be a subalgebra of $C(G \times G)_L$. Let $W_0 = \{(0, u, u, 0) : u \in \mathbf{R}^n\}$. Because of the definition of C, it is clear that W_0 is in the kernel of γ as action on C. Let P be the projection of W onto the orthogonal complement of W_0, so that $\gamma = \gamma \circ P$ as actions on C. Then according to theorem 8.11 of [**Rf3**] we have $C_L = C_{PLP}$ under the evident identification of elements of C^∞. But a quick calculation shows that

$$PLP(s, u, t, v) = (Ks, 0, 0, -Kv).$$

In particular, if we let Q denote the orthogonal projection on $W_1 = \{(s, 0, 0, v) : s, v \in \mathbf{R}^n\}$, we see that $Q(PLP)Q = PLP$. We are thus in a position to apply theorem 8.7 of [**Rf3**]. This says that we should consider the restrictions of γ and PLP to W_1. But under the evident identification, these are just the β and J of earlier. Then theorem 8.7 of [**Rf3**] states that $C_L^\gamma = C_J^\beta$ for the evident meaning of this notation. Since we saw earlier that Δ_J is a unital homomorphism from A_J into C_J^β, we see that we can now interpret it as a unital homomorphism from A_J into C_L^γ. If we compose this with the inclusion of C_L^γ into $C(G \times G)_L$, we see that Δ_J gives a unital injective homomorphism from A_J into $C(G \times G)_L$. It is easy to check that at the level of functions this homomorphism is still our original Δ.

To continue, we need to consider how our process for deformation quantization relates to tensor products of C*-algebras. We will always use the minimal tensor product for C*-algebras **[KR]**, and we will denote it by $\otimes$. When confusion might arise we will denote it by $\otimes_\mu$ and will denote the algebraic tensor product over the complex numbers by $\otimes_a$.

2.1 Proposition. *Let α and β be actions of vector groups V and W on C*-algebras A and B respectively. Let $\alpha \otimes \beta$ denote the corresponding action of $V \oplus W$ on $A \otimes B$. Let J and K be skew-symmetric operators on V and W respectively, so that $J \oplus K$ is a skew-symmetric operator on $V \oplus W$. Then*

$$(A \otimes B)^{\alpha\otimes\beta}_{J\oplus K} \cong A^\alpha_J \otimes B^\beta_K$$

under the evident identification of $A^\infty \otimes_a B^\infty$ as a dense subalgebra of both sides.

Proof. It is easily verified that on $A^\infty \otimes_a B^\infty$ the deformed products of the two sides agree, so that, in particular, $A^\infty \otimes_a B^\infty$ is a subalgebra of the left-hand side.Thus the issue is what happens to the norms. Now by definitions 4.8 and 4.9 of **[Rf3]** the norm on $(A \otimes B)^{\alpha\otimes\beta}_{J\oplus K}$ is determined by its action on $\mathcal{S}^{A\otimes B}(V \oplus W)$. But this contains $\mathcal{S}^A(V) \otimes_a \mathcal{S}^B(W)$ as a dense subspace in the evident way, and the latter is clearly a right rigged space **[Rf1]** over the pre-C*-algebra $A \otimes_a B$ (which is a dense subalgebra of $A \otimes_\mu B$). Thus the norms of those elements of $(A \otimes B)^{\alpha\otimes\beta}_{J\oplus K}$ which are actually in $A^\infty \otimes_a B^\infty$ are determined by their actions on the rigged space $\mathcal{S}^A(V) \otimes_a \mathcal{S}^B(W)$. As done before definition 4.8 of **[Rf3]**, denote the operator on $\mathcal{S}^A(V)$ corresponding to $a \in A^\infty$ by L^J_a, etc. Then for an element t of $A^\infty \otimes_a B^\infty$ with expression $t = \sum a_j \otimes b_j$, it is readily calculated that $L^{J\oplus K}_t = \sum L^J_{a_j} \otimes L^K_{b_j}$ as operators on $\mathcal{S}^A(V) \otimes_a \mathcal{S}^B(W)$. But by definition, A^α_J is faithfully represented on $\mathcal{S}^A(V)$, as is B^β_K on $\mathcal{S}^B(W)$. By considering induced representations as in theorem 5.1 of **[Rf1]**, it is then easily seen that the norm on $A^\alpha_J \otimes B^\beta_K$ is determined by the action of $A^\alpha_J \otimes_a B^\beta_K$ on $\mathcal{S}^A(V) \otimes_a \mathcal{S}^B(W)$, with $t = \sum a_j \otimes b_j$ acting as $\sum L^J_{a_j} \otimes L^K_{b_j}$. The desired conclusion now follows. □

2.2 Corollary. *With $A = C(G)$ and with J, K, α, γ as earlier in this section, we have*

$$(A \otimes A)^\gamma_L = A^\alpha_J \otimes A^\alpha_J$$

under the natural identification of $A^\infty \otimes_a A^\infty$ as a dense subalgebra of each side.

Now $A \otimes A = C(G \times G)$, and we have already seen how to consider Δ (i.e. Δ_J) as a continuous unital homomorphism from A_J into $C(G \times G)_L$. Combining this with Corollary 2.2, we see that Δ can be considered as a unital homomorphism from A_J into $A_J \otimes A_J$. Tracing through the various identifications made above, we see that this Δ, when restricted to $C^\infty(G)$, is just the original comultiplication. Since the latter is coassociative, it follows by continuity that Δ on A_J is coassociative. Thus we obtain the main result of this paper.

2.3 THEOREM. *Let $A = C(G)$, α and J be as above. Then the usual comultiplication, Δ, on A, when restricted to $C^\infty(G)$, determines a unital injective homomorphism from A_J into $A_J \otimes A_J$ which is coassociative.*

We remark that since A is commutative, and so nuclear, it follows from theorem 4.1 of [**Rf4**] that A_J is nuclear, so that all C*-tensor products with A_J coincide.

We now turn to consideration of the coidentity for Δ. Again, there is no need to deform the ordinary coidentity. Specifically, let ϵ denote the homomorphism from $C(G)$ into the complex numbers consisting of evaluating functions at the identity element of G. We want to show that it determines a homomorphism from A_J into the complex numbers. Note that ϵ is not equivariant for α for any action on the complex numbers. So to proceed, let π denote the restriction homomorphism from $C(G)$ onto $C(H)$. Note that α can be viewed as an action on $C(H)$, and that π is then equivariant. Thus by theorem 5.7 of [**Rf3**], π determines a homomorphism of C*-algebras, π_J, from $C(G)_J$ to $C(H)_J$, which is surjective by proposition 5.8 of [**Rf3**].

