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Chapter 4 - Integration

Throughout the first four sections of this chapter μ will always denote a real or complex valued measure on a measurable space (X, S), and B will denote a Banach space. If μ takes complex values, then B will be assumed to be over the field of complex numbers. We will write scalars on the right as well as on the left of elements of B. All measurable functions will be assumed to have values in B unless the contrary is explicitly stated. By measurable sets we will always mean μ-measurable, and similarly for locally measurable sets. We will not distinguish between μ and its extension to a complete measure on μ-measurable sets.

A. Integrable simple functions

4.1 Definition. An integrable simple function (ISF) with respect to μ is a simple μ-measurable function f whose carrier has finite $|\mu|$-measure. Thus an ISF can be represented as $\sum_{i=1}^{n} b_i \chi_{E_i}$ where the E_i are disjoint μ-measurable sets of finite $|\mu|$-measure, and the b_i are in B.

We would like to define the integral of f with respect to μ to be $\sum b_i \mu(E_i)$, but we must first show that this quantity is independent of the representation of f.

4.2 Lemma. If $\sum^{m} b_i \chi_{E_i} = \sum^{n} c_j \chi_{F_j}$ as functions, where the E_i are disjoint and the F_j are disjoint, then $\sum^{m} b_i \mu(E_i) = \sum^{n} c_j \mu(F_j)$.

Proof. We may assume that the b_i and c_j are non-zero. Since the two functions are equal, their carriers are equal, and so $\oplus E_i = \oplus F_j$. It follows that $E_i = \oplus\limits_{j} E_i \cap F_j$ for each i, and that $F_j = \oplus\limits_{i} E_i \cap F_j$ for each j. Thus we have

$$\sum_{i,j} b_i \chi_{E_i \cap F_j} = \sum_{i} b_i \chi_{E_i} = \sum c_j \chi_{F_j} = \sum_{i,j} c_j \chi_{E_j \cap F_j} .$$

Since the $E_i \cap F_j$ are disjoint, we must have $b_i = c_j$ if $E_i \cap F_j \neq \emptyset$. It follows that

$$\sum b_i \mu(E_i) = \sum_{i,j} b_i \mu(E_i \cap F_j) = \sum_{i,j} c_j \mu(E_i \cap F_j) = \sum c_j \mu(F_j). /\!/$$

If f is an ISF represented by $\sum b_i \chi_{E_i}$, and if E is a locally μ-measurable set, then $\chi_E f$ is clearly an ISF represented by $\sum b_i \chi_{E \cap E_i}$.

4.3 Definition. If f is an ISF which is represented as $\sum b_i \chi_{E_i}$ with the E_i disjoint, and if E is a locally measurable set, then the integral of f over E with respect to μ is defined to be $\sum b_i \mu(E \cap E_i)$. It will be denoted by $\int_E f(x)d\mu(x)$, or $\int_E fd\mu$. If $E = X$ we may write $\int fd\mu$ instead of $\int_X fd\mu$.

4.4 Lemma. If f and g are ISF, then so is $f+g$, and for any locally measurable set E we have $\int_E (f+g)d\mu = \int_E fd\mu + \int_E gd\mu$.

Proof. It is clear that $f+g$ is an ISF. If f and g are represented by $\sum b_i \chi_{E_i}$ and $\sum c_j \chi_{F_j}$ respectively, where the E_i are disjoint and so are the F_j, and where $\oplus E_i = \oplus F_i$ but we allow the b_i or the c_j to have value 0, then $f = \sum\limits_{i,j} b_i \chi_{E_i \cap F_j}$ and $g = \sum\limits_{i,j} c_j \chi_{E_i \cap F_j}$,

so that $f+g = \sum_{i,j} (b_i+c_j)\chi_{E_i \cap F_j}$. Since the $E_i \cap F_j$ are disjoint, we have

$$\int_E (f+g)d\mu = \sum_{i,j} (b_i+c_j)\mu(E \cap E_i \cap F_j)$$

$$= \sum_{ij} b_i\mu(E \cap E_i \cap F_j) + \sum_{i,j} c_j\mu(E \cap E_i \cap F_j) = \int_E fd\mu + \int_E gd\mu.\; /\!/$$

4.5 Corollary. If f is an ISF, and if f is represented by $\sum b_i\chi_{E_i}$ where the E_i are no longer required to be disjoint, then for any locally measurable set E it is still true that $\int_E fd\mu = \sum b_i\mu(E \cap E_i)$.

We will let the reader supply the proofs of the following simple properties of the integral of ISF:

4.6 Lemma. Let f be an ISF and let E be a locally measurable set.

a) If r is a scalar, then rf is an ISF, and $\int_E (rf)d\mu = r \int_E fd\mu$,

b) $\|f(\cdot)\|$ is an ISF,

c) If f is non-negative real-valued, then $\int_E fd|\mu| \geq 0$,

d) If f and g are real-valued ISF, and if $f \geq g$, then
$$\int_E fd|\mu| \geq \int_E gd|\mu|,$$

e) If f is non-negative and $F \subseteq E$ is locally measurable, then
$$\int_F fd|\mu| \leq \int_E fd|\mu|.$$

f) If $E = F \oplus G$ with F and G locally measurable, then
$$\int_E fd\mu = \int_F fd\mu + \int_G fd\mu.$$

We remark that property a) together with Lemma 4.4 shows that the ISF form a vector space, on which the integral is a linear functional.

The next result will be of crucial importance in extending the domain of the integral to a wider class of functions.

4.7 Lemma. If f is an ISF and if E is a locally measurable set, then

$$\|\int_E f d\mu\| \leq \int_E \|f(x)\| d|\mu|(x).$$

Proof. If $f = \sum b_i \chi_{E_i}$ with the E_i disjoint, then $\|f(\cdot)\| = \sum \|b_i\| \chi_{E_i}(\cdot)$, and so

$$\|\int_E f d\mu\| = \|\sum b_i \mu(E \cap E_i)\| \leq \sum \|b_i\| |\mu(E \cap E_i)|$$

$$\leq \sum \|b_i\| |\mu|(E \cap E_i) = \int_E \|f(x)\| d|\mu|(x). /\!/$$

4.8 Definition. On the vector space of ISF we define the function $\|\cdot\|_1$ by $\|f\|_1 = \int \|f(x)\| d|\mu|(x)$. We will call $\|f\|_1$ the L^1-norm of f.

4.9 Lemma. The function $\|\cdot\|_1$ is a seminorm on the vector space of ISF, that is, it satisfies all of the properties of a norm except that if $\|f\|_1 = 0$ it does not follow that $f = 0$ (we can only conclude that $f(x) = 0$ a.e.).

Proof. If f and g are ISF, then

$$\|f+g\|_1 = \int \|f(x) + g(x)\| d|\mu|(x) \leq \int (\|f(x)\| + \|g(x)\|) d|\mu|(x)$$

$$= \int \|f(x)\| d|\mu|(x) + \int \|g(x)\| d|\mu|(x) = \|f\|_1 + \|g\|_1.$$

We will let the reader verify that $\|\cdot\|_1$ also satisfies the other properties of a seminorm.//

By means of the seminorm $\|\cdot\|_1$, we define a semimetric, d, on the vector space of ISF by $d(f, g) = \|f-g\|_1$. Thus, as mentioned in Chapter 0, d is a function which satisfies all of the properties of a metric except that if $d(f, g) = 0$ it does not necessarily follow that $f = g$. This semimetric defines a topology, that is, a collection of open sets, on the vector space of ISF, but this topology need not be Hausdorff, that is, limits of sequences need not be unique (but we will find that they are unique a.e.). Furthermore, the usual definition of uniform continuity with respect to a metric is equally applicable to semimetrics, and in the present case we have:

4.10 Lemma. For any locally measurable set E, the function $f \mapsto \int_E f d\mu$ is a uniformly continuous function on the vector space of ISF.