Now because H is Abelian, α as an action on $C(H)$ can be expressed as

$$(\alpha_{(s,u)}f)(x) = f(\eta(-s+u)x).$$

Consequently, α acting on $C(H)$ has the subspace $W_0 = \{(s,s) : s \in \mathbf{R}^n\}$ in its kernel. Let P denote the projection on the orthogonal complement of W_0. Then $\alpha = \alpha \circ P$ on $C(H)$. But a simple calculation shows that $PJP = 0$. Then according to corollary 8.9 of [**Rf3**] we have $C(H)_J = C(H)$. Thus π_J is a homomorphism from $C(G)_J$ onto $C(H)$.

If we view $C(H)$ as a commutative (i.e. undeformed) quantum group with the standard comultiplication, it is easily seen that π_J is compatible with the comultiplications. Once we have established the coidentity and coinverse for $C(G)_J$, it will be clear that π_J is compatible with these also. We will thus obtain:

2.4 PROPOSITION. *The group H (qua the Hopf algebra $C(H)$) is a subgroup of the quantum group $C(G)_J$ (i.e. via the restriction map the Hopf algebra $C(H)$ is a quotient of the Hopf algebra $C(G)_J$).*

We remark that by theorem 7.7 of [**Rf3**] the kernel of π_J will be exactly a strict deformation quantization of the algebra of those continuous functions on G which have value 0 on H.

For our discussion of the coidentity on $C(G)_J$ we only need the fact that the restriction map gives a continuous map (namely π_J) from $C(G)_J$ onto $C(H)$. For we can then compose π_J with evaluation of functions on H at the identity element of G, which is clearly continuous. We will denote this composed homomorphism by ϵ_J or just ϵ.

Now at the level of functions in $C^\infty(G)$, both Δ and ϵ are the original comultiplication and coidentity for the undeformed algebra. But the relations characterizing a coidentity for a comultiplication on a Hopf algebra, namely

$$(\epsilon \otimes id)\Delta = id = (id \otimes \epsilon)\Delta,$$

where id denotes the identity operator on the algebra, do not involve the multiplication in the Hopf algebra. By continuity and density it follows that these relations will continue to be satisfied on A_J. Thus we have obtained:

2.5 PROPOSITION. *The usual coidentity, ϵ, on $A = C(G)$ consisting of evaluation of functions at the identity element of G, determines a homomorphism from A_J into the complex numbers which is a coidentity for the comultiplication Δ on A_J.*

We conclude this section by writing down for our special case the master unitary operator (i.e. the "Kac-Takesaki operator" as in [**MN**]) which is so often used in the discussion of quantum groups in the setting of operator algebras, and which has received a beautiful axiomatic treatment in [**BS**]. There are several conventions to be made. We will choose the ones corresponding to the right regular representation of a group. Thus we consider the operator W defined on functions on $G \times G$ by

$$(WF)(x, y) = F(xy, y).$$

This can be viewed at several levels. If $F = f \otimes g$ for $f, g \in C(G)$, then

$$(W(f \otimes g))(x, y) = f(xy)g(y) = ((\Delta f)(1 \otimes g))(x, y).$$

This suggests that for any Hopf algebra A with comultiplication Δ, we define an operator, W, on $A \otimes A$ by

$$W(a \otimes b) = (\Delta a)(1 \otimes b).$$

By using the coassociativity of Δ it is straight-forward to verify that W satisfies the pentagonal relation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}$$

which is at the heart of [**BS**]. (Here $W_{12} = W \otimes id$ on $A \otimes A \otimes A$, etc.) On functions on $G \times G$ the inverse of the original W is given by

$$(W^{-1}(f \otimes g))(x, y) = f(xy^{-1})g(y),$$

which suggests that on any Hopf algebra A the inverse of W should be given by

$$W^{-1}(a \otimes b) = ((id \otimes S)\Delta a)(1 \otimes b)$$

where S is the coinverse (i.e. antipode) of A; and indeed this is easily verified (see, e.g. [**Vn**]).

For the quantum groups defined in this section, these considerations suggest that W should be defined by

$$W(f \otimes g) = (\Delta f) \times_J (1 \otimes g),$$

for $f, g \in C^\infty(G)$ (where we write $\times_J$ instead of $\times_{J \oplus J}$), that is,

$$(W(f \otimes g))(x,x')$$
$$= \int (\Delta f)(\eta(-Ks)x\eta(-Ku), \eta(-Ks')x'\eta(-Ku'))g(\eta(-t')x'\eta(v'))$$
$$e(s \cdot t + u \cdot v + s' \cdot t' + u' \cdot v').$$

Since t and v do not appear in the integral other than in $e(\ldots)$, we can apply proposition 1.11 of **[Rf3]** to reduce this to (omitting primes)

$$W(f \otimes g)(x,y) = \int f(x\eta(-Ks)y\eta(-Ku))g(\eta(-t)y\eta(-v))e(s \cdot t + u \cdot v).$$

This suggests that more generally for $F \in C^\infty(G \times G)$ we set

$$(WF)(x,y) = \int F(x\eta(-Ks)y\eta(-Ku), \eta(-t)y\eta(-v))e(s \cdot t + u \cdot v).$$

Again, W will satisfy the pentagonal relation given above. In much the same way we are led to set

$$(W^{-1}F)(x,y) = \int F(x\eta(Ku)y^{-1}\eta(-Ks), \eta(-t)y\eta(-v))e(s \cdot t + u \cdot v).$$

A straight-forward calculation involving the techniques used earlier for reducing oscillatory integrals (e.g. proposition 1.11 of **[Rf3]**) shows that W^{-1} is indeed the inverse of W.

Now equip $C^\infty(G \times G)$ with the usual inner product using the Haar measure on G. Another straight-forward calculation shows that with respect to this inner product we have $W^* = W^{-1}$. Thus W is unitary, and so extends to a unitary operator on $L^2(G \times G)$, which we still denote by W. This new W will still satisfy the above pentagonal relation, and it is the master unitary operator which connects our examples with the theory in **[BS]**. All this suggests that there should be a quite interesting general theory of the deformation of the master unitary operators of the Baaj-Skandalis theory, related in particular to the twisting construction of Drinfeld, but this must await a later time.

3. The coinverse

As with the comultiplication and coidentity, we do not need to deform the usual coinverse (i.e. "antipode", or "antipodal map") on $C(G)$, namely the automorphism S of $C(G)$ defined by $(Sf)(x) = f(x^{-1})$. However, for a non-commutative Hopf algebra, such as our deformed algebras A_J, we do not expect a coinverse to be an automorphism, but rather an anti-automorphism, that is, an isomorphism of A_J onto $(A_J)^{op}$, where $(A_J)^{op}$ is A_J but with the opposite multiplication.

For the quantum groups studied in **[LS, W]** the coinverse is often only densely defined and not continuous. But for the quantum groups which we have been studying here the coinverse will be continuous. This is related to the fact that the Poisson duals of our Poisson groups G (for the Poisson brackets from α and J) are unimodular, as will be shown in **[Rf5]**.

To show that S is continuous, we want to invoke the functoriality of our deformation process. For this we need S to be equivariant for suitable actions. Now

$$(S(\alpha_{(s,u)}f))(x) = f(\eta(-s)x^{-1}\eta(u)) = (Sf)(\eta(-u)x\eta(s)).$$

This suggests that we define an action, β, of V on A by

$$(\beta_{(s,u)}f)(x) = f(\eta(-u)x\eta(s)),$$

since we see that S is then equivariant for α and β. Let us denote the deformed algebras for α and β by A_J^α and A_J^β respectively. Then by functoriality (theorem 5.7 of [**Rf3**]) S determines a homomorphism, which we also denote by S, from A_J^α to A_J^β. By proposition 5.8 of [**Rf3**] this homomorphism will, in fact, be an isomorphism. Since S on A is clearly its own inverse, we have $S \circ \beta_{(s,u)} = \alpha_{(s,u)} \circ S$, and so S also determines an isomorphism, S, from A_J^β to A_J^α, and it is clear that $S^2 = Id$ in the two evident senses.

Now for $f, g \in C^\infty(G)$ denote their deformed product using β by $f \times_J^\beta g$, etc. Then

$$(f \times_J^\beta g)(x) = \int f(\eta(Ku)x\eta(Ks))g(\eta(-v)x\eta(t))e(s \cdot t + u \cdot v),$$

which by proposition 1.13 of [**Rf3**] is

$$= \int f(\eta(u)x\eta(s))g(\eta(Kv)x\eta(-Kt))e(s \cdot t + u \cdot v) = g \times_J^\alpha f.$$

It is clear from this that A_J^β is just the opposite algebra of A_J^α, so that S can be viewed as determining a continuous anti-automorphism of A_J, as desired.

We must now verify that S on A_J satisfies the properties which say that it is a coinverse for Δ. Now the first property which is usually required is that $(S \otimes S) \circ \Delta = \sigma \circ \Delta \circ S$, where σ is the flip map determined by $\sigma(a \otimes b) = b \otimes a$. But this property does not involve the deformed product, and so clearly holds at the level of functions. By continuity it then holds on A_J.

When in a purely algebraic setting, the main property which is usually required of a coinverse [**D2**] is that $m \circ (id \otimes S) \circ \Delta = \iota \circ \epsilon$, where m is the product on the algebra, say A, and ι is the "identity element", i.e. the homomorphism from the complex numbers into A defined by $\iota(z) = z1_A$. One also requires the corresponding relation in which $id \otimes S$ is replaced by $S \otimes id$. But in the C*-algebra setting there can be serious problems with this requirement, because m need not be continuous. (E.g. consider the case in which A is the algebra of compact operators.) This is related to the fact that m is not an algebra homomorphism. As a consequence, the route which is usually taken is to introduce the analogue of Haar measure, and then to require that S satisfy suitable properties with respect to this Haar measure. But before we do this, let us see that S does satisfy the above algebraic condition on the dense subalgebra $C^\infty(G)$. To do this, we first determine what m is as a map from $C^\infty(G \times G)$ to $C^\infty(G)$. Now if $f, g \in C^\infty(G)$ then

$$(m(f \otimes g))(x) = (f \times_J g)(x) = \int f(\eta(-Ks)x\eta(-Ku))g(\eta(-t)x\eta(v))e(s\cdot t+u\cdot v).$$

Thus for $F \in C^\infty(G \times G)$ we should set

$$(mF)(x) = \int F(\eta(-Ks)x\eta(-Ku), \eta(-t)x\eta(v))e(s \cdot t + u \cdot v).$$

Notice that this integral exists (as an oscillatory integral), even though in general m will not be continuous for the C*-norm on A_J. It is easily seen that $mF \in C^\infty(G)$.

Then for $f \in C^\infty(G)$ we have

$$\begin{aligned} &m((id \otimes S)(\Delta f))(x) \\ &\quad = \int ((id \otimes S)(\Delta f))(\eta(-Ks)x\eta(-Ku), \eta(-t)x\eta(v))e(s \cdot t + u \cdot v) \\ &\quad = \int (\Delta f)(\eta(-Ks)x\eta(-Ku), \eta(-v)x^{-1}\eta(t))e(s \cdot t + u \cdot v) \\ &\quad = \int f(\eta(-Ks)x\eta(-(Ku + v))x^{-1}\eta(t))e(s \cdot t + u \cdot v). \end{aligned}$$

We now want to make the substitution $v \mapsto v - Ku$. The justification comes from the following lemma which we will also need later. For simplicity of notation we phrase it only in terms of single variables, but it then is clearly applicable to (s, u), etc. We use notation as in proposition 1.10 of **[Rf3]**.