Proof. This is an immediate application of Lemma 4.7, for if f and g are ISF, then we have

$$\|\int_E f d\mu - \int_E g d\mu\| = \|\int_E (f-g)d\mu\|$$

$$\leq \int_E \|f(x) - g(x)\|d|\mu|(x) \leq \int_X \|f(x) - g(x)\|d|\mu|(x) = \|f-g\|_1.//$$

The usual definition of a Cauchy sequence in a metric space is equally applicable to a semimetric space, and, of course, a semimetric space is said to be complete if every Cauchy sequence has a limit (which need not be unique). One can form the completion of a semimetric space in the usual

way by taking equivalence classes of Cauchy sequences, and it will follow that uniformly continuous functions from a semimetric space into a complete space will extend to the completion. The objective of the next sections is to apply these ideas to the vector space of ISF with the L^1-norm, and to the uniformly continuous functions consisting of taking the integrals over locally measurable sets.

B. **Integrable functions and convergence in mean**

4.11 __Definition__. A sequence, f_n, of ISF will be said to be a __mean Cauchy sequence__ if it is a Cauchy sequence with respect to the L^1-norm, that is, if $\lim_{m,n} \|f_m - f_n\|_1 = 0$.

The completion of the vector space of ISF with respect to the L^1-norm is by definition the collection of equivalence classes of mean Cauchy sequences of ISF, where two mean Cauchy sequences, f_n and g_n, are said to be equivalent if $\lim_n \|f_n - g_n\|_1 = 0$. An ISF f will be identified with the element of the completion which is represented by the constant sequence each of whose terms is f. The main objective of this section is to show that each element of the completion can be identified with a measurable function, which is unique a.e. Thus the completion of the vector space of ISF can be identified with a certain space of measurable functions, whose elements will be called integrable functions. The principle tool for making this identification is the Riesz-Weyl Theorem (Theorem 3.26). To place us in a position to apply this theorem, we need:

4.12 __Lemma__. If a sequence of ISF is a mean Cauchy sequence, then it is Cauchy in measure.

Proof. Let f_n be a mean Cauchy sequence of ISF, and let $\epsilon > 0$ be given. Let

$$E_{mn} = \{x \in X : \|f_n(x) - f_m(x)\| \geq \epsilon\}.$$

Because the f_n are ISF, it is clear that each E_{mn} is a measurable set of finite measure, so that each $\chi_{E_{mn}}$ is an ISF. We must show that $\lim_{m,n} |\mu|(E_{mn}) = 0$. But it is clear that $\chi_{E_{mn}} \leq \|f_n(\cdot) - f_m(\cdot)\|/\epsilon$, and so by Lemma 4.6d we have

$$|\mu|(E_{mn}) = \int \chi_{E_{mn}} d|\mu| \leq \int \|f_n(x) - f_m(x)\|/\epsilon \; d|\mu|(x) = \|f_n - f_m\|_1/\epsilon.$$

The result now follows from the fact that the f_n form a mean Cauchy sequence.$/\!/$

It follows from this proposition and the Riesz-Weyl theorem that if f_n is a mean Cauchy sequence of ISF, then there is a measurable function, f, to which f_n converges in measure, and f is unique a.e. We would like to identify f with the element of the completion of the vector space of ISF corresponding to the sequence f_n. But before we can do this, there are two things that we must check. To begin with, we must be sure that two equivalent Cauchy sequences converge in measure to the same function.

4.13 Lemma. If f_n and g_n are equivalent mean Cauchy sequences, and if f_n converges in measure to f, then so does g_n.

Proof. The sequence f_1, g_1, f_2, g_2,... of ISF is easily seen to
be a mean Cauchy sequence, and so is Cauchy in measure by Lemma 4.12. But
it has a subsequence which converges in measure to f. By the same
argument as that used at the end of the proof of Theorem 3.26 it follows
that the whole sequence, and so the sequence g_n, converges in measure to
f. //

We must also show that a given measurable function cannot represent
more than one element of the completion of the vector space of ISF. That
is, we must show that:

4.14 Key Lemma. If f_n and g_n are two mean Cauchy sequences of
ISF which converge in measure to a given measurable function f, then
they are equivalent sequences, that is, $\lim_n \|f_n - g_n\|_1 = 0$.

Proof. It is easily seen that it suffices to show that f_n and g_n
have subsequences which are equivalent. Now by Corollary 3.27 there are
subsequences, f_{n_k} and g_{n_k} respectively, which converge to f a.u. We
propose to show that these subsequences are equivalent. Let $h_k = f_{n_k} - g_{n_k}$.
It is easily seen that h_n is a mean Cauchy sequence of ISF which
converges a.u. to the function 0. The proof of this lemma reduces to
showing that $\|h_n\|_1$ converges to 0.

Let $\varepsilon > 0$ be given, and choose N so that if m, n $\geq$ N then
$\|h_n - h_m\|_1 < \varepsilon/8$. We show that if n $\geq$ N then $\|h_n\|_1 < \varepsilon$. To do this, it
suffices to show that $\|h_N\|_1 < \varepsilon/2$, for then $\|h_n\|_1 \leq \|h_n - h_N\|_1 + \|h_N\|_1 < \varepsilon$.

Let E be the carrier of h_N. Since h_N is an ISF, we have $|\mu|(E) < \infty$. Furthermore, h_N is bounded, that is, there is a constant, c, such that $\|h_N(x)\| \leq c$ for all x. Since h_n converges to 0 a.u., we can find a measurable set $F \subseteq E$ such that $|\mu|(E-F) < \varepsilon/4c$ and h_n converges uniformly to 0 on F. Then

$$\int_{E-F} \|h_N(x)\| d|\mu|(x) \leq \int c\chi_{E-F} d|\mu| = c|\mu|(E-F) < \varepsilon/4.$$

Now, since h_n converges to 0 uniformly on F and since $|\mu|(F) < \infty$, we can find an integer $m \geq N$ such that $\int_F \|h_m(x)\| d|\mu|(x) < \varepsilon/8$. But

$$\left| \int_F \|h_N(x)\| d|\mu|(x) - \int_F \|h_m(x)\| d|\mu|(x) \right| \leq \int_F \left| \|h_N(x)\| - \|h_m(x)\| \right| d|\mu|(x)$$

$$\leq \int_F \|h_N(x) - h_m(x)\| d|\mu|(x) < \|h_N - h_m\|_1 < \varepsilon/8.$$

It follows that $\int_F \|h_N(x)\| d|\mu|(x) < \varepsilon/4$. Thus

$$\|h_N\|_1 = \int_{E-F} \|h_N(x)\| d|\mu|(x) + \int_F \|h_N(x)\| d|\mu|(x) \leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2,$$

as desired.//

4.15 Definition. A measurable function f will be said to be Bochner-Lebesgue integrable (or Bochner integrable, or just integrable) if there is a mean Cauchy sequence of ISF which converges to f in measure. If f is real-valued, we will say simply that f is Lebesgue integrable.

Thus the integrable functions are those which correspond to the points of the completion of the vector space of ISF. Of course, the integral

of the ISF over any fixed locally measurable set, being a uniformly
continuous function, extends to a function on the collection of integrable
functions (because we are assuming that the integral takes values in a
Banach, that is complete, space). Specifically if f_n is a mean Cauchy
sequence of ISF which converges to f in measure, then $\int_E f_n d\mu$ is a
Cauchy sequence (see Lemma 4.10). Furthermore its limit depends only on f,
for, if g_n is another mean Cauchy sequence which converges to f in
measure, then f_n and g_n are equivalent Cauchy sequences by Lemma 4.14,
and so $\int_E f_n d\mu$ and $\int_E g_n d\mu$ will also be equivalent Cauchy sequences
(again see Lemma 4.10), and so will have the same limit. Thus we are free
to make:

4.16 Definition. If f is an integrable function, then its Bochner-
Lebesgue integral (or Bochner integral, or integral) over a locally
measurable set E is defined to be the limit of $\int_E f_n d\mu$ where f_n is any
mean Cauchy sequence of ISF which converges to f in measure. This
limit will be denoted by $\int_E f d\mu$ or $\int_E f(x) d\mu(x)$. If $E = X$, we will
denote the integral simply by $\int f d\mu$, or $\int f(x) d\mu(x)$. If f is real-
valued, then $\int_E f d\mu$ is called simply the Lebesgue integral of f over E.