3.1 LEMMA. *Let $F \in \mathcal{B}^A(V \times V)$, and let J be any skew-symmetric operator on V. Then*

$$\iint F(u + Jv, v)e(u \cdot v) = \iint F(u, v)e(u \cdot v).$$

PROOF. For ψ_m's as in proposition 1.10 of **[Rf3]**, we have

$$\iint F(u + Jv, v)e(u \cdot v) = \lim \iint F(u + Jv, v)\psi_m(u)\psi_n(v)e(u \cdot v).$$

Since the latter integrals are ordinary integrals, we can now make the desired change of variables, so that, in view of the fact that J is skew-symmetric, we obtain

$$\lim \iint F(u, v)\psi_m(u - Jv)\psi_n(v)e(u \cdot v).$$

But the functions $\psi_m(u - Jv)\psi_n(v)$ on $W = V \times V$ have value 1 on increasing balls and satisfy the hypotheses of proposition 1.6 of **[Rf3]**. Thus by the last part of that proposition the above limit is $\iint F(u, v)e(u \cdot v)$, as desired. □

We now apply this lemma to the integral considered just before it, to obtain

$$\int f(\eta(-Ks)x\eta(-v)x^{-1}\eta(t))e(s \cdot t + u \cdot v).$$

Since u no longer appears inside f, we can apply proposition 1.11 of **[Rf3]** to conclude that this integral

$$= \int f(\eta(-Ks)\eta(t))e(s \cdot t).$$

Another application of Lemma 3.1 together with corollary 1.12 of **[Rf3]** shows that this latter integral is $= f(e)$ where e is the identity element of G. A similar calculation works when $id \otimes S$ is replaced by $S \otimes id$. We have thus obtained:

3.2 PROPOSITION. *On $C^\infty(G)_J$ we have*

$$m \circ (id \otimes S) \circ \Delta = \iota \circ \epsilon = m \circ (S \otimes id) \circ \Delta,$$

so that S is a coinverse for Δ there.

4. Haar measure

Since we are dealing here only with compact quantum groups, not non-compact ones, we can expect that Haar measure will be continuous for the C*-norms. For the quantum groups constructed in the previous sections, we can, in fact, take Haar measure at the level of functions to be just the usual Haar measure on the group. We denote it by μ, so $\mu(f) = \int f$ for $f \in C^\infty(G)$. We need to show that this extends to a continuous positive linear functional on A_J satisfying suitable properties with respect to Δ.

The key is to note that, because Haar measure on a compact group is invariant for both right and left translation, μ is invariant with respect to the action α which we have been using. Thus we see that what we need is:

4.1 THEOREM. *Let α be an action of a vector group V on a C^*-algebra A, and let J be a skew-symmetric operator on V. Let μ be a positive linear functional on A which is α-invariant. Then the restriction of α to A^∞ determines a continuous positive linear functional, μ_J, on A_J, and*

$$\mu_J(a \times_J b) = \mu(ab)$$

for all $a, b \in A^\infty$. Furthermore, μ_J is α-invariant for the action α on A_J. If μ is faithful on A, then μ_J will be faithful on A_J. If μ is actually a (bounded) trace on A, then μ_J will be a trace on A_J.

PROOF. If A does not have an identity element, we can adjoin one and extend α and μ in the usual way. Thus we can assume from now on that A, and so also A_J, has an identity element, I. For convenience we will also assume that μ is a state, i.e. that $\mu(I) = 1$.

We begin with the following calculation using the α-invariance of μ. Let $a, b \in A^\infty$. Then

$$\mu(a \times_J b) = \mu(\int \alpha_{Ju}(a)\alpha_v(b)e(u \cdot v)) = \int \mu(\alpha_{Ju-v}(a)b)e(u \cdot v).$$

We can now apply Lemma 3.1 with the roles of u and v interchanged, to conclude that this integral

$$= \int \mu(\alpha_{-v}(a)b)e(u \cdot v) = \mu(ab),$$

where for the last equality we have applied corollary 1.12 of **[Rf3]**. From this it is clear that μ is positive as a linear functional on A_J^∞, and that the GNS Hilbert spaces for μ as a functional on A_J^∞ and A^∞ coincide. It also follows that μ is a trace on A_J^∞ if it is one on A^∞, and so will be one on A_J once continuity is established. We note that the α-invariance of μ_J will also be clear once continuity is established.

The crux then is to show continuity. We do this by showing that the action of A_J^∞ on the GNS Hilbert space is by bounded operators, and that the resulting representation is continuous. The latter actually follows from the former because by corollary 7.6 of **[Rf3]** A_J^∞ is a local algebra in A_J so that corollary 3.1.5 of **[Bl]** can be invoked. But the proof we give here yields both of these facts simultaneously.

Now according to definition 4.8 of **[Rf3]** the norm on A_J is determined by its action, L, on the A-rigged space $\mathcal{S}^A(V)$ defined by

$$(L_a f)(x) = \iint \alpha_{x+Ju}(a) f(x+v) e(u \cdot v),$$

where the inner product on $\mathcal{S}^A$ is defined by

$$\langle f, g \rangle_A = \int f(x)^* g(x).$$

We can then define an ordinary inner product on $\mathcal{S}^A$ by

$$\langle f, g \rangle_\mu = \mu(\langle f, g \rangle).$$

As explained in corollary 5.5 of **[Rf1]**, the action of A_J will give a representation by bounded operators on the corresponding Hilbert space, and in particular, this representation will be norm-decreasing. That is, we will have

$$|\langle f, L_a g \rangle_\mu| \leq \|a\| \|f\|_2 \|g\|_2,$$

where $\| \ \|_2$ is with respect to the scalar-valued inner product using μ. Choose $\varphi \in C_c^\infty(V)$ such that $\int \varphi(x)^2 = 1$. Define $f \in \mathcal{S}^A$ by $f(x) = \varphi(x) I$. Since μ is a state,

$$\|f\|_2^2 = \int \varphi(x)^2 \mu(I) = 1.$$

Furthermore,

$$\langle f, L_a f \rangle_\mu = \mu(\int f(x)^* (L_a f)(x)) = \mu(\int \varphi(x) \iint \alpha_{x+Ju}(a) \varphi(x+v) e(u \cdot v)),$$

which, by the α-invariance of μ,

$$= \int \varphi(x) \iint \mu(a) \varphi(x+v) e(u \cdot v),$$

which, by corollary 1.12 of **[Rf3]**,

$$= \int \varphi(x)^2 \mu(a) = \mu(a).$$

Consequently, from the earlier inequality,

$$|\mu(a)| = |\langle f, L_a f\rangle_\mu| \le \|a\|_J \|f\|_2^2 = \|a\|_J,$$

which is the desired continuity.

To complete the non-unital case we must invoke the fact that if B is an α invariant ideal in A, then B_J is an ideal in A_J by proposition 5.9 of **[Rf3]**.