The Lebesgue integral was defined by Lebesgue in his thesis in 1902
for real-valued functions on the real line. The generalization of the
Lebesgue integral to real-valued functions on an arbitrary measure space was
the work of a number of mathematics. The extension of the Lebesgue
integral to functions with values in a Banach space was developed by
Bochner in 1933.

For the purpose of identifying integrable functions with the completion of the vector space of ISF it was important to use convergence in measure, since a mean Cauchy sequence of ISF always converges in measure as we have already seen, but need not converge a.e., much less a.u. However, for the purposes of defining integrable functions and their integrals, these other types of convergence are equally satisfactory, as the following proposition shows.

4.17 Proposition. Let f be a measurable function. Then the following three conditions are equivalent: There exists a mean Cauchy sequence f_n of ISF which converges to f

1) in measure   2) a.u.   3) a.e.

In all three cases f is integrable, and $\int_E f_n d\mu$ converges to $\int_E f d\mu$.

Proof. The fact that 1) implies 2) follows from Corollary 3.27. The fact that 2) implies 3) follows from Proposition 3.17. Finally, suppose that f_n converges, to f a.e. Since f_n is a mean Cauchy sequence, and so Cauchy in measure by Lemma 4.12, it follows from the Riesz-Weyl theorem (Theorem 3.26) that there is a subsequence, f_{n_k} which converges a.u., and so both a.e. and in measure, to a function g. But f_{n_k} converges to f a.e., and so $f = g$ a.e., so that f_{n_k} converges to f in measure. Thus f_{n_k} is a sequence satisfying 1). The proof of the rest of Proposition 4.17 should be clear from the above considerations.//

C. Properties of the Bochner-Lebesgue integral

In this section we show that most of the properties which we proved at the beginning of this chapter for ISF actually hold for integrable functions. In fact, most of the proofs consist simply of combining the properties for ISF with the definition of integrable functions in terms of mean Cauchy sequences of ISF.

4.18 Proposition. Let f and g be integrable functions, and let E be a locally measurable set. Then

a) $f+g$ is integrable, and $\int_E (f+g)d\mu = \int_E fd\mu + \int_E gd\mu$.

b) If r is a scalar, then rf is integrable and $\int_E (rf)d\mu = r\int_E fd\mu$.

c) $\|f(\cdot)\|$ is integrable, and $\|\int_E fd\mu\| \leq \int_E \|f(x)\|d|\mu|(x)$. (Note that a function is μ-integrable iff it is $|\mu|$-integrable).

d) If f is real-valued and if $f \geq 0$ a.e., then $\int_E fd|\mu| \geq 0$.

e) If f and g are real-valued and if $f \geq g$ a.e., then
$\int_E fd|\mu| \geq \int_E gd|\mu|$.

f) If f is real-valued and $f \geq 0$ a.e., and if $F \subseteq E$ is locally measurable, then $\int_F fd|\mu| \leq \int_E fd|\mu|$.

g) If $E = F \oplus G$ with F and G locally measurable, then
$\int_E fd\mu = \int_F fd\mu + \int_G fd\mu$.

Proof. We have seen that all of the above properties hold for ISF. To prove a) let f_n and g_n be mean Cauchy sequences of ISF which converge a.e. to f and g respectively. (We use condition 3 of

Proposition 4.17.) Then $f_n + g_n$ is easily seen to be a mean Cauchy sequence of ISF which converges to $f+g$ a.e., and so

$$\int_E (f+g)d\mu = \lim_n \int_E (f_n+g_n)d\mu$$

$$= \lim_n \int_E f_n d\mu + \lim_n \int_E g_n d\mu = \int_E f d\mu + \int_E g d\mu.$$

The proofs of the other facts are quite similar, and we leave them to the reader. We remark that to prove part d) one must show that if f_n is a mean Cauchy sequence of real-valued ISF which converges to the non-negative function f a.e., then so is $|f_n(\cdot)|$, so that f can be approximated by a sequence of non-negative ISF.//

Properties a) and b) above show that the integrable functions form a vector space, and that the integral over any locally measurable set is a linear function on this vector space.

4.19 Definition. We will denote the vector space of μ-integrable B-valued functions by $\mathcal{L}^1(X, S, \mu, B)$, (or appropriate abbreviations of this, such as $\mathcal{L}^1$, when this involves no ambiguity).

4.20 Definition. On $\mathcal{L}^1$ we define a function, $\|\cdot\|_1$, by $\|f\|_1 = \int \|f(x)\| d|\mu|(x)$ for all $f \in \mathcal{L}^1$. We will call $\|f\|_1$ the $\underline{L^1\text{-norm}}$ of f.

4.21 Proposition. The function $\|\cdot\|_1$ is a seminorm on $\mathcal{L}^1$.

Proof. The proof that $\|\cdot\|_1$ is a seminorm is very straightforward, and similar to the proof of Lemma 4.9, and so we leave it to the reader.//

We would like to see under what conditions an integrable function f
has the property that $\|f\|_1 = 0$, but we first need some preliminary results
which are of some independent interest.

4.22 Proposition. If f is an integrable function, then the carrier
of f is σ-finite.

Proof. Let f_n be a mean Cauchy sequence of ISF which converges to
f a.e., and let E_n be the carrier of f_n. Since f_n is an ISF, E_n
has finite measure. But the carrier of f is contained in the union of
all the E_n together possibly with a null set.∥

4.23 Proposition. Let f be a non-negative integrable function, and
let E be a measurable set. If $f \geq \chi_E$ a.e., then $|\mu|(E) < \infty$. In
particular, χ_E is integrable.

Proof. It is clear that, except for a null set, E is contained in the
carrier of f, and so is σ-finite. Thus there is a sequence, E_n, of
subsets of E of finite measure which increases up to E, so that
$|\mu|(E_n)$ increases to $|\mu|(E)$ by Proposition 2.16. But χ_{E_n} is integrable
for each n and $f \geq \chi_{E_n}$ a.e., so that according to Proposition 4.18e
we have

$$|\mu|(E_n) = \int \chi_{E_n} d|\mu| \leq \int f d|\mu|.$$

It follows that

$$|\mu|(E) \leq \int f d|\mu|.∥$$

4.24 Definition. A sequence, f_n, of integrable functions is said to be Cauchy in mean (or a mean Cauchy sequence) if it is a Cauchy sequence with respect to the L^1-norm, that is, if $\lim\limits_{m,n}\|f_n-f_m\|_1 = 0$. A sequence, f_n, of integrable functions is said to converge in mean to an integrable function f if $\lim\limits_{n}\|f-f_n\|_1 = 0$.

4.25 Proposition. If a sequence of integrable functions is Cauchy in mean, then it is Cauchy in measure. If a sequence of integrable functions converges in mean to an integrable function f, then it converges in measure to f.

Proof. For mean Cauchy sequences the proof is the same as the proof of Lemma 4.12 except that we must use Proposition 4.23 to be sure that $\chi_{E_{mn}}$ is integrable, where E_{mn} is defined as in the proof of Lemma 4.12. The proof for the case of a mean convergent sequence is just a slight variation of the proof for the case of a mean Cauchy sequence.⫽

4.26 Proposition. Let f be an integrable function. Then $\|f\|_1 = 0$ if and only if $f = 0$ a.e.

Proof. It is clear that if $f = 0$ a.e. then $\|f\|_1 = 0$, for the mean Cauchy sequence of ISF each of whose terms is the function 0 will converge to f a.e. Conversely, if $\|f\|_1 = 0$, then the sequence each of whose terms is the function 0 converges to f in mean, and so in measure by Proposition 4.25. But this sequence also converges to 0 in measure, and so $f = 0$ a.e. by Proposition 3.21.⫽

It is easily seen that the collection of functions whose L^1-norm is 0 forms a subspace of $\mathcal{L}^1$, and that $\|\cdot\|_1$ defines an actual norm on the factor space obtained by factoring $\mathcal{L}^1$ by this space of functions whose L^1-norm is zero (that is, on the space obtained by identifying any two functions the L^1-norm of whose difference is zero). Proposition 4.26 shows that this factor space consists simply of the equivalence classes of integrable functions which agree a.e.

4.27 Definition. The normed space consisting of the equivalence classes of integrable functions which agree a.e. will be denoted by $L^1(X, S, \mu, B)$, (or appropriate abbreviations of this, such as L^1). We will denote the norm on L^1 again by $\|\cdot\|_1$, and refer to it as the L^1-norm.

Our motivation for defining $\mathcal{L}^1$ was to obtain a completion of the vector space of ISF. Thus we would expect $\mathcal{L}^1$, and so L^1, to be complete, so that L^1 is a Banach space. We will now begin to verify this fact. The next few results essentially parallel the usual proof of the fact that the completion of a metric space is in fact complete, but they also yield some other useful pieces of information. The first of these results essentially shows that the L^1-norm on $\mathcal{L}^1$ gives the same metric as that which we would have obtained by the usual process of extending the metric on a metric space to a metric on its completion. Note that we did not quite define the L^1-norm by this process.

4.28 Lemma. If f_n is a mean Cauchy sequence of ISF which converges to f in measure (or a.u., or a.e.), then f_n converges to f in mean.

Proof. It is easily seen that for each fixed n the sequence $\|f_m(\cdot) - f_n(\cdot)\|$ is a mean Cauchy sequence of ISF which converges to $\|f(\cdot) - f_n(\cdot)\|$ in measure (or a.u., or a.e.), so that

$$\|f - f_n\|_1 = \int \|f(x) - f_n(x)\| d|\mu|(x)$$

$$= \lim_m \int \|f_m(x) - f_n(x)\| d|\mu|(x) = \lim_m \|f_m - f_n\|_1.$$

If for a given $\varepsilon > 0$ we choose N large enough so that $\|f_m - f_n\|_1 < \varepsilon$ whenever $m, n > N$, then for $n > N$ it follows that $\|f - f_n\|_1 < \varepsilon.$ //

4.29 Corollary. The vector space of ISF is dense in $\mathcal{L}^1$ with respect to the L^1 norm.

This corollary is, of course, just a special case of the fact that a metric space is dense in its completion.

4.30 Theorem. $\mathcal{L}^1$, and so L^1, is complete.

Proof. Let f_n be a mean Cauchy sequence of elements of $\mathcal{L}^1$. Using Corollary 4.29 choose for each n an ISF g_n such that $\|f_n - g_n\|_1 < 1/n$. It is easily seen that g_n is a mean Cauchy sequence, and so by the Riesz-Weyl theorem (Theorem 3.26) g_n converges in measure to a measurable function, f, which must thus be integrable. By Lemma 4.28 g_n converges to f in mean. It is easily seen that this implies that f_n converges to f in mean (basically because f_n and g_n are equivalent Cauchy sequences). //

We conclude this section with some consequences of Corollary 4.29.

4.31 __Proposition.__ Let f be an integrable function. Then for every $\varepsilon > 0$ there is a measurable set E of finite measure such that

$$\int_{X-E} \|f(x)\| \, d|\mu|(x) < \varepsilon.$$

__Proof.__ Given $\varepsilon > 0$, we can, by Corollary 4.29, find an ISF, g, such that $\|f-g\|_1 < \varepsilon$. Let E be the carrier of g, so that E is a measurable set of finite measure. Then

$$\int_{X-E} \|f(x)\| \, d|\mu|(x) = \int_{X-E} \|f(x) - g(x)\| \, d|\mu|(x) \leq \|f-g\|_1 < \varepsilon.\,/\!/$$

4.32 __Proposition.__ Let f be an integrable function, and let E be a locally measurable set. Then

$$\int_E f \, d\mu = \int \chi_E f \, d\mu.$$

__Proof.__ It is easily verified that this equality is true whenever f is an ISF. It is also easily seen that both sides of the equality are continuous functions of f with respect to the L^1-norm. From the fact that the ISF are dense it follows that the equality holds for all f.$/\!/$

D. __The Indefinite Integral of an Integrable Function__

Let f be an integrable function. In this section we will study the properties of $\int_E f \, d\mu$ as a function of E.

4.33 __Definition.__ Let $f \in \mathcal{L}^1(X, S, \mu, B)$. Then the set function μ_f defined on the σ-field of locally measurable sets by

$$\mu_f(E) = \int_E f \, d\mu$$

(and so having values in B) will be called the _indefinite integral_ of f.

4.34 Theorem. For every integrable function f the set function μ_f is a measure.

Proof. We must show that μ_f is countably additive. We will show this by first verifying it when f is an ISF, and then, in the general case, by approximating f by ISF.

Assume now that f is an ISF. If $f = b\chi_F$ for some $b \in B$ and some measurable set F of finite measure, then $\mu_f(E) = b\mu(E \cap F)$ for every locally measurable set E. Thus the countable additivity of μ_f follows immediately from the countable additivity of μ. But every ISF is just a finite sum of terms of the form $b\chi_F$, and so the theorem is true for any ISF.

The proof in the general case depends on the following inequality:

4.35 Lemma. If f and g are integrable functions, then $\|\mu_f(E) - \mu_g(E)\| \le \|f-g\|_1$ for every locally measurable set E.

Proof. $\|\mu_f(E) - \mu_g(E)\| = \|\int_E f d\mu + \int_E g d\mu\| \le \int_E \|f(x) - g(x)\| d|\mu|(x)$ $\le \|f-g\|_1 .\text{//}$

We return to the proof of Theorem 4.34. Let a locally measurable set E be given, and let $E = \bigoplus_{i=1}^{\infty} E_i$, with the E_i locally measurable. We have already seen that μ_f is finitely additive (Proposition 4.18g), and so to prove countable additivity it suffices to show that $\mu_f(\bigoplus_{i=1}^{n} E_i)$

converges to $\mu_f(E)$ as n goes to ∞. Let $\epsilon > 0$ be given. Choose an ISF g such that $\|f-g\|_1 < \epsilon/3$. We have just seen that μ_g is countably additive, and so we can choose N so that if $n > N$ then $\|\mu_g(E) - \mu_g(\overset{n}{\underset{i=1}{\oplus}} E_i)\| < \epsilon/3$. Then for $n > N$ we have

$$\|\mu_f(E) - \mu_f(\overset{n}{\underset{i=1}{\oplus}} E_i)\|$$

$$\leq \|\mu_f(E) - \mu_g(E)\| + \|\mu_g(E) - \mu_g(\overset{n}{\underset{i=1}{\oplus}} E_i)\| + \|\mu_g(\overset{n}{\underset{i=1}{\oplus}} E_i) - \mu_f(\overset{n}{\underset{i=1}{\oplus}} E_i)\|$$

$$\leq \|f-g\|_1 \quad + \quad \epsilon/3 \quad + \quad \|f-g\|_1 \leq \epsilon. /\!/$$

The next proposition shows how to compute the total variation of μ_f.

4.36 Proposition. Let f be an integrable function. Then for each locally measurable set E we have

$$|\mu_f|(E) = \int_E \|f(x)\| d|\mu|(x).$$

Proof. Let E be given, and suppose that $E = \overset{n}{\underset{i=1}{\oplus}} E_i$. Then

$$\sum_{i=1}^n \|\mu_f(E_i)\| = \sum_{i=1}^n \|\int_{E_i} f d\mu\| \leq \sum_{i=1}^n \int_{E_i} \|f(x)\| d|\mu|(x) = \int_E \|f(x)\| d|\mu|(x).$$

It follows that $|\mu_f|(E) \leq \int_E \|f(x)\| d|\mu|(x)$.

To prove the reverse inequality we first show that it is true when f is an ISF, and then obtain the general case from the fact that the ISF are dense in $\mathcal{L}^1$.