Suppose now that μ is faithful. Let K denote the kernel of the GNS representation of A_J for μ_J. Because μ_J is α-invariant, K will be α-invariant. Consequently, K^∞ is dense in K. Now according to theorem 7.1 of **[Rf3]** we have $K^\infty \subseteq A^\infty$. But by the calculation done above, if $c \in K^\infty$ then

$$\mu(c^* c) = \mu_J(c^* \times_J c) = 0.$$

Thus $c = 0$ since μ is faithful. Consequently $K = \{0\}$, and so μ_J is faithful. □

We now return to letting μ denote Haar measure on G. We know now that it is a continuous trace on $C^\infty(G)_J$. For simplicity we denote its extension to A_J just by μ instead of μ_J. We would like to express the fact that μ is "left-invariant" for Δ. Now for ordinary Haar measure on the group G this says that for $f \in C(G)$ we have

$$\int f(xy)\, dy = \int f(y)\, dy.$$

Phrased in terms of Δ and μ, this says that

$$(id \otimes \mu)(\Delta f) = \iota(\mu(f)),$$

or

$$(id \otimes \mu) \circ \Delta = \iota \circ \mu.$$

But this property does not involve the product of functions, and so it obviously persists at the level of functions for the deformed product on $C^\infty(G)_J$. By continuity it holds also for A_J. (We remark that "slice maps" such as $id \otimes \mu$ are always well-defined and continuous on tensor products of C*-algebras, regardless of what tensor product is used since they always factor through the minimal tensor product.) In the same way, μ on A_J is also right-invariant for Δ, i.e.

$$(\mu \otimes id) \circ \Delta = \iota \circ \mu.$$

We now relate the coinverse S to the Haar measure μ on A_J. This is traditionally **[MN, Rf2, V, W]** expressed by the analogue of the fact that at the usual group level if we apply left translation by x^{-1} to

$$\int f(y) g(xy)\, dy$$

we obtain

$$\int f(x^{-1}y)g(y)\,dy,$$

with a similar relation for right translation. Phrased in terms of Δ, μ, and S, this says that

$$(id \otimes \mu)((1 \otimes f)(\Delta g)) = (id \otimes \mu)(((S \otimes id)\Delta f)(1 \otimes g)),$$

where 1 is the constant function 1. This relation clearly involves the product, but only as a bilinear map on $A \times A$, not as a linear map on $A \otimes A$, and so the analogous relation on $A_J \times A_J$ does have meaning, with both sides being continuous in f and g. We proceed to verify that it holds.

As before, we will denote the deformed product on $A_J \otimes A_J$ just by $\times_J$. Then for $f, g, h \in C^\infty(G)$ we have

$$(id \otimes \mu)((1 \otimes f) \times_J (g \otimes h)) = g\mu(f \times_J h) = g\mu(fh) = (id \otimes \mu)((1 \otimes f)(g \otimes h)),$$

where we have used Theorem 4.1. By continuity this relation will persist when $f \otimes h$ is replaced by $F \in C^\infty(G \times G)$. When we apply this for $F = \Delta g$ we obtain

$$((1 \otimes \mu)((1 \otimes f) \times_J \Delta g) = (1 \otimes \mu)((1 \otimes f)\Delta g).$$

In the same way we have

$$(id \otimes \mu)(((S \otimes id)(f \otimes g)) \times_J (1 \otimes h)) = (Sf)\mu(g \times_J h)$$
$$= (Sf)\mu(gh) = (id \otimes \mu)(((S \otimes id)(f \otimes g))(1 \otimes h)),$$

which by continuity gives

$$(id \otimes \mu)(((S \otimes id)\Delta f) \times_J (1 \otimes g)) = (id \otimes \mu)(((S \otimes id)\Delta f)(1 \otimes g)).$$

Thus we see that we have reduced the situation to the ordinary group case where the relation originated. It follows that it holds also in the deformed algebra at the level of functions, and so by continuity holds also in A_J.

We note next that the relation $\mu \circ S = \mu$ which is usually also required in the relation between the Haar measure and the coinverse of a quantum group obviously holds in our present situation, since it holds for the undeformed algebra and does not involve the product of functions.

Finally, we note that Haar measure on $C(G)$ is faithful, and so by Theorem 4.1 it is faithful on A_J also. (Haar measures on compact quantum groups are not always faithful, as discussed before theorem 5.7 of [**W**].)

We summarize the results of this section in:

4.2 THEOREM. *The Haar measure, μ, on $A = C(G)$ determines a Haar measure on the quantum group A_J, that is, a continuous linear functional μ_J such that*

(1)

$$(id \otimes \mu_J) \circ \Delta = \iota \circ \mu_J \qquad \textit{(left invariance)}$$
$$(\mu_J \otimes id) \circ \Delta = \iota \circ \mu_J \qquad \textit{(right invariance)},$$

(2)
$$(id\otimes\mu_J)((1\otimes a)(\Delta b)) = (id\otimes\mu_J)(((S\otimes id)\Delta a)(1\otimes b)),$$

(3)
$$\mu_J\circ S=\mu_J.$$

Furthermore, μ_J is a faithful trace on A_J.

5. Representations

The general representation theory for compact quantum groups has been extensively developed in **[W]**. We will make here only some brief comments about how this general theory relates to the specific class of quantum groups constructed earlier in this paper. Whenever we say "representation" we will always mean "finite-dimensional unitary representation".

Let π be a representation of the group G on a (finite-dimensional) Hilbert space Ξ, and let $B(\Xi)$ denote the C*-algebra of operators on Ξ. Then π can be viewed as a continuous function from G into $B(\Xi)$, and so as a unitary element of $B(\Xi)\otimes A$ where $A=C(G)$. The fact that π is a representation then translates into the relation
$$(id\otimes\Delta)(\pi)=\pi_{12}\pi_{13}$$
where $\pi_{12}=\pi\otimes 1_A\in B(\Xi)\otimes A\otimes A$, and $\pi_{13}=\sum T_k\otimes 1_A\otimes a_k$ if $\pi=\sum T_k\otimes a_k$. This is the motivation for one form of the usual definition (see definition 2.1 and section 5 of **[W]**) of a unitary representation of a quantum group A (i.e. a corepresentation of the corresponding Hopf algebra A of "functions on the quantum group"), namely a unitary element of $B(\Xi)\otimes A$ which satisfies the above relation.

We notice that the above relation involves the product in $B(\Xi)$ (in the "first entry") but not the product in A (since only a product with 1_A is involved). So it only depends on the comultiplication Δ. This suggests that if we have a *preferred* deformation (i.e. one in which the comultiplication Δ is not changed **[G, GGS1, GGS2]**) then every representation of the original group will immediately be a representation of the deformed quantum group. We only must exercise some care that the representation is actually associated with the algebra being deformed (e.g. a smooth algebra, and not just some C*-completion of it).