Suppose now that f is an ISF, and let $f = \sum^k b_i \chi_{F_i}$ where the F_i are disjoint. Let $F = \overset{k}{\oplus} F_i$. In the proof we must take into account the

definition of $|\mu|$, and so for each i suppose that measurable sets G_{ij} are given such that $E \cap F_i = \overset{n_i}{\underset{j}{\oplus}} G_{ij}$. Then

$$|\mu_f|(E) = |\mu_f|(E \cap F) \geq \sum_{i,j} \|\mu_f(G_{ij})\|$$

$$= \sum_{ij} \|\int_{G_{ij}} f d\mu\| = \sum_{ij} \|b_i\| |\mu(G_{ij})|$$

$$= \sum_{i} \|b_i\| (\sum_{j} |\mu(G_{ij})|).$$

It follows that

$$|\mu_f|(E) \geq \sum_{i} \|b_i\| |\mu|(F_i \cap E) = \int_E \|f(x)\| d|\mu|(x),$$

so that the theorem is true whenever f is an ISF.

To complete the proof we need the following Lemma:

4.37 Lemma. Let μ and ν be measures on a measurable space (X, S) with values in B (so that $\mu+\nu$ is defined by $(\mu+\nu)(E) = \mu(E) + \nu(E)$). Then $|\mu+\nu| \leq |\mu| + |\nu|$, in the sense that for each $E \in S$ we have $|\mu+\nu|(E) \leq |\mu|(E) + |\nu|(E)$. Consequently, $||\mu|(E) - |\nu|(E)| \leq |\mu-\nu|(E)$ for all $E \in S$.

Proof. If $E = \overset{n}{\underset{i}{\oplus}} E_i$, then

$$\sum_{}^{n} \|(\mu+\nu)(E_i)\| \leq \sum_{}^{n} \|\mu(E_i)\| + \sum_{}^{n} \|\nu(E_i)\| \leq |\mu|(E) + |\nu|(E).$$

The desired inequalities follow immediately.//

We now return to the proof of Theorem 4.36. Suppose now that f and g are integrable functions and that E is a locally measurable set. Then

$$\left| |\mu_f|(E) - |\mu_g|(E) \right| \leq |\mu_f - \mu_g|(E) = |\mu_{(f-g)}|(E)$$

$$\leq \int_E \|(f-g)(x)\| d|\mu|(x) \leq \int \|(f-g)(x)\| d|\mu|(x) = \|f-g\|_1.$$

It follows that $|\mu_f|(E)$ is a continuous function of f with respect to the L^1-norm for each fixed E, as is $\int_E \|f(x)\| d|\mu|(x)$. Since we have seen that the equality which we wish to prove is true for the ISF, and since these are dense in $\mathcal{L}^1$, it follows that the equality is true for all integrable functions. //

The indefinite integral of a μ-integrable function is closely related to the measure μ. The following definition and proposition provide one useful aspect of this relation.

4.38 Definition. Let m and μ be arbitrary measures on a measurable space (X, S). We say that m is strongly absolutely μ-continuous if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|m|(E) < \varepsilon$ for all $E \in S$ such that $|\mu|(E) < \delta$.

We will examine some related types of μ-continuity in chapter 7.

4.39 Proposition. Let μ be a scalar-valued measure and let f be a μ-integrable function. Then the indefinite integral, μ_f, of f (viewed as defined only as measurable sets) is strongly absolutely μ-continuous.

Proof. Again the proof involves approximating f by ISF. Let $\epsilon > 0$ be given. Choose an ISF g such that $\|f-g\|_1 < \epsilon/2$. Since g is an ISF it is bounded, so there is a constant, c, such that $\|g(x)\| \le c$ for all x. Let $\delta = \epsilon/(2c)$. Suppose now that E is a measurable set such that $|\mu|(E) < \delta$. Then

$$|\mu_f|(E) = \int_E \|f(x)\| d|\mu|(x) \le \int_E \|f(x) - g(x)\| d|\mu|(x) + \int_E \|g(x)\| d|\mu|(x)$$

$$\le \|f-g\|_1 + c|\mu|(E) \le \epsilon. /\!/$$

A typical application of this proposition can be found in the proof of Theorem 4.40.

E. Some convergence theorems

In this section we derive some theorems which are very useful in determining whether a sequence of integrable functions converges in mean to a given integrable function. As a corollary of the first of these theorems we will obtain a very useful characterization of integrable functions.

4.40 Theorem. (The Lebesgue Dominated Convergence Theorem). Let f_n be a sequence of integrable function which converges a.e. to a (necessarily measurable) function f. If there exists a real-valued integrable function g such that $\|f_n(x)\| \le g(x)$ a.e. for each n, then the f_n form a mean Cauchy sequence, f is integrable, and f_n converges to f in mean.

Proof. Let $\epsilon > 0$ be given. By Proposition 4.31 choose a measurable set E of finite measure such that $\int_{X-E} g(x) d|\mu|(x) < \epsilon/6$. (Note that $g \ge 0$ a.e. so that we don't need to take absolute values). Then for all

m and n we have

$$\int_{X-E} \|f_n(x) - f_m(x)\| d|\mu|(x) \leq \int_{X-E} \|f_n(x)\| d|\mu|(x) + \int_{X-E} \|f_m(x)\| d|\mu|(x)$$

$$\leq 2 \int g(x) d|\mu|(x) < \epsilon/3.$$

We saw in the last section that μ_g is strongly absolutely μ-continuous, so we choose $\delta > 0$ such that if $|\mu|(G) < \delta$ then $|\mu_g|(G) < \epsilon/6$. Since E has finite measure, it follows from Egoroff's theorem that the sequence f_n converges to f a.u. on E. Thus we can choose a measurable set $F \subseteq E$ such that $|\mu|(E-F) < \delta$ and the sequence f_n converges to f uniformly on F. Because of the way in which δ was chosen, and because $|\mu|(E-F) < \epsilon$, we have for all m and n

$$\int_{E-F} \|f_n(x) - f_m(x)\| d|\mu|(x) \leq \int_{E-F} \|f_n(x)\| d|\mu|(x) + \int_{E-F} \|f_m(x)\| d|\mu|(x)$$

$$\leq 2\int_{E-F} g(x) d|\mu|(x) = 2|\mu_g|(E-F) \leq \epsilon/3.$$

Finally, since the sequence f_n converges to f uniformly on F, we can find N such that if m, n > N, then $\|f_n(x) - f_m(x)\| \leq \epsilon/(3|\mu|(F))$ for all $x \in F$. Then for all m, n > N we have

$$\int_F \|f_n(x) - f_m(x)\| d|\mu|(x) \leq \int_F \epsilon/(3|\mu|(F)) d|\mu|(x) = \epsilon/3.$$

It follows that for all m, n > N we have

$$\|f_n - f_m\|_1 = (\int_{X-E} + \int_{E-F} + \int_F) \|f_n(x) - f_m(x)\| d|\mu|(x) \leq \epsilon,$$

so that the f_n form a mean Cauchy sequence as desired.

Since $\mathcal{L}^1$ is complete, the sequence f_n converges in mean to some integrable function. By Corollary 3.27 a subsequence of the f_n converges a.e. to this same function, and so this function must equal f a.e. Thus f is integrable and f_n converges to f in mean.∥

Bochner did not define integrable B-valued functions in the way that we did in Definition 4.15. Rather, he assumed that the theory of the Lebesgue integral for real-valued functions was known, and he defined integrable B-valued functions to be measurable functions which are dominated in norm by an integrable real-valued function. The next theorem shows that his definition is equivalent to Definition 4.15. But the main reason for our interest in this theorem is that it gives a very useful characterization of integrable functions (see for example the proof of Theorem 5.4).

4.41 Theorem. Let μ be a scalar-valued measure on (X, S), and let f be a μ-measurable B-valued function. If there is a real-valued μ-integrable function g such that $\|f(x)\| \leq g(x)$ a.e., then f is μ-integrable.

Proof. We must produce a mean Cauchy sequence of ISF which converges to f a.e. Since f is measurable, we can find a sequence, f_n, of simple measurable functions which converges to f a.e. For each n let

$$h_n(x) = \begin{cases} f_n(x) & \text{if } \|f_n(x)\| \leq 2g(x) \\ 0 & \text{if } \|f_n(x)\| > 2g(x). \end{cases}$$

Equivalently, if we let $E_n = \{x : 2g(x) - \|f_n(x)\| \geq 0\}$, then

$h_n = f_n \chi_{E_n}$. Since E_n is clearly locally measurable, it follows that each h_n is also a simple measurable function. It is easily seen that the sequence h_n converges to f a.e., and we have just arranged matters so that for each n we have $\|h_n(x)\| \le 2g(x)$ for all x. But it is easily seen by using Proposition 4.23 that this inequality implies that each h_n is integrable, which is something which did not need to be true for the f_n. It follows from the Lebesgue dominated convergence theorem (Theorem 4.40) that h_n is a mean Cauchy sequence and that f is integrable.//

The remaining theorems of this section involve the order properties of the real numbers, and so do not generalize to vector-valued functions. However these theorems are very useful for working with vector-valued functions, as we will see in the next chapter.

4.42 Theorem. (The Monotone Convergence Theorem). Let f_n be a sequence of real-valued integrable functions which is non-decreasing a.e. (that is, for each n we have $f_n(x) \ge f_{n-1}(x)$ a.e.). If the sequence of the norms of the f_n is bounded, that is, if there is a constant, c, such that $\|f_n\|_1 \le c$ for all n, then f_n is a mean Cauchy sequence, and there exists an integrable function f such that f_n converges to f a.e. and in mean. In particular, $\int f_n d\mu$ converges to $\int f d\mu$. The same result holds if instead the f_n are non-increasing a.e.

Proof. If the sequence f_n is non-increasing, then the sequence $-f_n$ is non-decreasing, and so it suffices to consider only the case in which the sequence f_n is non-decreasing. Then the numbers $\int f_n d|\mu|$ are

non-decreasing, and are clearly bounded above by c, and so they form a

Cauchy sequence. But $f_n - f_m$ is either positive a.e. or negative a.e.,

depending on whether or not n is larger than m, and so we have

$$\| f_n - f_m \|_1 = \int | f_n - f_m | d|\mu| | = |\int (f_n - f_m) d|\mu| | = |\int f_n d|\mu| - \int f_m d|\mu| |.$$

Thus f_n is a mean Cauchy sequence also. Since $\mathcal{L}^1$ is complete,

there exists an $f \in \mathcal{L}^1$ to which the sequence f_n converges in mean, and

hence in measure. But then by Corollary 3.27 there must be a subsequence

of the f_n which converges to f a.e. Since the f_n are nondecreasing,

it follows that the sequence f_n itself also converges to f a.e.//

4.43 Corollary. If f_n is a sequence of real-valued integrable func-

tions which is non-decreasing a.e. and which converges a.e. to a function

f, and if the sequence of the norms of the f_n is bounded, then f is

integrable and the sequence f_n converges to f in mean. In particular,

$\int f_n d\mu$ converges to $\int f d\mu$. The same result holds if instead the sequence

f_n is non-increasing a.e.

Proof. By Theorem 4.42 the sequence f_n will converge a.e. and in mean

to an integrable function h, which must of course equal f a.e.//

By making the appropriate definition of the integrals with respect to a non-negative measure of arbitrary non-negative extended real-valued functions which are measurable in a natural sense, we can conveniently state a corollary of the Monotone Convergence Theorem which is useful in certain situations. For a typical application of this corollary see the proof of Theorem 6.9.

4.44 Definition. Let μ be a non-negative measure, and let f be a non-negative extended real-valued function on X. Let E be the set where f takes the value ∞. We say that f is μ-measurable if E is μ-measurable and if f is μ-measurable in the usual sense on $X - E$. If f is μ-measurable, then we define the integral of f with respect to μ as follows. If $\mu(E) > 0$ then we set $\int f d\mu = \infty$. If $\mu(E) = 0$, then we can view f as being undefined on the null set E, and we then set $\int f d\mu = \infty$ if f is not integrable in the usual sense, whereas if f is integrable, then we let $\int f d\mu$ be the usual integral of f.

4.45 Corollary. If μ is a non-negative measure and if f_n is a non-decreasing sequence of μ-measurable non-negative extended real-valued functions which converges a.e. to a non-negative extended real-valued function f, then f is μ-measurable and the sequence $\int f_n d\mu$ converges to $\int f d\mu$.

Proof. We will let the reader verify that f must be μ-measurable, since in the applications we make of this corollary in these notes we will always know in advance that f is measurable. We remark next that if g and h are μ-measurable non-negative extended real-valued functions such that $g \geq h$ a.e. then it follows from Theorem 4.41 and Proposition 4.18e that $\int g d\mu \geq \int h d\mu$. Thus if $\int f_n d\mu = \infty$ for at least one n, then it is

clear that the sequence $\int f_n d\mu$ converges to $\infty = \int f d\mu$. On the other hand, if all of the f_n are integrable, then either the increasing sequence $\int f_n d\mu$ is unbounded, in which case it is clear again that it converges to $\infty = \int f d\mu$, or else it is bounded, in which case we can apply Corollary 4.43 to conclude that f also is integrable and that $\int f_n d\mu$ again converges to $\int f d\mu$. //

We remark that the corresponding statement for the case in which the sequence f_n is non-increasing is false in general unless at least one of the f_n is integrable (in which case Corollary 4.43 is applicable), for essentially the same reason as the fact that Proposition 2.17 is false unless at least one of the E_n has finite measure.

4.45 Theorem. (Fatou's Lemma). If μ is a non-negative measure, and if f_n is a sequence of non-negative integrable functions, then

$$\int_E (\liminf f_n)d\mu \leq \liminf \int f_n d\mu.$$

In particular, if the right hand is finite, then $\liminf f_n$ is integrable.

Proof. Let $g_n = \inf\{f_i : n \leq i < \infty\}$. Now g_n is the limit as m goes to ∞ of the decreasing sequence $\inf\{f_i : n \leq i \leq m\}$. But from Proposition 3.7 it follows that the infinum of a finite number of measurable functions is measurable. Thus g_n is measurable for each n. Since $\liminf f_n = \lim g_n$, we see that $\liminf f_n$ is also measurable. Now g_n is a non-decreasing sequence, and so $\lim \int g_n d\mu = \int \liminf f_n d\mu$ by Corollary 4.44. But $g_n \leq f_n$, and so $\int g_n d\mu \leq \int f_n d\mu$, for all n. Because the sequence $\int g_n d\mu$ is non-decreasing, we obtain the inequality

$$\liminf \int f_n d\mu \geq \lim \int g_n d\mu = \int \liminf f_n d\mu$$

as desired. //

Exercises

1. Show that if f is a real-valued continuous function on $[a, b]$, then its Riemann integral on $[a, b]$ is equal to its Lebesgue integral on $[a, b]$.

2. Let μ be Lebesgue measure on the whole real line, and let $B = L^1(\mu)$. Define a function f from $[0, 1]$ to B by $f(t) = \chi_{[t, t+1]}$ (or, more precisely, the equivalence class thereof). Show that f is continuous, and so, by Exercise 2 of Chapter 3, is also measurable. Show that f is μ-integrable, and compute $\int_{[0,1]} f d\mu$ (that is, determine what function in $L^1(\mu)$ it is).

3. Show that $\lim\limits_{a \to \infty} \int_0^a (\sin x)/x \, dx$ exists and is finite, but that $(\sin x)/x$ is not Lebesgue integrable on $[0, \infty)$.

4. Let B and B' be Banach spaces, and let T be a bounded linear transformation from B to B', that is, a linear transformation such that there is a constant K such that $\|T(b)\|_{B'} \leq K\|b\|_B$ for all $b \in B$. Let μ be a scalar measure on (X, S). Show that if f is a B-valued μ-measurable function, then $T \circ f$ is a B'-valued μ-measurable function, and that if f is integrable then so is $T \circ f$ and $\int_E (T \circ f) d\mu = T(\int_E f d\mu)$.