We apply this observation to the quantum groups constructed in the previous sections, in which $A=C(G)$. Now it is known that every (finite-dimensional unitary) representation of a Lie group G will be smooth, i.e. will be an element of $B(\Xi)\otimes C^\infty(G)$. Thus by the remarks in the previous paragraph it will immediately give a representation of the quantum group A_J. Furthermore, intertwining operators between representations of G will immediately be intertwining operators (as defined in definition 2.2 of **[W]**) between the corresponding representations of A_J. If we now apply the theory of orthogonality relations and the semi-simplicity of the representation theory of compact quantum groups as developed in [W] (notably theorem 4.5 of [W]), it becomes clear that we obtain from the irreducible representations of G representations for all of the equivalence classes of irreducible representations of the quantum group A_J. Thus nothing very interesting has happened so far.

The point at which (for preferred deformations) the representation theory becomes more interesting, is in the consideration of inner tensor products of representations, because their definition depends crucially on the product in the Hopf algebra of the quantum group, and it is this product which is deformed under preferred deformations. To be specific, let π and ρ be representations of a quantum group A on Hilbert spaces Ξ and Z. Then by definition (equation 2.15 of [W]) their inner tensor product, which we will denote by $\pi \boxtimes \rho$, is the element

$$\pi \boxtimes \rho = \pi_{13}\rho_{23}$$

of $B(\Xi) \otimes B(Z) \otimes A$. Notice its dependence in the "third entry" on the product in A.

For quantum groups in general, $\pi \boxtimes \rho$ need not be equivalent to $\rho \boxtimes \pi$, and the decomposition of $\pi \boxtimes \rho$ into irreducible representations can be complicated. So it is of interest to determine what happens for the quantum groups which we have been studying.

A traditional method for dealing with representations of groups is by means of their characters. This also works for compact quantum groups. For $\pi \in B(\Xi) \otimes A$ a representation of a quantum group A, its character, χ_π, is defined (see [W], before the statement of theorem 5.8) to be

$$\chi_\pi = (Tr \otimes id)(\pi),$$

where Tr is the usual trace on $B(\Xi)$. Thus χ_π is an element of A. If we have chosen matrix units $\{m_{jk}\}$ for $B(\Xi)$, so that $\pi = \sum m_{jk} \otimes a_{jk}$ for certain elements a_{jk} of A, then

$$\chi_\pi = \sum a_{jj}.$$

If $\rho = \sum n_{pq} \otimes b_{pq}$ for matrix units in $B(Z)$, then

$$\pi \boxtimes \rho = \sum m_{jk} \otimes n_{pq} \otimes a_{jk}b_{pq},$$

and the $m_{jk} \otimes n_{pq}$'s will form matrix units for $B(\Xi \otimes Z)$. Thus we obtain the traditional formula

$$\chi_{\pi\boxtimes\rho} = \sum a_{jj}b_{pp} = \chi_\pi\chi_\rho.$$

Since in general A is not commutative, we will have $\chi_\pi\chi_\rho \neq \chi_\rho\chi_\pi$ in general, giving one indication of why $\pi \boxtimes \rho$ need not be equivalent to $\rho \boxtimes \pi$.

Let us now consider what happens for the quantum groups constructed in the earlier sections of this paper. Thus we have $A = C(G)$ and the corresponding quantum groups A_J. If π is a representation of G, then when it is expressed as $\sum m_{jk} \otimes a_{jk}$ the a_{jk}'s will be the usual (smooth) coefficient functions. This expression will then also be that for π as a representation of A_J. From this it is clear that the character of π as a representation of A_J will be given by the same function in A^∞ as the usual character of π as a representation of G.

Now characters of representations of groups are central functions, i.e. functions f such that $f(xyx^{-1}) = f(y)$ for $x, y \in G$. Let us see what happens to the deformed product when central functions are involved. Accordingly, let f and g be smooth functions on G, and assume that f is central. Then

$$\begin{aligned}(f \times_J g)(x) &= \int f(\eta(-Ks)x\eta(-Ku))g(\eta(-t)x\eta(v))e(s \cdot t + u \cdot v) \\ &= \int f(\eta(-K(s+u))x)g(\eta(-t)x\eta(v))e(s \cdot t + u \cdot v) \\ &= \int (\alpha_{J(s+u,0)}f)(x)(\alpha_{(t,v)}g)(x)e(s \cdot t + u \cdot v).\end{aligned}$$

We want to make the change of variables $s \mapsto s - u$. This can be justified by letting T be the operator on V defined by $T(s, u) = (s + u, 0)$. Then, if we omit the variable x, the above integral is

$$= \int (\alpha_{JT(s,u)}f)\,(\alpha_{(t,u)}g)\,e(s \cdot t + u \cdot v),$$

which by proposition 1.13 of **[Rf3]**,

$$= \int (\alpha_{J(s,u)}f)\,(\alpha_{T^t(t,v)}g)\,e(s \cdot t + u \cdot v).$$

But $T^t(t, v) = (t, t)$, and so by an application of proposition 1.11 of **[Rf3]** the above

$$= \int (\alpha_{(Ks,0)}f)\,(\alpha_{(t,t)}g)\,e(s \cdot t),$$

which, on putting back the variable x, is

$$= \int f(\eta(-Ks)x)g(Ad^{-1}_{\eta(t)}(x))\,e(s \cdot t),$$

where Ad denotes the usual operation of conjugation. We then see that if also g is central, then application of corollary 1.12 of **[Rf3]** yields $(fg)(x)$. That is, we have obtained:

5.1 PROPOSITION. *Let G be a compact Lie group, and let the product $\times_J$ on $C^\infty(G)$ be defined as in section 1. If f and g are smooth functions on G and if f is central, then*

$$(f \times_J g)(x) = \iint f(\eta(-Ks)x)g(Ad^{-1}_{\eta(t)}(x))\,e(s \cdot t).$$

If also g is central, then

$$f \times_J g = fg.$$

5.2 COROLLARY. *Let π and ρ be representations of the compact Lie group G, and so of A_J. Then the character of the representation $\pi \boxtimes \rho$ of A_J is $\chi_\pi \chi_\rho$ (pointwise product).*

PROOF. From the earlier comments, $\chi_{\pi \boxtimes \rho} = \chi_\pi \times_J \chi_\rho$, which by proposition 5.1 is $= \chi_\pi \chi_\rho$. □