5. Show that if f is a μ-integrable scalar-valued function and $b \in B$, then the B-valued function $g(x) = f(x)b$ is μ-integrable, and $\int g d\mu = (\int f d\mu)b$.

6. Let f be a B-valued function which is integrable with respect to Lebesgue measure on $[0, 1]$. If $\int_0^t f d\mu = 0$ for all $t \in [0, 1]$, what can you conclude about f?

7. Show that if f is a μ-integrable B-valued function and if $\int f(x)_g(x)d\mu(x) = 0$ for all scalar-valued ISF g, then $f = 0$ a.e.

8. A left-continuous non-decreasing real-valued function α defined on R is said to be absolutely continuous if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\sum_{i=1}^{n} |\alpha(b_i) - \alpha(a_i)| < \varepsilon$ for every finite disjoint collection $\{[a_i, b_i) : i = 1,\ldots,n\}$ of interval for which $\sum_{i=1}^{n} (b_i - a_i) < \delta$. If α is any left-continuous non-decreasing function, show that α is absolutely continuous if and only if the corresponding Stieltjes-Lebesgue measure is strongly absolutely continuous with respect to Lebesgue measure.

9. Find a sequence f_n of integrable functions which converges a.e. to an integrable function f but such that $\int f_n d\mu$ does not converge to $\int f d\mu$. This shows that some condition such as domination by an integrable function is necessary in the Lebesgue dominated convergence theorem.

10. Evaluate $\lim_{n \to \infty} \int_{-\infty}^{\infty} (1+x^2+n(x \sin(1/x))^2)^{-1}dx$. If you knew only the theory of the Riemann integral how would you go about doing this?

11. Show that $(x^3 \sin x)/(x^{10} + \log(2+|x|))^{1/2}$ is Lebesgue integrable on $(-\infty, \infty)$.

12. The Mean Value Theorem: This theorem is stated in terms of convex sets, and so we will need to discuss first the rudiments of the theory of convex sets.

   a) A subset A of a vector space is called convex if the line segment joining any two points of A lies entirely in A, that is, whenever $a, b \in A$ then $ta + (1-t)b \in A$ for all $0 \leq t \leq 1$. Show that the intersection of any collection of convex sets in convex, so every set A of a vector space is contained in a smallest convex set called

12. (cont) the convex hull of A, which we will denote by c(A). Show that

$$c(A) = \{ \sum_{i=1}^{n} t_i a_i : a_i \in A, t_i \geq 0, \sum_{i=1}^{n} t_i = 1\}.$$

If A is contained in a Banach space and if $\overline{c}(A)$ denotes the closure of c(A), show that $\overline{c}(A)$ is the smallest closed convex set containing A. It is called the closed convex hull of A.

b) The Mean Value Theorem. Let μ be a non-negative measure, and let f be a B-valued μ-integrable function. Then for every measurable set E of strictly positive finite measure we have

$$\mu_f(E)/\mu(E) = (1/\mu(E)) \int_E f d\mu \in \overline{c}(er_E f) \subseteq \overline{c}(\text{range } f|_E),$$

(where $er_E f$ is as defined in exercise 6 of Chapter 3). This is the analogue of the fact that if a real valued continuous function f on [a, b] has values in the interval [m, M], then $(1/(b-a)) \int_a^b f(x)dx \in [m, M]$. Hint: As usual, prove this first for ISF and then approximate.

c) Prove conversely that if μ is a non-negative measure, f is a B-valued μ-integrable function, E is a locally measurable set and K is a closed subset of B, then if $(\mu_f(F)/\mu(F)) \in K$ for every measurable $F \subseteq E$ such that $0 < \mu(F) < \infty$, it follows that $er_E f \subseteq K$. In particular, if E is measurable then $f(x) \in K$ for almost all $x \in E$.

13. a) Show that if f is a μ-integrable B-valued function then the range of μ_f, that is $\{\mu_f(E) : E$ is a locally measurable set$\}$, is a relatively compact subset of B (that is, has compact closure).

13. b) Show that if μ is non-atomic then the closure of the range of μ_f

is convex. Hint: Use exercise 8 of Chapter 1.

c) Show that if the range of f is countable, then the range of μ_f

is closed (so that if μ is non-atomic the range of μ_f is convex).

Hint: Use exercise 15 of Chapter 1.

(A classical theorem of Liapounoff states that the range itself of a

non-atomic measure with values in a finite dimensional vector space

is convex, but no reasonably simple proof of this fact is known. The

corresponding statement is false in the infinite dimensional case.

In fact in exercise 19.f) we will give an example of a function, f,

which is Bochner integrable with respect to Lebesgue measure, μ,

but such that the range of μ_f is not convex (and so also not closed).

d) Find an example of a finite real-valued Borel measure whose range is

not convex.

14. If μ is a non-negative measure on (X, S) and if m is a B-valued

measure on (X, S) having the property that $m(E) = 0$ whenever $\mu(E) = 0$

(e.g. an indefinite integral), then for every locally measurable set E

define a subset A_E (or more precisely, $A_E(m)$) of B by

$A_E = \{m(F)/\mu(F) : F \subseteq E$ and $0 < \mu(F) < \infty\}$, called the "average range"

of m on E. Show that if f is a μ-integrable B-valued function, then

μ_f locally almost has compact average range, that is, for every measurable

set E with $\mu(E) < \infty$ and for every $\varepsilon > 0$ there exists $F \subseteq E$ with

$\mu(E-F) < \varepsilon$ such that A_F is relatively compact. This and the next

three exercises are closely related to the Radon-Nikodym theorem which

we will consider in Chapter 7. Hint: Use the results of exercise 5 of

14. (cont) Chapter 3, the mean value theorem (exercise 12 above), and the fact that if A is compact then so is $\overline{c}(A)$. To prove this last result, show that $\overline{c}(A)$ is totally bounded if A is.

15. A subset C of a vector space is called a cone (with vertex at 0) if $tc \in C$ for every $c \in C$ and every scalar $t \geq 0$. If A is a subset of a vector space, then the cone generated by A is defined to be $\{ta : a \in A$ and t is a non-negative scalar$\}$. Show that if μ is a non-negative measure and if f is a μ-integrable B-valued function, then locally μ_f somewhere has compact direction, that is, for every measurable set E with $0 < \mu(E) < \infty$ there exists $F \subseteq E$ and a compact subset K of B not containing 0 such that $\mu(F) > 0$ and $\mu_f(G)$ is in the cone generated by K for every $G \subseteq F$.

16. With μ, f and A_E as in exercise 14, show that μ_f locally somewhere has average range of small diameter, that is, for every measurable set E of finite measure and every $\varepsilon > 0$ there exists a measurable set $F \subseteq E$ such that $\mu(F) > 0$ and the diameter of A_F is less than ε. Recall that the diameter of a set $K \subseteq B$ is $\sup\{\|b-b'\| : b, b' \in K\}$.

17. Let μ be Lebesgue measure on $[0, 1]$, and let $B = L^1([0, 1], \mu)$. Define a function, $\dot{m}$, on the σ-field of Lebesgue measurable subsets of $[0, 1]$ with values in B, by $m(E) = \chi_E$ (or, more precisely, the equivalence class thereof). Show that m is a measure. Compute the total variation of m. Is m strongly absolutely μ-continuous? In spite of this, show that m cannot be the indefinite integral of any μ-integrable function, because it does not satisfy the properties described in exercises 14, 15 and 16 above. Also show that the closure of the range of m is not convex. (Compare with exercise 13.b) above.)

18. a) Let μ be a non-atomic non-negative measure, and let f be a μ-integrable B-valued function. Show that for any locally measurable set E the closure of $A_E(\mu_f)$ is a convex set which, in fact, is equal to $\overline{c}(er_E f)$. Hint: Use exercise 8 of Chapter 1.

   b) Show that if m is the measure of exercise 17, then the closure of $A_E(m)$ is not convex.

   c) Find a real-valued function, f, integrable with respect to Lebesgue measure μ such that $A_R(\mu_f)$ is not closed.

   d) Find a measure μ and a real-valued μ-integrable function for which the conclusion of part a) fails.