Since the inner product of characters can be used to determine multiplicities of representations [W], and since we have seen that the restriction to $C^\infty(G)$ of the GNS inner product for Haar measure on A_J is the same as the usual inner product, it follows that the multiplicities of irreducible representations in $\pi \boxtimes \rho$ as representations of A_J will be the same as those for the group G. It also follows that $\rho \boxtimes \pi$ will be equivalent to $\pi \boxtimes \rho$. (In particular, the character ring of A_J will be isomorphic to that of A.) But this equivalence is not altogether trivial. In the case of groups, one demonstrates this equivalence by noticing that the usual flip operator, σ, from $\Xi \otimes Z$ to $Z \otimes \Xi$ defined by $\sigma(\xi \otimes \zeta) = \zeta \otimes \xi$ is an intertwining operator. But Woronowicz has shown (proposition 2.4 of [W]) that for a compact matrix quantum group A, if the flip operator is an intertwining operator for all pairs of representations, then the algebra A must be commutative, putting us back in the usual group case. Thus for the compact quantum groups defined in this paper, we can not expect the flip operator to be an intertwining operator. So it becomes interesting to try to find the intertwining operators.

For this purpose it is notationally easier to use another (equivalent) view of representations. Specifically, if π is a representation of G on Ξ, we view π as the linear map from Ξ to $\Xi \otimes A$ defined by

$$(\pi(\xi))_x = \pi_x(\xi).$$

If ρ is a representation on Z, then $\pi \boxtimes \rho$ is given on elementary tensors by the obvious extension of the deformed product on A, namely

$$(\pi \boxtimes \rho)(\xi \otimes \zeta)_x = \int \pi_{\eta(-Ks)x\eta(-Ku)}\xi \otimes \rho_{\eta(-t)x\eta(v)}\zeta \, e(s \cdot t + u \cdot v)$$

$$= \int (\pi_{\eta(-Ks)} \otimes \rho_{\eta(-t)})(\pi_x \otimes \rho_x)(\pi_{\eta(-Ku)} \otimes \rho_{\eta(v)})(\xi \otimes \zeta) \, e(s \cdot t + u \cdot v).$$

This suggests that we define the operator $R_{\pi\rho}$ in $B(\Xi \otimes Z)$ by

$$R_{\pi\rho} = \int \pi_{\eta(Ks)} \otimes \rho_{\eta(t)} \, e(s \cdot t).$$

To see what this operator does, consider the restriction of π and ρ to the subgroup H. Then Ξ and Z will decompose into weight spaces, that is, isotypic components for the various characters of H. Since we are taking H to be $\mathbf{R}^n/L$ via η, where L is the lattice $\mathbf{Z}^n$, we can identify as before the characters of H with the elements of L. Suppose now that ξ and ζ are in weight spaces corresponding to p and q

in L respectively, so that $\pi_{\eta(w)} = e(w \cdot p)\xi$ and similarly for ζ. Then

$$\begin{aligned} R_{\pi\rho}(\xi \otimes \zeta) &= \int \pi_{\eta(Ks)}\xi \otimes \rho_{\eta(t)}\zeta\, e(s \cdot t) \\ &= \int e(Ks \cdot p + t \cdot q + s \cdot t)\, \xi \otimes \zeta \\ &= e(Kp \cdot q)\, \xi \otimes \zeta. \end{aligned}$$

From this it is clear that $R_{\pi\rho}$ is unitary and that its inverse corresponds to $-K$. We also note that the above formula bears considerable analogy with proposition 2.22 of **[Rf3]**.

If we now look back at the formula for $\pi \boxtimes \rho$, we see that it can be rewritten in terms of $R_{\pi\rho}$ and the usual inner tensor product $\pi \otimes \rho$ as

$$(\pi \boxtimes \rho)_x = R_{\pi\rho}(\pi \otimes \rho)_x R_{\pi\rho}^{-1}.$$

Let $\sigma_{\pi\rho}$ denote the usual flip operator from $\Xi \otimes Z$ to $Z \otimes \Xi$, etc. A simple calculation using the skew-symmetry of K shows that

$$\sigma_{\rho\pi} R_{\rho\pi} \sigma_{\pi\rho} = R_{\pi\rho}^{-1}.$$

Then we will have, omitting x's,

$$R_{\pi\rho}^{-1}(\pi \boxtimes \rho)R_{\pi\rho} = \pi \otimes \rho = \sigma_{\rho\pi}(\rho \otimes \pi)\sigma_{\pi\rho}$$

$$= \sigma_{\rho\pi} R_{\rho\pi}^{-1}(\rho \boxtimes \pi) R_{\rho\pi} \sigma_{\pi\rho} = R_{\pi\rho}\sigma_{\pi\rho}(\rho \boxtimes \pi)\sigma_{\rho\pi} R_{\pi\rho}^{-1},$$

so that

$$\rho \boxtimes \pi = \sigma_{\rho\pi} R_{\pi\rho}^{-2}(\pi \boxtimes \rho) R_{\pi\rho}^{2} \sigma_{\rho\pi}.$$

Thus $\sigma_{\rho\pi} R_{\pi\rho}^{-2}$ is the desired intertwining operator.

In conclusion, we remark that by putting together the various ingredients above, it is not difficult to see that our quantum groups A_J will be compact matrix quantum groups in the sense of Woronowicz **[W]**.

6. Examples

The group $SU(2)$ contains no tori of dimension ≥ 2, and so can not provide an example of a quantum group constructed by our method. But the group $\mathbf{T} \times SU(2)$ does contain a 2-torus. We will see that the associated quantum groups have some interesting features. (We also remark that these quantum groups are not among those constructed in **[LS]**, and appear to be new.)

For a while it is notationally just as convenient to treat any group G of the form $\mathbf{T} \times M$ where M is a compact Lie group containing a 1-torus C. As our earlier H we will then take $H = \mathbf{T} \times C$. The Lie algebra of H will be identified with $\mathbf{R}^2$. We let ϕ denote the exponential map from $\mathbf{R}$ to C, with kernel the integers. Then if we view $\mathbf{T}$ as the circle group in the complex plane, the exponential map η for H can be written as $\eta(s,t) = (e(s), \phi(t))$. The action α of $V = \mathbf{R}^4$ on $C(G)$ is then defined by

$$\begin{aligned}(\alpha_{(s,u,t,v)}f)(\zeta, m) &= f(\bar{e}(s)\zeta e(t), \phi(-u)m\phi(v)) \\ &= f(e(t-s)\zeta, \phi(v-u)Ad_{\phi(-v)}(m)).\end{aligned}$$

We let $K = \theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for some real number θ, and set $J = K \oplus (-K)$ as before. It is clear that elements of V of the form $(s, 0, s, 0)$ are in the kernel of α. Let P be the projection onto the orthogonal complement of these elements, so that $\alpha = \alpha \circ P$. Then according to theorem 8.11 of [**Rf3**], $A_J = A_{PJP}$. Now

$$P(s,u,t,v) = \theta((u+v)/2, (t-s)/2, -(u+v)/2, (t-s)/2).$$

From this it is clear that vectors of the form $(s, u, s, -u)$ are in the kernel of PJP. Let Q be the projection on the orthogonal complement of these vectors. Note that $PQ = Q$. Thus $PJP = QJQ$, so that $A_J = A_{QJQ}$. Then according to theorem 8.7 of [**Rf3**], if we let $V' = QV$, and let α' and J' be the restrictions of α and QJQ to V', we have $A_J = A_{J'}$ in the evident sense. It is clear that one orthonormal basis for V' is $(1/\sqrt{2}, 0, -1/\sqrt{2}, 0)$, $(0, 1/\sqrt{2}, 0, 1\sqrt{2})$. If we identify $\mathbf{R}^2$ with V' by means of this basis, we see that α' is defined on $\mathbf{R}^2$ by

$$(\alpha'_{(s,u)}f)(\zeta, m) = (\alpha_{(s,u,-s,u)/\sqrt{2}}f)(\zeta, m) = f(\bar{e}(\sqrt{2}s)\zeta, Ad_{\phi(-u/\sqrt{2})}(m)).$$

It is easily calculated that in this basis $J' = 2\theta K$. Thus the deformed product is defined by

$$\begin{aligned}&(f \times_{J'} g)(\zeta, m) \\ &\quad = \int f(\bar{e}(2\theta\sqrt{2}u)\zeta, Ad_{\phi(\theta\sqrt{2}s)}(m))g(\bar{e}(\sqrt{2}t)\zeta, Ad_{\phi(-v/\sqrt{2})}(m))e(st+uv).\end{aligned}$$

We can make the change of variables to bring this to the simpler form

$$(f \times_{J'} g)(\zeta, m) = \int f(\bar{e}(2\theta u)\zeta, Ad_{\phi(2\theta s)}(m))g(\bar{e}(t)\zeta, Ad_{\phi(-v)}(m))e(st+uv).$$

(Justification is given by proposition 1.13 of [**Rf3**].) Thus if we change notation once again and let α be the action of $V = \mathbf{R}^2$ on A defined by

$$(\alpha_{(s,u)}f)(\zeta, m) = f(\bar{e}(s)\zeta, Ad_{\phi(u)}(m)),$$

and if we let $J = 2\theta K$, then our deformed algebra is just A_J. Of course, the comultiplication, coidentity, and coinverse, are just the usual ones.

Let us now consider our special case of $G = \mathbf{T} \times SU(2)$, and take C to be the diagonal matrices in $SU(2)$. For $u \in \mathbf{R}$ we let $\phi(u)$ be the diagonal matrix in $SU(2)$ whose top-left entry is $e(u)$. The elements of $SU(2)$ are parametrized by the pairs of complex numbers (z, w) such that $|z|^2 + |w|^2 = 1$, with the corresponding matrix having (z, w) as top row. It is easily seen that under this parametrization $Ad_{\phi(u)}(z, w) = (z, e(2u)w)$. Thus our final action is defined by

$$(\alpha_{(s,u)}f)(\zeta, (z, w)) = f(\bar{e}(s)\zeta, (z, e(2u)w)).$$

(Note that $(0, 1/2)$ is in the kernel of α.) Thus the orbits in G are 2-tori unless $w = 0$, in which case they are circles. But the Poisson bracket on the circles will be 0, so that, for $\theta \neq 0$, the symplectic leaves are just the single points of G for which $w = 0$, together with the 2-tori. But according to example 10.2 of **[Rf3]**, when we quantize the tori, we will obtain quantum tori. More specifically, according to proposition 5.8 of **[Rf3]**, quantum tori will be quotient algebras of the C*-algebra A_J. In particular, for θ irrational we will obtain simple quantum tori which are not of type I. Thus A_J itself will not be of type I. This shows that the algebra structure of our quantum groups is quite non-trivial. The possibility that a compact quantum group could fail to be of type I (because it is related to quantum tori) was first observed in **[L, LS]**.

Now let us consider $SU(2) \times SU(2)$, with H taken to be the product of the two diagonal circle groups. Then

$$H \subseteq \mathbf{T} \times SU(2) \subseteq SU(2) \times SU(2),$$

so that the inclusion of $\mathbf{T} \times SU(2)$ in $SU(2) \times SU(2)$ will be equivariant for the original action α corresponding to H. Thus the quantum group $(SU(2) \times SU(2))_J$ (in the evident sense) will have the quantum group $(\mathbf{T} \times SU(2))_J$ as a subgroup (i.e. the C*-algebra of the latter will be a quotient of that of the former). In particular, $(SU(2) \times SU(2))_J$ also will fail to be of type I for irrational θ. (Again, this example is not one of those discussed in **[LS]**, and seems to be new.)

If instead we consider $SU(3)$, we can embed $\mathbf{T} \times SU(2)$ by sending $SU(2)$ to $SU(2) \oplus I$ and $\mathbf{T}$ to the diagonal elements with 1 in the top-left entry, and this inclusion will be equivariant for the action α associated with $H \subseteq \mathbf{T} \times SU(2)$. So again the quantum group $(\mathbf{T} \times SU(2))_J$ will be a subgroup of the quantum group $SU(3)_J$, and the latter will fail to be of type I when θ is irrational. (This time, $SU(3)_J$ is one of the examples which is treated in **[LS]**.)

By slightly modifying the arguments at the beginning of this section, we can show that $(\mathbf{T} \times SO(3))_J$ also fails to be of type I when θ is irrational. Since either $\mathbf{T} \times SU(2)$ or $\mathbf{T} \times SO(3)$ can be embedded as a subgroup of any connected compact non-commutative Lie group bigger than $SU(2)$, we see by similar arguments that all such groups can be deformed into quantum groups which are not of type I. Of course, we can also use tori of higher dimension in most of these groups.

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