19. There are several ways of defining the integral of a function with values in a Banach space which are more general than the Bochner integral (in the sense that certain functions which are not measurable according to our definition can still be integrated), but which are less well understood than the Bochner integral. We will not try to give a precise description of these other definitions, but in this exercise and the next one we will give some suggestive examples of one of them (another can be found in exercise of Chapter 5).

   Let μ be Lebesgue measure on $[0, 1]$, and let $B = L^{\infty}([0, 1], \mu)$ (see exercise 7 of Chapter 3). Define a function, f, from $[0, 1]$ to B by $f(t) = \chi_{[0,t]}$ (or, more precisely, the equivalence class thereof).

   a) Show that f is not measurable.

   b) We would nevertheless like to have a meaning for $\int_E f d\mu$ for every μ-measurable set E. To obtain a feeling for what the value of this

19. b)(cont) integral should be, view f as a function with values in

B' = $L^1([0, 1], \mu)$, and show that now it is measurable, and in fact

integrable (see exercise 2). Thus μ_f is well defined as a measure

with value in B'.

c) Show that $\mu_f(E)$ is a continuous function (more precisely, the

equivalence class of $\mu_f(E)$ contains a continuous function) for every

measurable set E. Thus μ_f can be viewed as having values in

C([0, 1]) and so in B (since C([0, 1]) can be identified with a

closed subspace of $L^\infty([0, 1], \mu)$). Show that viewed in this way μ_f

is still a measure. What is its total variation?

d) Viewing f as having values in B, one might then expect $\int_E f d\mu$

to be $\mu_f(E)$ viewed as an element of B. The way to justify this is

as follows: For g $\in$ B and h $\in$ B' note that gh is integrable,

and let $<g, h> = \int gh \, d\mu$. Show that for any h $\in$ B' the function

t $\mapsto$ $<f(t), h>$ is measurable (so we say that f is weakly measurable

for the duality $<$, $>$). Then verify that

$$<\mu_f(E), h> = \int_E <f(t), h> d\mu(t)$$

for every measurable set E and every h $\in$ B'. Hint: Show it first

when h is an ISF. To do this note that g $\mapsto$ $<g, h>$ is a bounded

linear functional and use exercise 4. (We thus say that f is weakly

integrable for the duality, $<$, $>$, between B and B', and that

the weak integral of f over any set E is the vector $\mu_f(E)$. Thus

μ_f viewed as having values in B can be considered to be the indefinite

integral of the weakly integrable function f.)

19. e) Viewing μ_f as having values in $C([0, 1])$, show that it is not

the indefinite integral of any Bochner integrable function with values

in $C([0, 1])$, (basically because thus function would have to be f).

This illustrates the fact that a measure with values in a certain

space (e.g. $C([0, 1])$) which is not an indefinite integral, may be

an indefinite integral in at least a weak sense if the space in which

it is viewed as taking its values is enlarged (e.g. to B). Note that

the range of μ_f is relatively compact. (Why?)

f) Show that the range of μ_f, even when viewed as being in B', is not

closed. Hint: Show that the function $\alpha(t) = t/2$ is not in the

range of μ_f but is in the convex hull of the range of μ_f (and so

must be in the closure of the range of μ_f). Thus Liapounoff's theorem

(see comments in exercise 13.c)) is not true for even the indefinite

integrals of continuous Bochner integrable functions. (It would be

interesting to have a characterization of those functions which are in

the range of μ_f.)

20. Let μ be Lebesgue measure on $[0, 1]$, and let f_1, f_2,... be the

characteristic functions of the intervals $[0, 1/2]$, $[1/2, 1]$, $[0, 1/3]$,

$[1/3, 2/3]$, $[2/3, 1]$, $[0, 1/4]$,... (you may have discovered this

sequence of sets in answer to exercise 4.c) of Chapter 3). Let f be

the function on $[0, 1]$ with values in ℓ^∞ whose value at $t \in [0, 1]$

is the sequence $(f_1(t), f_2(t),...)$.

a) Show that f is not measurable. Hint: Egoroff's theorem is helpful.

b) Nevertheless we would like to integrate f. In fact we would expect

20. b)(cont) that

$$\int_E f d\mu = (\int_E f_1 d\mu, \int_E f_2 d\mu, \ldots)$$

for any measurable set E. If we define $\mu_f(E) = \int_E f d\mu$ by the above formula, show that it is a measure, and has finite total variation.

c) We justify the definition of the integral of f suggested above in a manner entirely analogous to the justification provided in exercise 19.d). Given $a \in \ell^\infty$ and $b \in \ell^1$ note that their pointwise product, ab, is in ℓ^1, and let $<a, b> = \sum_{i=1}^\infty a_i b_i$. Show that for every $b \in \ell^1$ the function $t \to <f(t), b>$ is measurable (so we say that f is weakly measurable for the duality $< , >$). Then verify that

$$<\mu_f(E), b> = \int_E <f(t), b> d\mu(t)$$

for every measurable set E and every $b \in \ell^1$. (We thus say that f is weakly integrable for the duality $< , >$, and that the weak integral of f over any set E is the vector $\mu_f(E)$.)

d) A Banach space of some interest is the subspace of ℓ^∞ consisting of all sequences which converge to 0. This space is traditionally denoted by c_0. Show that the range of μ_f is contained in c_0, and in fact is contained in a compact subset of c_0. But show that μ_f cannot be the indefinite integral of a function with values in c_0. This gives another example of the phenomenon described in exercise 19.e).

21. Let μ be Lebesgue measure on $\mathbb{R}$. The Fourier transform of any function $f \in \mathcal{L}^1(\mu, c)$ is defined to be the function $\mathcal{F}(f)$ on $\mathbb{R}$ whose value at $t \in \mathbb{R}$ is

$$\mathcal{F}(f)(t) = \int f(x)\exp(ixt)dx.$$

a) Prove that the Fourier transform of any function in $\mathcal{L}^1$ is a bounded uniformly continuous function.

b) Prove that if both f and $x \mapsto xf(x)$ are in $\mathcal{L}^1$, then $\mathcal{F}(f)$ is differentiable and

$$\frac{d\,\mathcal{F}(f)}{dt}(s) = \mathcal{F}(x \mapsto ixf(x))(s).$$

22. Let G be a topological group, such as $\mathbb{R}$, and let B be a Banach space. Then a representation of G on B is a map, R, from G to the set of bounded invertible linear operators on B (the definition of a bounded operator was given in exercise 4 above) such that $R(s+t) = R(s)R(t)$ for all $s, t \in G$ (composition of operators). The representation R is called strongly continuous if $s \mapsto R(s)b$ is a (norm) continuous function on G for each $b \in B$, and it is called uniformly bounded if there is a constant c such that $\|R(s)b\| \le c\|b\|$ for all $t \in G$ and $b \in B$.

a) If $G = \mathbb{R}$, if μ is Lebesgue measure on $\mathbb{R}$ if $B = L^1(\mu, B')$ where B' is any Banach space, and if R is defined by $(R(s)f)(t) = f(t-s)$, show that R is a uniformly bounded continuous representation of R. Hint: Use exercise 14.a) of Chapter 1.

22. b) Let R be a uniformly bounded strongly continuous representation of R on a Banach space B, and let μ be Lebesgue measure on R. For each $f \in L^1(\mu, c)$ define a function, R_f, on B by

$$R_f(b) = \int f(s)(R(s)b)d\mu(s)$$

for every $b \in B$. Show that R_f is a bounded linear operator on B.

c) With R and R_f as in part b, show that there is a sequence, f_n, of elements of $L^1(\mu)$ of norm one such that $\lim R_{f_n} b = b$ for all $b \in B$. Hint: Try an "approximate δ-function".

This exercise will be further developed in the exercises of Chapters 5 and 6.